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AN ENTIRE FUNCTION WHICH HAS WANDERING DOMAINS

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Abstract

Let $f(z)$ denote a rational or entire function of the complex variable z and $f_n(z)$, $n = 1, 2, \cdots$, the n-th iterate of f. Provided f is not rational of order 0 or 1, the set $\mathfrak{C}$ of those points where $\{f_n(z)\}$ forms a normal family is a proper subset of the plane and is invariant under the map $z \to f(z)$. A component G of $\mathfrak{C}$ is a wandering domain of f if $f_k(G) \cap f_n(G) = \emptyset$ for all $k \geqq 1$, $n \geqq 1$, $k \neq n$. The paper contains the construction of a transcendental entire function which has wandering domains.

The theory of the iteration of a rational or entire function $f(z)$ of the complex variable z deals with the sequence of natural iterates $f_n(z)$ defined by

$$f(z) = z, \quad f_1(z) = f(z), \quad f_{n+1}(z) = f_1(f_n(z)), \qquad n = 0, 1, 2, \cdots.$$

In the theory developed by Fatou (1919, 1926) and Julia (1918) an important part is played by the set $\mathfrak{F} = \mathfrak{F}(f)$ of these points of the complex plane where $\{f_n(z)\}$ is not a normal family. Unless $f(z)$ is a rational function of order 0 or 1, (which we henceforth exclude) the set $\mathfrak{F}(f)$ is a non-empty perfect set, whose complement $\mathfrak{C} = \mathfrak{C}(f)$ consists of an at most countably infinite collection of (open) components G_i, each of which is a maximal domain of normality of $\{f_n\}$.

It is shown by Fatou (1919, 1926) that $\mathfrak{F}(f)$ is completely invariant under the mapping $z \to f(z)$, i.e. if α belongs to $\mathfrak{F}(f)$ then so do $f(\alpha)$ and every solution β of $f(\beta) = \alpha$. It follows that $\mathfrak{C}(f)$ is also completely invariant and, in particular, for each component G_i of $\mathfrak{C}(f)$ there is just one component G_j such that $f(G_i) \subset G_j$. By definition, the component G_0 of $\mathfrak{C}(f)$ is a *wandering domain* of f if

$$f_k(G_0) \cap f_n(G_0) = \emptyset \qquad \text{for all} \qquad 1 \leqq k, n < \infty, k \neq n.$$

No examples of wandering domains for either entire or rational functions seem to be known and indeed Jacobson (1969) raises the question whether they can occur at all for rational f. Pelles also discusses the notion.

In Baker (1963) an entire function $g(z)$ was constructed as follows:

Let $C = (4e)^{-1}$ and $\gamma_1 > 4e$. Then define inductively

$$(1) \qquad \gamma_{n+1} = C\gamma_n^2\left(1+\frac{\gamma_n}{\gamma_1}\right)\left(1+\frac{\gamma_n}{\gamma_2}\right)\cdots\left(1+\frac{\gamma_n}{\gamma_n}\right), \qquad n = 1,2,\cdots.$$

Then $1 < \gamma_1 < \gamma_2 < \cdots$ and [c.f. Baker (1963): lemmas 1 and 2]

$$(2) \qquad g(z) = Cz^2 \prod_{n=1}^{\infty}\left(1+\frac{z}{\gamma_n}\right)$$

is an entire function which satisfies

$$(3) \qquad |g(e^{i\theta})| < \tfrac{1}{4}, \qquad 0 \leqq \theta \leqq 2\pi,$$

$$(4) \qquad \gamma_{n+1} < g(\gamma_n) < 2\gamma_{n+1}, \qquad n = 1,2,\cdots,$$

$$(5) \qquad g(\gamma_n^{1/2}) < \gamma_{n+1}^{1/2}, \qquad n = 1,2,\cdots,$$

and

$$(6) \qquad g(\gamma_n^2) > \gamma_{n+1}^2, \qquad n = 1,2,\cdots.$$

Moreover, if A_n denotes the annulus

$$(7) \qquad A_n : \gamma_n^2 < |z| < \gamma_{n+1}^{1/2},$$

then by Baker (1963) Theorem 1, there is an integer N such that for all $n > N$ the mapping $z \to g(z)$ maps A_n into A_{n+1}, so that $g_k(z) \to \infty$ uniformly in A_n as $k \to \infty$. Since by (3) $g_k(z) \to 0$ uniformly for $|z| \leqq 1$, it is clear that each A_n, $n > N$, belongs to a multiply connected component C_n of $\mathfrak{C}(g)$ and that C_n does not meet $\{z : |z| \leqq 1\}$, which belongs to a component of $\mathfrak{C}(g)$ which we designate C_0. It is natural to ask whether the C_n, $n > N$, are all different, but this question was left unanswered in Baker (1963). The solution is given by the

THEOREM. For $n > N$ *the components* C_n *of* $\mathfrak{C}(g)$ *described above are all different and each is a wandering domain of* g.

PROOF. Suppose that there are two values of $n > N$ for which A_n belong to the same component of $\mathfrak{C}(g)$. Suppose $n = m > N$ and $n = m + l$, $l > 0$, are such values. Then there is a path Γ in $\mathfrak{C}(g)$ which joins a point of A_m to a point of A_{m+l}. The path Γ must meet A_{m+1}, which therefore belongs to the same component of $\mathfrak{C}(g)$ as A_m. So we may take $l = 1$. By the complete invariance of $\mathfrak{C}(g)$ the path $g_k(\Gamma)$ lies in $\mathfrak{C}(g)$ and it joins A_{m+k} to A_{m+k+1}, $k = 1,2,\cdots$. Thus all A_n, $n > m$, belong to the same component of $\mathfrak{C}(g)$, which is therefore multiply-connected and unbounded.

It suffices to show that for all sufficiently large n the annuli A_n and A_{n+2} cannot be joined in $\mathfrak{C}(g)$. Now for all sufficiently large $n(> N_0)$ we have, since $\gamma_n \to \infty$ in (1) that,

$$(8) \qquad 4\gamma_n^2 < \gamma_{n+1}^{1/2}.$$

Take any $n > \text{Max}(N, N_0)$ and assume that A_n, A_{n+2} can be joined in $\mathfrak{C}(g)$. Then $z_1 = 2\gamma_n^2 \in A_n$ and $z_2 = \frac{1}{2}\gamma_{n+3}^{1/2} \in A_{n+2}$. There is then a simple polygon joining z_1 and z_2 in $\mathfrak{C}(g)$ and so z_1, z_2 belong to a simply-connected subdomain, say H, of $\mathfrak{C}(g)$. H may be mapped conformally by $z = \psi(t)$ onto $|t| < 1$ so that $\psi(0) = z_1$ and $\psi(u) = z_2$ where u is some value for which $|u| < 1$.

Since $g_k(z) \to \infty$ locally uniformly, as $k \to \infty$ for $z \in A_n$, the same is true locally uniformly in the component G of $\mathfrak{C}(g)$ to which A_n belongs. Thus for each integer $p > 0$, $g_p(G)$ is a domain in which $G_k(z) \to \infty$ locally uniformly, so $g_p(G)$ does not meet the component G_0 of $\mathfrak{C}(g)$ which includes the disc $\{z : |z| \leqq 1\}$, as $g_k(z) \to 0$ in G_0. Thus in G, and in particular in H, $g(z)$ omits the values 0, 1. Similarly the functions $F_p(t) = g_p\{\psi(t)\}$ omit the values 0, 1 in $|t| < 1$. By Schottky's theorem there is a constant B, independent of p, such that

$$(9) \qquad \begin{aligned} |g_p(z_2)| &= |F_p(u)| \\ &\leqq \exp\left[\left(\frac{1}{1-|u|}\right)\left\{(1+|u|)\log\max(1, |F_p(0)|) + 2B\right\}\right] \end{aligned}$$

Now $g_p(z_1)$ is positive and $\to\infty$ as $p \to \infty$. so for all sufficiently large p (9) gives, noting $F_p(0) = g_p(z_1)$,

$$|g_p(z_2)| \leqq k\,|g_p(z_1)\Big|^L,$$

where L, K are constants which depend on z_1, z_2 but not on p. Thus for all sufficiently large p we have

$$(10) \qquad 0 < g_p(\tfrac{1}{2}\gamma_{n+3}^{1/2}) \leqq K\{g_p(2\gamma_n^2)\}^L.$$

By (8), however, we have

$$2\gamma_n^2 < \gamma_{n+1}^{1/2} < \gamma_{n+1},$$

and every iterate g_k is positive and increasing on the positive real axis, so for $k \leqq 1$

$$\begin{aligned} g_k(2\gamma_n^2) < g_k(\gamma_{n+1}) &= g_{k-1}\{g(\gamma_{n+1})\} \\ &< g_{k-1}(2\gamma_{n+2}) < g_{k-1}(\tfrac{1}{2}\gamma_{n+3}^{1/2}, \end{aligned}$$

using (4) and (8). For all sufficiently large x one has $g(x) > Kx^L$ and so for all sufficiently large k

$$g_k(\tfrac{1}{2}\gamma_{n+3}^{1/2}) = g\{g_{k-1}(\tfrac{1}{2}\gamma_{n+3}^{1/2})\} > g\{g_k(2\gamma_n^2)\} > K\{g_k(2\gamma_n^2)^L\},$$

which contradicts (10). Thus the first assertion of the theorem is established: for $n > N$ the components C_n of $\mathfrak{C}(g)$ which contain A_n are all different, and each is a bounded domain.

It follows at once that each C_n is a wandering domain for g. If this is not the case, then there exist integers $n > N$, $k > 0$, $l > 0$ such that $g_k(C_n)$ meets $g_{k+l}(C_n)$, i.e. since $g_k(C_n) \subset C_{n+k}$, $g_l(G') \subset G'$, where $G' = C_{n+k}$. The sequence $\{g_{ln}(z)\}$, $n = 1, 2, \cdots$ is bounded in G', taking values only in G'. But this contradicts the fact that the whole sequence $\{g_k\}$, $k = 1, 2, \cdots$, tends locally uniformly to ∞ in G', as in every C_n, $n > N$.

The theorem is now established and clears up the problem of the existence of wandering domains, at least in the case of entire functions. It adds a little to the discussion of Baker (1963) where it was shown that, if for entire g the set $\mathfrak{C}(g)$ has a multiply-connected component, G, then there are just two alternatives, namely:

I. G is unbounded and completely invariant and every other component of $\mathfrak{C}(f)$ is simply-connected, or

II. All components of $\mathfrak{C}(f)$ are bounded and infinitely many of them are multiply-connected.

It was conjectured in Baker (1963) that alternative II occurred in the case of the g of our theorem and this is now established. It is interesting to note [c.f Baker (1963)] that truncating the infinite product in (2) gives a polynomial

$$P(z) = Cz^2 \prod_{n=1}^{k} \left(1 + \frac{z}{\gamma_n}\right)$$

such that alternative I applies to $\mathfrak{C}(P)$ which has an unbounded and multiply-connected component.

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