Correspondence

DEAR EDITOR,

I read Bob Burn's Note 87.23 in the March 2003 *Gazette* with considerable interest, since I based a Listener mathematical crossword (28 August 1999) on integer triangles with angles of 60° or 120°.

Several solvers subsequently asked me whether there were formulae like the well-known ones applying to Pythagorean triples, i.e. $c^2 = a^2 + b^2$ with $\gcd(a, b) = 1$ if and only if $c = u^2 + v^2$, $a = 2uv$, $b = u^2 - v^2$, where $\gcd(u, v) = 1$ and $u - v$ is not divisible by 2. Well, there are!

$c^2 = a^2 + b^2 - ab$ with $\gcd(a, b) = 1$ if and only if $c = u^2 + uv + v^2$, $a = u^2 + 2uv$, $b = u^2 - v^2$ where $\gcd(u, v) = 1$ and $u - v$ is not divisible by 3.

If a, b have the same sign, then angle C is 60°; if they have opposite signs, then C is 120°.

If c, a, b is a triple, then so too are $c, -b, a - b$ and $c, b - a, -a$. Angle C will be 60° in two of these cases and 120° in the other case. The new values of (u, v) are $(-u - v, u)$ and $(v, -u - v)$.

For instance:	7, 8, 3	→	7, −3, 5	and	7, −5, −8
	13, 15, 8	→	13, −8, 7	and	13, −7, −15
	19, 21, 5	→	19, −5, 16	and	19, −16, −21

Given $c \neq 1$, a solution of $c^2 = a^2 + b^2 - ab$ with $\gcd(a, b) = 1$ exists if and only if c is a prime number of the form $6m + 1$ or a product of such primes. If c has 2 or any prime of the form $6m - 1$ (or a power of 2 or any such prime) as a factor, then a and b must share that factor.

I used this result and some other rather more sophisticated results to create the puzzle, which took me and my computer (and *Mathematica*) the best part of eight weeks way back in 1998. The genesis of the puzzle sprang from consideration of a rather more general problem: what are the solutions of $c^2 = a^2 + b^2 - 2abk$, where k is a rational number between --1 and 1?

Yours sincerely,

RODDY FORMAN (aka RADIX)
Summer Fields, Oxford OX2 7EN

DEAR EDITOR,

I enjoyed Tom Apostol's Note 87.02 in the March 2003 *Gazette* on Sylvester's theorem, but believe that his treatment of the matter was unduly complicated.

I define a *queue* of any positive integer N as a sequence of consecutive positive integers whose sum is N (unlike Apostol, I allow the sequence to have just one element). So, for example, the integer 90 has exactly 6 queues:

90

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$29 + 30 + 31$

$16 + 17 + 18 + 19 + 20$

$6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14$

$2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13$

$21 + 22 + 23 + 24$

How many queues does the integer $N = 2^a \times 3^b \times 5^c \times \ldots$ have?

Suppose N has a queue of r integers with median $s/2$: $rs = 2N$. Then r and s have opposite parity (since we are summing *consecutive* integers) and $r < s$ (since we are summing *positive* integers). So every odd divisor q of N corresponds to a queue and vice-versa: if $q < \sqrt{2N}$ then the queue has q elements, whereas if $q > \sqrt{2N}$ then the queue has median $q/2$. If $N = 90$, as in the example above, the 6 queues correspond to $q = 1, 3, 5, 9, 15, 45$.

Let's now look at $N = 36$, whose odd divisors are 1, 3, 9. So 36 has exactly 3 queues:

36	($q = 1 < \sqrt{72}$, so the queue has length 1)
23, 24, 25	($q = 3 < \sqrt{72}$, so the queue has length 3)
1, 2, 3, 4, 5, 6, 7, 8	($q = 9 > \sqrt{72}$, so the queue has median 4.5)

More generally, the number of queues equals $(b + 1)(c + 1)\ldots$

Note that powers of 2 have exactly one queue and odd primes have exactly two.

Yours sincerely,

RODDY FORMAN (aka RADIX)
Summer Fields, Oxford OX2 7EN

DEAR EDITOR,

Martyn Cundy's appreciation of 'Donald Coxeter - Master of many dimensions' will be echoed by many in the Association; this was shown by the number of references to him at this year's Conference. After his death his family requested that friends should not send flowers or dwell in sadness but instead reflect upon Donald's long and productive life.

I had to use a Latin dictionary to confirm my guess that 'praecox' means 'precocious'. Perhaps Martyn Cundy intentionally forbore from coining the name 'Praecoxeter', preferring to leave it to the reader.

I loved geometry at school, but much of the geometry taught and researched at Cambridge in the 1950s was, to me, uninspiring, so I switched to group theory for my PhD. When Coxeter's *Introduction to Geometry* appeared in 1961 I was sent a free copy by the publishers who were more generous to academics in those days. I thought 'here's a mathematician who still does proper geometry, with figures; I'd like to meet him', so I applied for an assistant professorship at the University of Toronto for one year, and eventually stayed few a second year to give Donald's lecture courses whilst he was an sabbatical leave. He brought me back on to the path of geometry, not so much a straight and narrow path as a primrose path of dalliance.

A biography by the Toronto writer Siobhan Roberts will be published by Penguin in 2005. She has already written an article about Donald in a Toronto magazine, and I am sure we can look forward to a fascinating account of his life.

Yours sincerely

JOHN RIGBY

Flat 5 Cathedral Court, Cathedral Green, Llandaff, Cardiff CF5 2EB

DEAR EDITOR,

With reference to Note 86.25, I recently signed up for membership of the Association, and ordered back numbers of the *Gazette* from 500 to date. They arrived the day before Good Friday, which was both fortuitous and fortunate for me. My son Philip was to be married on the Saturday, and at the reception I was able to quote a suitably amended version of part of Hymne to Hymen. The poem was received with much hilarity, which pleased me immensely. (Many of the young people were at least somewhat *au fait* with mathematics and mathematical terms.)

I thank you, Blanche Descartes and C. A. B. Smith.

MICHAEL D. HIRSCHHORN

School of Mathematics, UNSW, Sydney 2052, Australia

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