



RESEARCH ARTICLE

# A note on the splittings of finitely presented Bestvina–Brady groups

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## Abstract

We show that when a finitely presented Bestvina–Brady group splits as an amalgamated product over a subgroup  $H$ , its defining graph contains an induced separating subgraph whose associated Bestvina–Brady group is contained in a conjugate of  $H$ .

## 1. Introduction

Let  $\Gamma$  be a finite simplicial graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma)$ . The associated *right-angled Artin group* (RAAG), denoted by  $A_\Gamma$ , is generated by  $V(\Gamma)$ , and two generators  $v$  and  $w$  commute whenever they are connected by an edge. A common question in group theory is: when does a group split as an amalgamated product or an HNN extension over a subgroup? For RAAGs, the splittings over infinite cyclic subgroups and abelian subgroups were characterized by Clay [3] and by Groves and Hull [6], respectively. Recently, Hull [7] generalized the splittings of RAAGs over abelian subgroups to non-abelian subgroups.

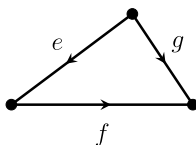
Let  $\phi: A_\Gamma \rightarrow \mathbb{Z}$  be a homomorphism that sends all the generators to 1. The kernel of  $\phi$  is called the *Bestvina–Brady group* and is denoted by  $BB_\Gamma$ . We only focus on finitely presented Bestvina–Brady groups, which is equivalent to saying that the flag complexes on the defining graphs are simply connected [1]. The author in [2] characterized the splittings of finitely presented Bestvina–Brady groups over abelian subgroups. In this note, we prove a result for the splittings of finitely presented Bestvina–Brady groups over non-abelian subgroups.

**Theorem 1.1.** *Let  $\Gamma$  be a finite simplicial graph with no cut vertices and whose associated flag complex is simply connected. Suppose that  $BB_\Gamma$  splits as an amalgamated product over a subgroup  $H$ . Then  $\Gamma$  contains an induced subgraph  $\Lambda$  that separates  $\Gamma$  and  $BB_\Lambda$  is contained in a conjugate of  $H$ .*

In other words, Theorem 1.1 says that if  $BB_\Gamma$  acts on a tree which is not a line, then there is an induced subgraph  $\Lambda$  of  $\Gamma$  such that  $\Lambda$  separates  $\Gamma$  and  $BB_\Lambda$  fixes an edge of  $T$ .

If  $\Gamma$  contains an induced subgraph  $\Lambda$  such that  $\Gamma \setminus \Lambda$  has more than one connected component, then  $A_\Gamma$  splits over  $A_\Lambda$  and  $BB_\Gamma$  splits over  $BB_\Lambda$ . In the language of Bass–Serre Theory, all the vertex groups and edge groups of this splitting for  $A_\Gamma$  are finitely presented, but this is not always the case for the corresponding splitting for  $BB_\Gamma$ ; see Example 3.3. We remark that  $\Gamma$  contains a cut vertex if and only if  $BB_\Gamma$  splits as a free product.

The proof of Theorem 1.1 uses basic facts about groups acting on trees, and the idea of the proof is similar to those in [6], [2], and [7].



**Figure 1.** A directed triangle.

## 2. Preliminaries

### 2.1. Bestvina–Brady groups

Let  $\Gamma$  be a finite simplicial graph. The main result in [1] states that  $\Gamma$  is connected if and only if  $\text{BB}_\Gamma$  is finitely generated, and that the flag complex on  $\Gamma$  is simply connected if and only if  $\text{BB}_\Gamma$  is finitely presented. In the latter situation, Dicks and Leary found an explicit presentation:

**Theorem 2.1.** ([5, Corollary 3]) *Let  $\Gamma$  be a finite simplicial directed graph. If the flag complex on  $\Gamma$  is simply connected, then  $\text{BB}_\Gamma$  is generated by all the directed edges of  $\Gamma$ , and the relators are of the form  $ee^{-1}$ , where  $e^{-1}$  denotes the edge  $e$  with the opposite orientation, and  $ef = g = fe$ , where  $e, f$ , and  $g$  form a directed triangle; see Figure 1.*

### 2.2. Group acting on trees

Let  $G$  be a group acting on a tree  $T$  without inversions. We always assume that actions are minimal and nontrivial. An element  $g \in G$  is called *elliptic* if it fixes a point in  $T$ ; otherwise, it is called *hyperbolic*. When  $g \in G$  is elliptic, the set of points fixed by  $g$  is a subtree of  $T$  and is denoted by  $\text{Fix}(g)$ . When  $g \in G$  is hyperbolic, it fixes a line in  $T$  on which it acts by translation. This line is called the *axis* of  $g$  and is denoted by  $\text{Axis}(g)$ .

**Lemma 2.2.** ([4, Lemma 1.1, Corollary 1.5], [6, Lemma 1.1]) *Let  $G$  be a group acting on a tree, and let  $g$  and  $h$  be commuting elements in  $G$ .*

- (1) *If  $h$  is hyperbolic, then  $\text{Axis}(h) \subseteq \text{Fix}(g)$ .*
- (2) *If both  $g$  and  $h$  are hyperbolic, then  $\text{Axis}(g) = \text{Axis}(h)$ .*
- (3) *If both  $g$  and  $h$  are elliptic, then  $\text{Fix}(g) \cap \text{Fix}(h) \neq \emptyset$ .*

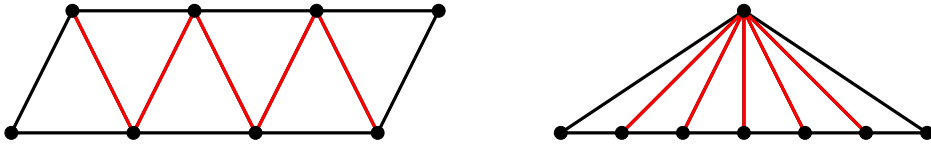
## 3. Proof of Theorem 1.1

Throughout this section, we identify edges of a graph with elements in the associated Bestvina–Brady group.

**Definition 3.1.** *Let  $\Gamma$  be a finite simplicial graph. A **triangle path**  $P_\Delta$  between two distinct edges  $e$  and  $f$  in  $\Gamma$  is a sequence of triangles  $\Delta_1, \dots, \Delta_n$  such that*

- *the edges  $e$  and  $f$  are contained in  $\Delta_1$  and  $\Delta_n$ , respectively;*
- *the triangles  $\Delta_i$  and  $\Delta_{i+1}$  share a unique edge for each  $i = 1, \dots, n - 1$ ;*
- *the triangles  $\Delta_i$  and  $\Delta_j$  do not share a common edge if  $j \neq i - 1$  or  $j \neq i + 1$ .*

*The edge shared by  $\Delta_i$  and  $\Delta_{i+1}$  is called an **intermediate edge**.*



**Figure 2.** Two triangle paths and their intermediate edges (red edges). The graph on the right illustrates that all the triangles in a triangle path can share a common vertex.

Two examples of triangle paths and their intermediate edges are given in Figure 2. Notice that triangle paths between two edges may not be unique, and all the triangles in a triangle path can share a common vertex.

Recall that a subgraph  $\Lambda$  of  $\Gamma$  separates two vertices (or two edges) if these two vertices (or edges) lie in different connected components of  $\Gamma \setminus \Lambda$ .

**Lemma 3.2.** *Let  $\Gamma$  be a finite simplicial graph without cut vertices and whose associated flag complex is simply connected. Let  $\Lambda$  be an induced subgraph of  $\Gamma$ . If every triangle path between  $e_1$  and  $e_2$  in  $E(\Gamma)$  has an intermediate edge in  $E(\Lambda)$ , then  $\Lambda$  separates  $e_1$  from  $e_2$ .*

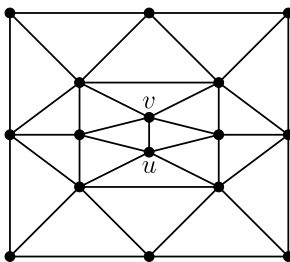
*Proof.* Since  $\Gamma$  has no cut vertices and the associated flag complex is simply connected, every edge of  $\Gamma$  is contained in a triangle, and there is a triangle path between any two edges in  $\Gamma$ . Let  $e_1 = (u_1, v_1)$  and  $e_2 = (u_2, v_2)$ . Suppose that  $\Lambda$  does not separate  $e_1$  from  $e_2$ . Then, without loss of generality, there is an edge path  $p$  between the vertices  $u_1$  and  $u_2$ . This edge path  $p$  is contained in some triangle path  $P_\Delta$  between  $e_1$  and  $e_2$ , and therefore, every vertex of  $p$ , possibly except for the two end vertices, is a vertex of an intermediate edge of  $P_\Delta$ . Since  $\Lambda$  contains an intermediate edge of  $P_\Delta$ , removing  $\Lambda$  will disconnect the path  $p$ . Thus, the path  $p$  cannot exist. Hence, the subgraph  $\Lambda$  separates  $e_1$  from  $e_2$ .  $\square$

We now prove the main theorem.

*Proof of Theorem 1.1.* Let  $\text{BB}_\Gamma$  act on a tree  $T$ , and let  $e_h \in E(\Gamma)$  be hyperbolic. Since  $\text{BB}_\Gamma$  splits as an amalgamated product, the tree  $T$  is not a path. Let  $\Lambda$  be the induced subgraph of  $\Gamma$  such that  $E(\Lambda)$  consists of all the elliptic edges of  $\Gamma$  that fix  $\text{Axis}(e_h)$  pointwise. Let  $e$  be an edge of  $\text{Axis}(e_h)$ . Then  $\text{BB}_\Lambda$  fixes  $e$  and is contained in a conjugate of  $H$ .

We now show that  $\Lambda$  separates  $\Gamma$ . We claim that there is an edge  $f \in E(\Gamma) \setminus E(\Lambda)$  such that every triangle path between  $e_h$  and  $f$  has an intermediate edge in  $E(\Lambda)$ . Suppose to the contrary that for every edge  $f' \in E(\Gamma) \setminus E(\Lambda)$ , there is a triangle path  $P_\Delta = \{\Delta_1, \dots, \Delta_n\}$  between  $e_h$  and  $f'$  such that none of its intermediate edges is in  $E(\Lambda)$ . Denote by  $\{f_1, \dots, f_{n-1}\}$  the set of intermediate edges of  $P_\Delta$ , where  $f_i$  is the edge shared by  $\Delta_i$  and  $\Delta_{i+1}$ . Since  $e_h$  and  $f_1$  are contained in  $\Delta_1$ , they are commuting elements. Since  $e_h$  is hyperbolic, the element  $f_1$  is also hyperbolic. Otherwise, it follows from Lemma 2.2 (1) that  $f_1 \in E(\Lambda)$ . Therefore, Lemma 2.2 (2) implies  $\text{Axis}(e_h) = \text{Axis}(f_1)$ . Similarly, since  $f_1$  is hyperbolic and commuting with  $f_2$ , the element  $f_2$  is also hyperbolic and has the axis  $\text{Axis}(f_2) = \text{Axis}(f_1)$ . Continuing with the same argument, we have that the edges  $e_h, f_1, \dots, f_{n-1}, f'$  are all hyperbolic with the same axis  $\text{Axis}(e_h)$ . Now, every edge in  $E(\Gamma) \setminus E(\Lambda)$  is hyperbolic and has the axis  $\text{Axis}(e_h)$ , which is fixed by  $E(\Lambda)$  pointwise. Thus, the set  $E(\Gamma)$  fixes  $\text{Axis}(e_h)$ , contradicting the fact that  $T$  is not a path. This proves the claim. Therefore, the subgraph  $\Lambda$  separates  $e_h$  from  $f$  by Lemma 3.2.

Next, suppose that every edge of  $\Gamma$  is elliptic. Since the action of  $\text{BB}_\Gamma$  on  $T$  has no global fixed points, it follows from [8, p.64, Corollary 2] that there are two edges  $e_\alpha$  and  $e_\beta$  in  $E(\Gamma)$  such that the intersection  $\text{Fix}(e_\alpha) \cap \text{Fix}(e_\beta)$  is empty. Let  $L$  be the geodesic in  $T$  between  $\text{Fix}(e_\alpha)$  and  $\text{Fix}(e_\beta)$ , and let  $e$  be an edge of  $L$ . Let  $\Lambda$  be an induced subgraph of  $\Gamma$  such that every edge of  $\Lambda$  fixes  $e$ . Then,  $\text{BB}_\Lambda$  fixes  $e$  and is contained in a conjugate of  $H$ . We now show that  $\Lambda$  separates  $\Gamma$ . Let  $P_\Delta = \{\Delta_1, \dots, \Delta_n\}$  be a triangle path between  $e_\alpha$  and  $e_\beta$ . Denote by  $\{f_1, \dots, f_{n-1}\}$  the set of intermediate edges of  $P_\Delta$ , where  $f_i$  is the edge shared by  $\Delta_i$  and  $\Delta_{i+1}$ . For convenience, we write  $f_0 = e_\alpha$  and  $f_n = e_\beta$ . Since  $f_i$  and  $f_{i+1}$  are contained in the



**Figure 3.** A splitting of a finitely presented Bestvina–Brady group over a finitely generated but not finitely presented subgroup.

triangle  $\Delta_{i+1}$ , they are commuting elliptic elements. It follows from Lemma 2.2 (3) that the intersection  $\text{Fix}(f_i) \cap \text{Fix}(f_{i+1})$  is nonempty for  $i = 0, \dots, n - 1$ . Thus, there is a path  $L'$  in  $T$  from  $\text{Fix}(f_0) = \text{Fix}(e_\alpha)$  to  $\text{Fix}(f_n) = \text{Fix}(e_\beta)$  lying entirely in  $\bigcup_{i=0}^n \text{Fix}(f_i)$ . Since  $T$  is a tree, we have  $L' = L$ . Then the edge  $e$  belongs to  $\text{Fix}(f_i)$  for some  $i$ . That is, there is an intermediate edge  $f_i$  of  $P_\Delta$  that belongs to  $E(\Lambda)$ . Since the choice of the triangle path  $P_\Delta$  between  $e_\alpha$  and  $e_\beta$  is arbitrary, the subgraph  $\Lambda$  separates  $e_\alpha$  from  $e_\beta$  by Lemma 3.2.  $\square$

We end this section with one example.

**Example 3.3.** Let  $\Gamma$  be the graph shown in Figure 3. Let  $W$  be the set of vertices that are adjacent to either  $u$  or  $v$  but different from  $u$  and  $v$ . Let  $\Lambda$ ,  $\Gamma_1$ , and  $\Gamma_2$  be the induced graphs on  $W$ ,  $V(\Gamma) \setminus \{u, v\}$ , and  $W \cup \{u, v\}$ , respectively. Then  $\Lambda$  is an induced separating subgraph of  $\Gamma$  and  $\text{BB}_\Gamma \cong \text{BB}_{\Gamma_1} *_{\text{BB}_\Lambda} \text{BB}_{\Gamma_2}$ . However, the groups  $\text{BB}_\Gamma$  and  $\text{BB}_{\Gamma_2}$  are finitely presented, while  $\text{BB}_{\Gamma_1}$  and  $\text{BB}_\Lambda$  are finitely generated but not finitely presented.

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