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ON THE LOG CANONICAL INVERSION OF ADJUNCTION

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Dedicated to V. Shokurov on his 60th birthday

Abstract We prove a result on the inversion of adjunction for log canonical pairs that generalizes Kawakita's result to log canonical centres of arbitrary codimension.

Keywords: inversion of adjunction; log canonical pairs; minimal model program

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The minimal model programme (MMP) is an ambitious programme that aims to generalize to higher-dimensional varieties many of the results in the classification of surfaces obtained by the Italian school of algebraic geometry in the early twentieth century. Log canonical singularities are the largest class of singularities for which the minimal model programme is expected to hold. Let (X, Δ) be a *pair* consisting of a normal variety X and an effective $\mathbb{Q}$-divisor $\Delta = \sum \delta_i \Delta_i$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Consider a *log resolution* of (X, Δ), i.e. a proper birational morphism $\mu \colon Y \to X$ such that Y is smooth, $\mathrm{Exc}(\mu)$ is a divisor and $\mu_*^{-1}\Delta + \mathrm{Exc}(\mu)$ has simple normal crossings. Write $K_Y + \Delta_Y = \mu^*(K_X + \Delta)$; then, $\Delta_Y = \sum b_i B_i$ is uniquely determined and (X, Δ) is *log canonical* (*lc*) (respectively, *Kawamata log terminal* (*klt*)) if $b_i \leqslant 1$ (respectively, $b_i < 1$) for all i. Similarly, (X, Δ) is *purely log terminal* (*plt*) if $0 \leqslant \delta_i \leqslant 1$, and for all log resolutions $\mu \colon Y \to X$ we have that $b_i < 1$ for all i such that B_i is μ-exceptional. Suppose that $\Delta = S + B$, where S is a prime divisor and $\nu \colon S^\nu \to S$ is the normalization; there then exists a uniquely defined $\mathbb{Q}$-divisor $\mathrm{Diff}(B)$ on S^ν such that $(K_X + S + B)|_{S^\nu} = K_{S^\nu} + \mathrm{Diff}(B)$ (see [**8**, § 16]). Note that if S' is the strict transform of S on Y and $\bar{\mu} \colon S' \to S^\nu$ is the induced morphism, then $\mathrm{Diff}(B) = \bar{\mu}_*((\Delta_Y - S')|_{S'})$.

It is of fundamental importance in the minimal model programme to relate the singularities of the pair $(X, S + B)$ to those of the pair $(S^\nu, \mathrm{Diff}(B))$. If we know that the pair $(X, S + B)$ is log canonical (respectively, plt), then it is easy to see that the pair $(S^\nu, \mathrm{Diff}(B))$ is also log canonical (respectively, klt). This process is known as adjunction. The inverse of adjunction, on the other hand, is the process of deducing information about the singularities on the ambient variety $(X, S + B)$ from information on the singularities of the divisor $(S^\nu, \mathrm{Diff}(B))$. These results are much more subtle and useful.

It is known that if $(S^\nu, \mathrm{Diff}(B))$ is klt, then $(X, S+B)$ is plt on a neighbourhood of S (see [**8**, 17.6]), and, by a more recent result of Kawakita, it is known that if $(S^\nu, \mathrm{Diff}(B))$ is lc, then $(X, S+B)$ is lc on a neighbourhood of S (see [**6**]). The proof of both results relies heavily on Kawamata–Viehweg vanishing.

The purpose of this short paper is to give a proof of a generalization of Kawakita's theorem on the inversion of adjunction to higher co-dimensional subvarieties. Our proof is based on the results of [**3**] and recovers a new proof of Kawakita's theorem. Our argument closely follows ideas of Shokurov (see [**13**]), but avoids the use of the minimal model programme for log canonical pairs. Note, moreover, that there are some similarities between this proof and the arguments in [**6**, **8**]. See also [**12**] for a related result in characteristic $p > 0$.

Before stating the main theorem we must introduce some notation. Let (X, Δ) be a pair; a subvariety $V \subset X$ is then a *log canonical centre* if, considering all log resolutions $\mu\colon Y \to X$ such that $K_Y + \Delta_Y = \mu^*(K_X + \Delta_X)$, where $\Delta_Y = \sum b_i B_i$, we have that $\max\{b_i \mid \mu(B_i) = V\} = 1$. Recall that, in this case, (X, Δ) is log canonical on a neighbourhood of the generic point of V (see [**8**, 17.1.1]). We denote by S a component B_i as above, such that $b_i = 1$ and $\mu(B_i) = V$. Let $\nu\colon W \to V$ be a birational morphism from a normal variety W, let $\Delta_S := (\Delta_Y - S)|_S$, and assume that $\bar{\mu}\colon S \to W$ is a morphism. We then define an effective $\mathbb{Q}$-divisor $B_W = \sum (1 - t_i) P_i$ on W as follows: for any codimension 1 point P_i on W, let

$$t_i = \sup\{\tau \mid (S, \Delta_S + \tau\bar{\mu}^* P_i) \text{ is lc over } \eta_{P_i}\},$$

where η_{P_i} denotes the generic point of P_i. Note that the t_i are rational numbers (positive or negative). It is known that the following hold (see [**1**, **7**]).

(1) The numbers t_i are independent of the log resolution $\mu\colon Y \to X$ and of the choice of the divisor S.

(2) If $W = V^\nu$ is the normalization of V, then $1 - t_i \geqslant 0$ for all $P_i \in V^\nu$, and the strict inequality only holds for finitely many codimension 1 points $P_i \in W$.

(3) If S is the only component of Δ_Y of coefficient 1, then (W, B_W) is klt.

(4) When $\dim V = \dim X - 1$, we let S be the strict transform of V and we have that $B_{V^\nu} = \mathrm{Diff}(\Delta - V)$, where $V^\nu \to V$ is the normalization.

(5) If $\eta\colon W' \to W$ is a birational morphism of normal varieties, then $\eta_* B_{W'} = B_W$, so we have a b-divisor $\boldsymbol{B} = \boldsymbol{B}(V; X, \Delta)$ defined by $\boldsymbol{B}_W = B_W$ (see [**4**] for the definition and properties of b-divisors).

(6) If $S \to W$ satisfies the standard normal crossing assumptions of [**9**, 8.3.6], then $\boldsymbol{B}$ descends to W in the sense that, for any birational morphism $\eta\colon W' \to W$, we have that $\eta^*(K_W + \boldsymbol{B}_W) = K_{W'} + \boldsymbol{B}_{W'}$ (see [**9**, 8.4.9]; see also [**2**]).

(7) If (W, B_W) is sub-log canonical (i.e. if $t_i \geqslant 0$ for any $P_i \in W$) for any sufficiently high model (or, equivalently, for any model W such that $S \to W$ satisfies the standard normal crossing assumptions), then we say that $(V^\nu, \boldsymbol{B})$ is log canonical.

We prove the following generalization of Kawakita's result.

Theorem 1. *Let V be a log canonical centre of a pair $(X, \Delta = \sum \delta_i \Delta_i)$, where $0 \leqslant \delta_i \leqslant 1$. Then, (X, Δ) is log canonical on a neighbourhood of V if and only if $(V^\nu, \boldsymbol{B}(V; X, \Delta))$ is log canonical.*

Proof. If (X, Δ) is log canonical on a neighbourhood of V, then it is well known and easy to see that $(V^\nu, \boldsymbol{B}(V; X, \Delta))$ is log canonical. Therefore, we assume that $(V^\nu, \boldsymbol{B}(V; X, \Delta))$ is log canonical and we prove that (X, Δ) is log canonical on a neighbourhood of V. Let $\mu\colon Y \to X$ be a divisorial log terminal (dlt) model (see [**10**, 3.1]) of (X, Δ), so if we write $\mu^*(K_X + \Delta) = K_Y + \Delta_Y$, then

(1) Y is $\mathbb{Q}$-factorial,

(2) $\Delta_Y = \sum b_i B_i \geqslant 0$,

(3) $(Y, \Delta'_Y = \sum_{b_i \leqslant 1} b_i B_i + \sum_{b_i > 1} B_i)$ is dlt and

(4) every exceptional divisor appears in Δ'_Y with coefficient greater than or equal to 1.

We may assume that $\Delta'_Y = S + \Gamma$, where S is an irreducible component of $\Delta_Y^{=1} = \sum_{b_i=1} B_i$ that dominates V and $\Sigma = \Delta_Y - S - \Gamma$. Note that $f(\Sigma) \not\supset V$. We now fix a sufficiently ample divisor H on Y and we run the $(K_Y + S + \Gamma)$-MMP with scaling of H over X (see [**3**, 3.10]). Let $\phi_i\colon Y_i \dashrightarrow Y_{i+1}$ be the induced sequence of flips and divisorial contractions and let $\mu_i\colon Y_i \to X$ be the induced morphisms. For any divisor G on Y, we let G_i be its strict transform on Y_i. There then exists a non-increasing sequence of rational numbers $s_i \geqslant s_{i+1}$ that is either

- finite with $s_{N+1} = 0$ or
- infinite with $\lim s_i = 0$

such that $K_{Y_i} + S_i + \Gamma_i + sH_i$ is nef over X for all $s_i \geqslant s \geqslant s_{i+1}$. For $i \geqslant i_0$, we may assume that each ϕ_i is a flip. By a well-known discrepancy computation, we may also assume that $S_i \dashrightarrow S_{i+1}$ is an isomorphism in codimension 1 for all $i \geqslant i_0$ (see the arguments in the proofs of Steps 1 and 2 of [**5**, 4.2.1]). For any $t > 0$, there exists a $\mathbb{Q}$-divisor Θ_t on Y such that $\Theta_t \sim_{\mathbb{Q}} \Gamma + tH$ and $(Y, S + \Theta_t)$ is plt. Note that if $t < s_i$, then $(Y_i, S_i + \Theta_{t,i})$ is plt (this is because plt singularities are preserved by steps of the minimal model programme).

Suppose that, for some $i \geqslant 0$, we have that $S_i \cap \Sigma_i \neq \emptyset$; then,

$$(\mu_i^*(K_X + \Delta))|_{S_i} = (K_{Y_i} + S_i + \Gamma_i + \Sigma_i)|_{S_i} = K_{S_i} + \mathrm{Diff}(\Gamma_i + \Sigma_i)$$

is not log canonical. Let $\bar{\mu}_i\colon S_i \to V^\nu$ be the induced morphism. We may replace $S_i \to V^\nu$ by a birational model $\tilde{\mu}\colon \tilde{S} \to W$ satisfying the standard normal crossing assumptions. If $g\colon \tilde{S} \to S_i$ is the induced morphism and we write $K_{\tilde{S}} + \Delta_{\tilde{S}} = g^*(K_{S_i} + \mathrm{Diff}(\Gamma_i + \Sigma_i))$, then there exists a component of $\Delta_{\tilde{S}}$ of coefficient greater than 1. After possibly replacing W by an appropriate birational model, we may assume that the image of this component is a codimension 1 point $P_i \in W$. But then $t_i < 0$, so $1 - t_i > 1$ and, hence, (W, B_W) is not log canonical. This proves the theorem.

Therefore, we assume that $S_i \cap \Sigma_i = \emptyset$ for all $i \geqslant 0$, and we derive a contradiction. Note that if this is the case, then any curve contained in S_i intersects Σ_i trivially and, hence, ϕ_i does not contract S_i. For any $m \gg 0$ such that $m\Sigma$ is an integral divisor, let $i \gg 0$ be the integer such that $s_i > 1/m \geqslant s_{i+1}$. Note that, since

$$H_i - m\Sigma_i - S_i \sim_{\mathbb{Q},X} K_{Y_i} + \Theta_{1/m,i} + (m-1)\left(K_{Y_i} + S_i + \Gamma_i + \frac{1}{m}H_i\right),$$

where $(Y_i, \Theta_{1/m,i})$ is klt and $K_{Y_i} + S_i + \Gamma_i + H_i/m$ is nef over X, by Kawamata–Viehweg vanishing (see [**11**, 2.70]), we have that $R^1(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i - S_i) = 0$ and, hence,

$$(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i) \to (\bar{\mu}_i)_*\mathcal{O}_{S_i}(H_i)$$

is surjective. On the other hand, for $m \gg 0$, the subsheaves

$$(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i) = (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(H_{i_0} - m\Sigma_{i_0}) \subset (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(H_{i_0})$$

are contained in $\mathcal{I}_{\mu_{i_0}}(\Sigma_{i_0}) \cdot (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(H_{i_0})$. Since $V \cap \mu_{i_0}(\Sigma_{i_0}) \neq \emptyset$ and $S_{i_0} \dashrightarrow S_i$ is an isomorphism in codimension 1, the induced homomorphism

$$(\mu_i)_*\mathcal{O}_{Y_i}(H_i - m\Sigma_i) \to (\bar{\mu}_{i_0})_*\mathcal{O}_{S_{i_0}}(H_{i_0}) = (\bar{\mu}_i)_*\mathcal{O}_{S_i}(H_i)$$

is not surjective. This is the required contradiction. □

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