Maximum and Minimum.

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[ABSTRACT.]

The object of this note was to point out that in using the method of limits to find a geometrical maximum or minimum it is not correct to conduct all the reasoning at the final stage when the limit has been reached, and to call attention to the form of statement which lays stress on the fact that the reasoning should be based on the consideration of the quantities involved while they are yet finite. Examples were given from one or two well-known books for students where the fallacious method of proof is adopted. Two of these examples follow :—

(1.) "The maximum or minimum straight line from a given point to a circle is the normal through the point."

FIGURE 17.

"Let AP be the minimum line drawn from A, and AQ a consecutive position. Then in the limit AP = AQ, ∴ the triangle APQ is isosceles. And since the angle PAQ is indefinitely small, each of the angles APQ, AQP is ultimately a right angle. And PQ being in the limit the direction of the tangent at P, AP is normal at P."

If "consecutive" means that the lines are coincident, then all the reasoning concerns a triangle which has already vanished and whose properties while it was finite were not examined. If "consecutive" simply means neighbouring, then the same reasoning would prove that any line is normal to a curve. For, instead of "Let AP be the minimum line," read "Let AP be any line drawn from A to the curve and AQ a consecutive position," and so on as before. Indeed the writer in one of the books considered falls into this snare in an equally simple case.

These objections do not apply if we say—If any line AP be taken in the neighbourhood of the minimum line and on one side of it, an equal line AQ can be found on the other side of it. Then APQ is an isosceles triangle, and the bisector of PAQ is perpendicular to the chord PQ. This is true of any such pair of equal lines, and hence is true of the coincident pair at the minimum position; and the bisector which is now coincident with AP and AQ is still perpendicular to PQ which is now a tangent.

(2.) "Of all quadrilaterals which can be formed from four straight lines of given lengths, the maximum is that which can be inscribed in a circle."

FIGURE 18 (a).

"Let ABCD be the position of maximum area. Take ABC′D′ a consecutive position keeping AB fixed. Let AD, BC meet in O. Then since AD = AD′,

∴ ultimately the angle ADD′ is a right angle;

∴ also ODD′ is a right angle, and OD = OD′ ultimately.

Similarly	OC = OC′ ultimately.
And	CD = C′D′ ;
∴ angle	DOC = D′OC′,
and triangle	OCD = OC′D′;
∴ angle	DOD′ = COC′.

Again, in the limit the area ABCD = ABC′D′;

∴ triangle OAB = area OC′BAD′.

Taking away the common part OBAD′, we get the triangle OAD′ = OBC′, and an angle AOD′ of the one = BOC′ of the other;

∴ OA . OD′ = OB . OC′;

∴ in the limit OA . OD = OB . OC ;

∴ A, B, C, D are concyclic."

Now if this proof had read :—

Let ABCD be *any quadrilateral whatever* formed by the four given lines. Take ABC′D′ a consecutive position keeping AB fixed—and so on as before, we should reach the conclusion that A, B, C, D are concyclic, which is obviously wrong.

Whatever be the fallacy in the second reading of our proof, it is present in the first.

It is asserted that triangle OAD′ = OBC′. But it must be remembered that the whole of the reasoning is being conducted at the final stage of the approach of the one figure to the position of the other and when each of these triangles has become zero; and although there is some circumlocution, the fact that each is zero is the only ground for asserting that their ratio is equal to 1, and it is not a valid ground.

Put shortly, the proof is this :—

FIGURE 18 (*b*).

Let ABCD be the position of maximum area. Then the flat triangle AOD = BOC and their angles at O are equal,

$$\therefore\ OA \cdot OD = OB \cdot OC.$$

$$\therefore\ A,\ B,\ C,\ D \text{ are concyclic.}$$

It obviously applies to any quadrilateral.

The following is not open to the same objection :—

Let ABC′D′, ABC″D″ be two equal areas on opposite sides of the maximum position.

Bisect D′D″ and C′C″ and let AD, BC meet in O. Then AD′OD″, BC′OC″ are kites having OD′ = OD″, OC′ = OC″, and D′C′ = D″C″.

∴ triangles OD′C′, OD″C″ are congruent.

∴ after taking away the common ∠ D″OC′, ∠ D′OD″ = C′OC″

∴ their halves ∠ AOD″ and ∠ BOC′ are equal.

Also, since triangles OD′C′, OD″C″ are congruent and the quadrilaterals are equal,

$$\therefore\ OD'ABC' = OD''ABC''.$$

Take away OD″ABC′ and the kites are proved equal in area and so are their halves, AOD″ and BOC′.

It follows that $OA \cdot OD'' = OB \cdot OC'$.

This is true for every such pair of equal quadrilaterals, and therefore for the coincident pair, when D′D″ coincide on OA and C′C″ coincide on OB.

∴ for the maximum position

$$OA \cdot OD = OB \cdot OC,$$

i.e., ABCD is cyclic.

An application of Sturm's Functions.

By J. D. Höppner.

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