RADIATION AND GRAVITATIONAL EQUATIONS OF MOTION

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1. Introduction. Among the classical field theories, general relativity theory occupies a somewhat peculiar place. Unlike those of most other field theories, the field equations in relativity theory are non-linear. This implies that many facts, well known in linear theories, have no analogues in general relativity theory, and conversely. The equations of motion of the sources of the gravitational field are contained in the field equations, a fact which does not apply for the motion of an electron in the electromagnetic field. Conversely, it is difficult to define the notion of a *wave* (familiar in electrodynamics) in relativity theory; for, the linear principle of superposition is crucial for the existence of waves (at least in the sense that the notion of a wave is normally used).

In many other respects, however, close analogues between general relativity theory and other classical field theories exist in spite of the discrepancies mentioned above. It is in part the object of this paper to investigate such analogies.

Since the gravitational field manifests itself in the motion of its sources, the problem of finding the equations of motion is of fundamental importance. This problem was solved some time ago [1], [2]. The general method for obtaining the equations of motion is to introduce an approximation procedure when solving the field equations. Each step of this approximation can be performed only if we impose upon the field certain restrictions (like adding dipoles). These restrictions yield, at the end of the approximation procedure, the differential equations of motion. We have to refer the reader for all the details and also the notation to the paper [2] mentioned above.

Now, in [2] there is at every step of the procedure a certain ambiguity for choosing the solutions of the field equations, which is restricted by assuming a set of co-ordinate conditions. Changing these co-ordinate conditions alters the equations of motion. However, the different equations which can be obtained are physically identical and are merely different mathematical representations of the motion in different co-ordinate systems.

This idea is already partly contained in [2]. There it is shown that rejection of the co-ordinate conditions at one stage of the approximation does not affect the differential equations of motion in the next one [2, §13]. Moreover, at every stage of the approximation the most general solution that can be obtained by rejecting the co-ordinate conditions, is given [2, (9.2) and (13.5)]. However, it is not shown that these solutions can also be obtained by a co-ordinate transformation from the old ones, and are thus equivalent to the old ones.

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Thus we are led to investigate the influence of co-ordinate transformations upon the equations of motion. This leads, strangely enough, to the destruction of analogies with linear field theories. It is seen that the analogue to the electromagnetic radiation of an accelerated electron exists only formally in general relativity theory. In the case of a gravitational radiation one can perform a co-ordinate transformation and then one regains the solution without radiation (cf. also [3]).

The group of transformations under which the gravitational field equations are covariant, is very general. This implies that the form of the equations of motion depends on the co-ordinates used. There is no physical meaning in the phrase: "the equations of motion of two particles" without reference to the frame in which they apply. It is seen that one can always find a co-ordinate system in which the motion is simply Newtonian. In such a system, however, the metric is very complicated. This is in agreement with a recent statement of Bergmann and Brunings [4] that the co-ordinate system can be chosen so that the equations of motion have *any* form we wish.

Furthermore, it is possible to transfer the whole approximation procedure of [2] into one concerning the co-ordinate system. Thus the condition of integrability of the field equations becomes a condition on the co-ordinate system. We are led to a new version of the usual approximation method: We can enforce integrability not only by adding dipoles but the simple procedure of changing the co-ordinate system.

2. Co-ordinate transformations. We have already mentioned that the general solution of the field equations (by rejecting the co-ordinate conditions) has been calculated [2, (9.2) and (13.5)]. It may be written in the following form:

$$(2.1)\qquad \left\{\begin{aligned} \underset{k-2}{\gamma^*_{00}} &= \underset{k-2}{\gamma_{00}} \\ \underset{k-1}{\gamma^*_{0m}} &= \underset{k-1}{\gamma_{0m}} + \underset{k-1}{a_{0,m}} \\ \underset{k}{\gamma^*_{mn}} &= \underset{k}{\gamma_{mn}} + \underset{k}{a_{m,n}} + \underset{k}{a_{n,m}} - \delta_{mn}\underset{k}{a_{r,r}} + \delta_{mn}\underset{k-1}{a_{0,0}} \end{aligned}\right.$$

and

$$(2.2)\qquad \left\{\begin{aligned} \underset{k}{\gamma^*_{00}} &= \underset{k}{\gamma_{00}} + \underset{k}{b_{r,r}} \\ \underset{k+1}{\gamma^*_{0m}} &= \underset{k+1}{\gamma_{0m}} + \underset{k+1}{b_{0,m}} + \underset{k}{b_{m,0}} \\ \underset{k}{\gamma^*_{mn}} &= \underset{k}{\gamma_{mn}} + \underset{k}{b_{m,n}} + \underset{k}{b_{n,m}} - \delta_{mn}\underset{k}{b_{r,r}} \end{aligned}\right.$$

The functions a_0, a_m and b_0, b_m are arbitrary.

We shall investigate whether the general solutions γ^* can be obtained from the particular γ's by a co-ordinate transformation. Let the transformation be

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$$x^{\beta} = x^{\beta}(x^{\alpha *}) = T^{\beta}(x^{\alpha *}) \tag{2.3}$$

Then, we calculate the transformed γ's. The transformation of the metric tensor is (assuming the summation convention)

$$g^*_{\alpha\beta} = T^{\gamma}{}_{|\alpha}\ T^{\theta}{}_{|\beta}\ g_{\gamma\theta} \tag{2.4}$$

When applying these equations we have to be careful that we take the co-ordinates of the same world point as arguments in all these functions. That is, in addition to the tensorial transformation we have to perform a substitution of the variables x by x^*, according to (2.3).

We develop the tensor $g_{\alpha\beta}$ into a power series with respect to the parameter λ, as this has been done in [2, (5.6)], but now we keep *all* the terms instead of only alternating ones. In the usual solution we assume that the lowest term different from zero is of the order λ^2 (apart from the constant ones $\underset{0}{g}_{\alpha\beta} = \eta_{\alpha\beta}$). We shall confine ourselves to co-ordinate systems where this same property holds. This means that in every co-ordinate system that we admit for consideration, we have a flat Minkowskian metric as a zero approximation of the gravitational field. With this assumption the expanison of the metric tensor becomes

$$\left\{\begin{aligned} g_{mn} &= -\delta_{mn} + \lambda^2 \underset{2}{h}_{mn} + \lambda^3 \underset{3}{h}_{mn} + \dots \\ g_{0m} &= \lambda^2 \underset{2}{h}_{0m} + \lambda^3 \underset{3}{h}_{0m} + \dots \\ g_{00} &= 1 + \lambda^2 \underset{2}{h}_{00} + \lambda^3 \underset{3}{h}_{00} + \dots \end{aligned}\right. \tag{2.5}$$

It was assumed in the original approximation procedure that the motion is "slow", so that one could introduce the "comma-differentiation" [2, (5.3)] with respect to time. If we want to retain in the starred co-ordinate system the assumption that motion is "slow", then we have to assume that also the derivatives of T with respect to x^{0*} are of a higher order in λ than those with respect to x^{m*}; in other words, we have to use the "comma-differentiation" for the transformation function T, too.

Then we can write

$$\left\{\begin{aligned} g^*_{mn} &= T^{\alpha}{}_{,m}\ T^{\beta}{}_{,n}\ g_{\alpha\beta} \\ g^*_{m0} &= \lambda\, T^{\alpha}{}_{,m}\ T^{\beta}{}_{,0}\ g_{\alpha\beta} \\ g^*_{00} &= \lambda^2 T^{\alpha}{}_{,0}\ T^{\beta}{}_{,0}\ g_{\alpha\beta} \end{aligned}\right. \tag{2.6}$$

Equations (2.6) apply quite independently of whether a power development in λ is used for T, or not.

We assume now that the transformation T^{α} in (2.3) is an infinitesimal co-ordinate transformation, i.e. that it is of the type

$$\left\{\begin{aligned} T^m &= x^{m*} + \lambda^k \underset{(k)}{T^m}(x^{\rho *}) \\ T^0 &= x^{0*} + \lambda^j \underset{(j)}{T^0}(x^{\beta *}) \end{aligned}\right. \tag{2.7}$$

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Then, it is easy to see that it is possible to obtain the formulae (2.1) and (2.2) by an appropriate choice of T^β in (2.7). We may indicate the type of calculations involved by taking the following example

$$\text{(2.8)} \qquad \begin{cases} x^m = T^m = x^{m*} + \lambda^k \underset{(k)}{T^m}(x^{\beta *}), \\ x^0 = T^0 = x^{0*}. \end{cases}$$

If we insert this together with the expansion of the g's (2.5) into (2.6), we get

$$\text{(2.9)} \qquad \begin{cases} g^*_{mn} = g_{mn} + \lambda^k(\delta_{lm}\underset{(k)}{T^s}{}_{,n} + \delta_{sn}\underset{(k)}{T^l}{}_{,m})\underset{0}{g_{ls}} + O(k+1) \\ g^*_{m0} = g_{m0} + \underset{(k)}{T^s}{}_{,0}\, \underset{0}{g_{ms}}\, \lambda^{k+1} \qquad + O(k+2) \\ g^*_{00} = g_{00} \qquad + O(k+2) \end{cases}$$

In the above formulae, everything is expressed in the starred co-ordinates. True, one should perform the substitution of the arguments of the occurring functions according to (2.8). This substitution, however, cannot give a contribution to the expressions in (2.9) up to the considered order. Hence we can write in these equations either the starred or the unstarred co-ordinates as arguments. This is particularly so because our zero approximation to the metric tensor in both co-ordinate systems is $\underset{0}{g_{\mu\nu}} = \eta_{\mu\nu}$.

We can express equation (2.9) in terms of the γ's. A straightforward calculation yields

$$\text{(2.10)} \qquad \begin{cases} \underset{k}{\gamma^*_{mn}} = \underset{k}{\gamma_{mn}} + \delta_{mn} \underset{k}{T^s}{}_{,s} - \underset{k}{T^m}{}_{,n} - \underset{k}{T^n}{}_{,m} \\ \underset{k}{\gamma^*_{00}} = \underset{k}{\gamma_{00}} - \underset{k}{T^s}{}_{,s} \\ \underset{k+1}{\gamma^*_{0m}} = \underset{k+1}{\gamma_{0m}} - \underset{k}{T^s}{}_{,0} \end{cases}$$

This set of equations represents the change of the variables γ under a co-ordinate transformation (2.8). Only the kth and higher approximations are influenced. In a similar way it is seen that the transformation (2.7) with

$$\text{(2.11)} \qquad \underset{k}{T^s} = -\underset{k}{a_s}; \qquad \underset{k-1}{T^0} = \underset{k-1}{a_0}$$

yields the expressions (2.1); whereas choosing

$$\text{(2.12)} \qquad \underset{k}{T^s} = -\underset{k}{b_s}; \qquad \underset{k+1}{T^0} = \underset{k+1}{b_0}$$

yields (2.2).

These results show that co-ordinate transformations produce all the changes of the γ's which have been found possible in [2] by rejecting the co-ordinate conditions in the kth step of the approximation procedure. Thus, if we have the usual solution, all the different solutions which result from the arbitrariness in the approximation procedure, can be obtained simply by an appropriate co-ordinate transformation, and conversely.

3. Gravitational radiation. It has been shown [5] that the terms omitted in the usual power series for $\gamma_{\alpha\beta}$ in [2, (5.6)] (i.e. the even terms in γ_{0m}, the odd ones in γ_{mn}, γ_{00}) are analogous to the ones representing radiation in electromagnetic theory. We shall adopt here, therefore, the name *radiation terms* for those terms.

One could expect from this analogy that it is possible to deduce similar effects in relativity theory as corresponding to the radiation damping force in electrodynamics. It has been seen [5] that the term which should originate these effects must be of the form

$$\underset{3}{\gamma_{00}} = 0; \quad \underset{4}{\gamma_{0m}} = -4\overset{1}{m}\frac{d^2\eta^m}{d\tau^2} - 4\overset{2}{m}\frac{d^2\zeta^m}{d\tau^2}. \tag{3.1}$$

Starting with this assumption, Hu [6] calculated the equations of motion up to the 9th order. As an illustration he considered two particles of equal mass m moving along circular orbits around each other. The distance r between these two particles may thus be considered as constant up to the order of the Newtonian equations of motion. Then, Hu obtained the result that the total energy defined in Newtonian mechanics as

$$E = \tfrac{1}{2}\,(mv^2 - 2Km^2/r) \tag{3.2}$$

is *increased* by the radiation "damping" force. This result is rather strange from the point of view of Newtonian mechanics, according to which the energy can only be radiated out at the loss of the total energy E.

Our remarks on co-ordinate transformations contained in the last section give the clue for the proper interpretation of Hu's result. It is easily seen that the term (3.1) which was chosen to start the radiation expansion, can be obtained from the usual solution of [2] by putting in (2.8)

$$\underset{3}{T^m} = 4\overset{1}{m}\dot{\eta}^m + 4\overset{2}{m}\dot{\zeta}^m. \tag{3.3}$$

Thus the term starting the radiation expansion is of just such a form that it can be created by a co-ordinate transformation (2.8). Therefore, it also can be wiped out by the corresponding inverse co-ordinate transformation. But in this new co-ordinate system there are no radiation terms, the new metric tensor is that one which we had before the radiation terms were inserted, the equations of motion are the original ones (without the radiation) and thus, we regain the old solution of the relativistic field equations without radiation terms. It may be noted that the co-ordinate system containing the radiation terms with the particular assumption (3.1) does not even require a departure from the usual co-ordinate conditions [2, (9.8)], since $\underset{4}{\gamma_{0m,m}} = \underset{3}{\gamma_{00,0}} = 0$.

We may investigate now whether there are other possibilities for inserting radiation terms. For, generalizing our argument, we are not forced to start radiation terms with the choice of $\underset{4}{\gamma_{0m}}$ as this was done in (3.1). We can ask whether it is possible to start the omitted terms in the original development for the γ's at any stage of the approximation procedure, say at the $2k$th one.

The prescriptions of [2] imply that one never must add arbitrarily to a field variable any additional poles or higher harmonics, except when this is unavoidable. On the other hand, the equation giving the first radiation terms is one of the following:

$$\underset{2k+1}{\gamma_{00,ss}} = 0; \quad \text{or} \quad \underset{2k}{\gamma_{0m,ss}} = 0; \quad \text{or} \quad \underset{2k+1}{\gamma_{mn,ss}} = 0. \tag{3.4}$$

If we want to take for one of these γ's a solution $\not\equiv 0$ which is nowhere singular in space (including infinity), we see that the only possibility is γ equal to a function of τ. It is readily seen that the particular choice (3.1) suggested by the electromagnetic analogy is indeed of the required form, since η, ζ are functions of τ only. However, a straightforward investigation yields the result that all terms of this type can be created or annihilated by suitable co-ordinate transformations.

One might object to this method that a co-ordinate transformation annihilates the radiation terms only in the lowest approximation where they first appear, but nothing is known as to higher order terms. However, it is possible to carry out the approximation procedure in the new co-ordinate system (without the radiation in the lowest approximation where it was first inserted) and thus to obtain by direct application of the method of [2] the terms originated by the preliminary insertion of the radiation terms and the subsequent co-ordinate transformation. These additional terms could be of two kinds: *either* they are time-functions only and thus may be got rid of by a new co-ordinate transformation of higher order;—*or* they could be singular. If they were singular, this would amount to inserting arbitrarily singular terms at a certain stage in the approximation procedure. The solution of Einstein's field equations would, then, proceed without radiation up to a certain approximation in a suitably chosen co-ordinate system, and then suddenly an additional singular term would be added, which, true enough, could by no means be got rid of and would give a contribution to the equations of motion. However, the approximation procedure, at least as outlined in [2], stands or falls with the prescription that no arbitrary singular terms be inserted at any stage of the procedure. Therefore, if a radiation term should lead to a singular term in higher approximations, after it had been wiped out by a co-ordinate transformation in the approximation where it had first been inserted, it has to be excluded for that very reason. If it leads to additional time-functions only, then those can be annihilated by new (regular!) co-ordinate transformations. Thus, if we add "radiation terms" at a certain stage of the approximation, they are *either* meaningless *or* make the approximation procedure inconsistent.

As already indicated in the introduction to this paper, these results really should have been expected *a priori*. At each step the approximation to the gravitational field variables is only determined up to certain additional terms, and for the sake of the consistency of the method, it *must* be that the different solutions are changed into each other by mere co-ordinate transformations. As we agreed throughout our work to introduce only such radiation terms which are consistent with all the requirements of the approximation procedure, we *can* really introduce no other solutions than those already found.[1]

4. The equations of motion. We have shown that one can change the relativistic equations of motion in form by performing co-ordinate transformations. In particular, we were able in the last section to create and annihilate radiation terms in the equations of motion as well as in the expansions for the field variables.

Now, we may ask: What is generally the influence of a co-ordinate transformation of the type (2.7) upon the relativistic equations of motion? Is it perhaps possible to adjust the co-ordinate system at every step of the approximation procedure in such a way that the motion has always a certain standard form? Intuitively, one could expect that a co-ordinate transformation can change the equations of motion to any form we like. However, considering only infinitesimal transformations, we cannot *a priori* be sure that this is true.

We have already seen before that radiation terms are irrelevant. Thus, we may as well stick to the power series [2, (5.6)]. In order to apply a co-ordinate transformation we assume that the field equations are solved up to the order $2k+1$. Then we know the following quantities:

$$\underset{2}{\gamma_{00}} \cdots \underset{2k}{\gamma_{00}};\ \underset{3}{\gamma_{0m}} \cdots \underset{2k+1}{\gamma_{0m}};\ \underset{4}{\gamma_{mn}} \cdots \underset{2k}{\gamma_{mn}} \tag{4.1}$$

and the equations of motion of the corresponding order are

$$\lambda^4 \underset{4}{\overset{i}{C}}_m(\eta, \zeta) + \ldots + \lambda^{2k} \underset{(2k)}{\overset{i}{C}}_m(\eta, \zeta) = 0 \qquad (i = 1, 2). \tag{4.2}$$

After this step we consider two cases. In the first one we go on in the usual manner of [2], but in the second case we perform an infinitesimal co-ordinate transformation.

Thus in the *first* case we shall calculate the field variables in the old co-ordinate system of [2] up to the $(2k+4)$th order, and similarly we proceed with the equations of motion. The latter will be

$$\lambda^4 \underset{4}{\overset{i}{C}}_m(\eta, \zeta) + \ldots + \lambda^{2k+4} \underset{2k+4}{\overset{i}{C}}_m(\eta, \zeta) = 0 \qquad (i = 1, 2). \tag{4.3}$$

In the *second* case, we proceed in a different way. We perform an infinitesimal transformation before going on with the approximation procedure:

$$x^m = x^{m*} + \lambda^{2k} \underset{(2k)}{T^m}(x^*). \tag{4.4}$$

[1] A more elaborate discussion of this subject by means of a slightly different approach has been given earlier [3].

This changes the values of the field variables (4.1) according to (2.12), where everything is now expressed in x^*. As was shown before, the effect of this co-ordinate transformation (4.4) is the same as choosing different solutions of the field equations at the step *before* obtaining (4.1). Thus, in the "new" co-ordinate system we have, up to the order $2k$, the following expressions for the field variables and equations of motion:

$$(4.5)\quad \begin{cases} \underset{2}{\gamma_{00}} \ldots \underset{2k-2}{\gamma_{00}}, \quad \underset{2k}{\gamma_{00}} - \underset{2k}{T^r}_{,r} \\ \underset{3}{\gamma_{0m}} \ldots \underset{2k-1}{\gamma_{0m}}, \quad \underset{2k+1}{\gamma_{0m}} - \underset{2k}{T^m}_{,0} \\ \underset{4}{\gamma_{mn}} \ldots \underset{2k-2}{\gamma_{mn}}, \quad \underset{2k}{\gamma_{mn}} - \underset{2k}{T^m}_{,n} - \underset{2k}{T^n}_{,m} + \delta_{mn}\underset{2k}{T^s}_{,s} \end{cases}$$

$$(4.6)\qquad \lambda^4 \overset{i}{\underset{4}{C}}(\eta^*\zeta^*) + \ldots + \lambda^{2k} \overset{i}{\underset{(2k)}{C}}(\eta^*\zeta^*) = 0 \qquad (i = 1, 2).$$

In the above equations it is understood that one has to replace the original arguments x in all the occurring functions by x^* (and hence also η, ζ by η^*, ζ^* respectively).

With the values (4.5) for the field variables we can go on with the approximation procedure as in the first case. After performing two more steps in the approximation method we obtain the equations of motion of the order $2k + 4$, now expressed entirely in the *new* co-ordinates. It is to be expected that they will be formally different from those obtained by the procedure performed in the first case above.

To investigate this question we calculate the new equations of motion after taking the new solutions (4.5) for the γ's up to the $2k$th step. Since one has to go through two stages of the approximation method, this is quite a laborious undertaking.

We can simplify the computational work involved by making some special assumptions. We may note that we need the behaviour of the transformation (4.4) in any case only in the neighbourhood of the world-lines of the particles. Thus we can develop the expression for $\underset{2k}{T^\mu}$ around the world-lines into a Taylor series. Herein we assume that the first and second space derivatives shall vanish. Moreover, we assume that only $\underset{2k}{T^m}$ is different from zero, whereas $\underset{2k}{T^0}$ vanishes. Thus we have, near the first world-line, the following co-ordinate transformation[2]

$$(4.7)\qquad x^m = x^{m*} + \lambda^{2k} \overset{1}{\underset{(2k)}{T^m}}(x^*).$$

Then, the only γ which is influenced up to the order $2k + 1$ is γ_{0m}. It becomes, according to (2.10),

$$(4.8)\qquad \underset{2k+1}{\gamma^*_{0m}} = \underset{2k+1}{\gamma_{0m}} - \overset{1}{\underset{2k}{T^m}}_{,0}.$$

[2] The index "1" above T means that this transformation is different from the identity only in the neighbourhood of the *first* world-line.

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When proceeding to the higher approximations, we are interested only in terms that contain $\underset{2k}{T^m}$; the remaining ones are just those which we would have got without the co-ordinate transformation. Keeping only terms that contain T, we obtain the difference between the equations of motion in the new and in the old co-ordinate systems. We may note that we can use the standard co-ordinate conditions [2, (9.8)] throughout; for, (4.8) satisfies these conditions in the neighbourhood of the first world-line and for the subsequent steps we are free to choose any co-ordinate conditions we like.

Starting our calculations, we have first to compute $\underset{2k+3}{\Lambda^*_{0n}}$. The result is

$$\underset{2k+3}{\Lambda^*_{0m}} = \underset{2k+3}{\Lambda_{0m}} + \tfrac{1}{2}\,\phi_{,ms}\,\underset{2k}{\overset{1}{T}}{}^{s}{}_{,0}. \tag{4.9}$$

We may note that the expression (4.9) is obtained from calculations made in [2]. For, we observe that in our problem $\underset{2k}{\overset{1}{T}}{}^m$ can only combine with terms of the order *two* so as to yield expressions of the order $2k + 3$. Hence it is seen that we obtain in $\underset{2k+3}{\Lambda^*_{0m}}$ the same contribution from $\underset{2k}{\overset{1}{T}}{}^m$ as we have in $\underset{5}{\Lambda_{0m}}$ from $-\underset{3}{\gamma_{0m}}$. Thus the result (4.9) is obtained immediately from [2, (A.5.2)].

To find now $\underset{2k+3}{\gamma_{0m}}$ we can, of course, use the standard co-ordinate conditions. Then, we obtain near the first world-line

$$\underset{2k+3}{\gamma^*_{0n}} = \underset{2k+3}{\gamma_{0n}} + \tfrac{1}{2}\,\phi_{,n}\,\underset{2k}{\overset{1}{T}}{}^{l}{}_{,0}\,\{x^l - \eta^l\} - \tfrac{1}{2}\,\phi\,\underset{2k}{\overset{1}{T}}{}^{n}{}_{,0}. \tag{4.10}$$

The next step is to calculate $\underset{2k+4}{\Lambda^*_{mn}}$. We obtain it in a similar way as $\underset{2k+3}{\Lambda^*_{0m}}$ above from the calculations in [2]. Using [2, (A.12.3)], we get the following result:

$$\left\{\begin{aligned} 2\underset{2k+4}{\Lambda^*_{mn}} = {} & 2\underset{2k+4}{\Lambda_{mn}} - \phi_{,0mn}\,\underset{2k}{\overset{1}{T}}{}^{l}{}_{,0}\,\{x^l - \eta^l\} - \underline{\phi_{,mn}\,\underset{2k}{\overset{1}{T}}{}^{l}{}_{,00}\,\{x^l - \eta^l\}} \\ & - 2\underset{2k}{\overset{1}{T}}{}^{l}{}_{,0}\,\underset{3}{\gamma_{0l,mn}} + \underset{2k}{\overset{1}{T}}{}^{l}{}_{,0}\,\underset{3}{\gamma_{0n,ml}} + \underset{2k}{\overset{1}{T}}{}^{m}{}_{,0}\,\phi_{,0n} \\ & + \phi_{,0m}\underset{2k}{\overset{1}{T}}{}^{n}{}_{,0} - 2\delta_{mn}\underset{2k}{\overset{1}{T}}{}^{l}{}_{,0}\,\phi_{,0l} + \underline{\phi_{,m}\,\underset{2k}{\overset{1}{T}}{}_{,00}} \\ & + \underline{\phi_{,n}\,\underset{2k}{\overset{1}{T}}{}^{m}{}_{,00}} + \underset{2k}{\overset{1}{T}}{}^{s}{}_{,0}\,\underset{3}{\gamma_{0m,ns}} + \phi_{,mn}\,\underset{2k}{\overset{1}{T}}{}^{l}{}_{,0}\,\dot{\eta}^l. \end{aligned}\right. \tag{4.11}$$

The next step is to find the surface integrals

$$\underset{2k+4}{\overset{1}{C}}{}_m = \int \underset{2k+4}{\Lambda^*_{mn}}\,n_n\,ds. \tag{4.12}$$

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Only the underlined terms in (4.11) can give a contribution, since only these approach infinity like $1/r^2$ near the first particle. Evaluating these integrals yields the following result:

$$(4.13) \qquad \underset{2k+4}{\overset{1}{C^*_m}} = \underset{2k+4}{\overset{1}{C_m}} + 4\underset{2k}{\overset{1}{T^m}}{}_{,00}\,\overset{1}{m}.$$

Thus, the equations of motion have in the new co-ordinate system the following form:

$$(4.14) \qquad \lambda^4 \underset{4}{\overset{1}{C_m}} + \ldots + \lambda^{2k+4}\{\underset{2k+4}{\overset{1}{C_m}} + 4\underset{2k}{\overset{1}{T^m}}{}_{,00}\,\overset{1}{m}\} = 0.$$

Our work above was referring to one world-line only. However, we can arrange very well that our transformation term $\underset{2k}{\overset{1}{T^m}}$ is zero near the second world-line so as not to influence the surface integrals around the second particle. Conversely, we can assume another transformation $\underset{2k}{\overset{2}{T^m}}$ of a type similar to the first one, which changes the surface integrals for the second particle only. Then, both transformations together yield the following equations of motion of the order $2k + 4$:

$$(4.15) \qquad \lambda^4 \underset{4}{\overset{i}{C_m}} + \ldots + \lambda^{2k+4}[\underset{2k+4}{\overset{i}{C_m}} + 4\underset{2k}{\overset{i}{T^m}}{}_{,00}\,\overset{i}{m}] = 0 \qquad (i = 1, 2).$$

The old functions η, ζ are functions of time only. Hence it is seen that it is always possible to choose transformations so that the square brackets vanish. This means that we can always choose a transformation $\underset{2k}{T^\mu}$ so that the equations of motion of the order $2k + 4$ do not contain any terms of that order at all. Thus we see that our restrictive assumptions for the admissible co-ordinate transformations are still sufficiently wide to allow us to construct a co-ordinate system where the co-efficient of λ^{2k} in the equations of motion of the order $2j > 2k$ vanishes.

This argument can be repeated. Let us assume that we have solved the field equations up to the order $2j > 2$ and obtained the equations of motion of the same order. Then, we can preform a co-ordinate transformation choosing $\underset{2}{\overset{1.2}{T^m}}$ so that the terms connected with the power λ^6 vanish in the new equations of motion. That will change all of the $\underset{2k}{C_m}$'s with $k<j$. Then, we transform the equations of motion again, choosing $\underset{4}{T}$ so that the new $\underset{8}{C}$'s in that new co-ordinate system vanish, etc. Finally, we shall end up with differential equations of motion of the $2j$th order, but containing only $\underset{4}{C_m}$. This means that it is always possible to construct a co-ordinate system so that the equations of motion are just Newtonian.[3]

[3] A detailed investigation of all these statements may be found in [8].

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This is the standard form to which the differential equations of motion can be reduced. We cannot go further and reduce e.g. the two particles to being at rest with respect to each other, because we have explicitly assumed that our admissible co-ordinate transformations be different from the identity transformation at most by a term proportional to λ^2.

It is also possible to arrive at the same result by expressing η, ζ in (4.3) in the new system by means of (4.4) instead of calculating the equations of motion anew in the transformed co-ordinate system.

5. Conclusions. We have seen that it is always possible to set up such a co-ordinate system that the relativistic equations of motion of any order have Newtonian form. We shall now investigate what conclusions can be drawn from this statement.

First of all, we have to emphasize that our foregoing mathematical deductions do *not* imply that the motion is the same as it would be unrelativistically in such a specially chosen co-ordinate system. Only the *form* of the differential equations of motion is Newtonian; we must keep in mind, however, that the metric in this case is by no means of Newtonian character near the singularities. If we wish that the metric field be of Newtonian character near its sources, then the motion is non-Newtonian and the equations of motion are as calculated in [2].

So far, this does not yield any new ideas. We may note, however, that the above statement about the possible Newtonian form of the equations of motion can be formulated in a slightly different way. For, we observe that it is the same thing as saying that, at every step of the approximation procedure, we can reach the vanishing of the corresponding surface integrals by choosing the co-ordinate system two steps before in an appropriate way. This shows that we really have found a new version of the method in [2] for solving Einstein's field equations, which is equivalent to the one introducing and annihilating dipoles.

Let us formulate this conception somewhat more precisely. Assume that the field equations are to be solved by making the usual expansion [2, (5.6)] of the field variables with respect to the parameter λ. Suppose the field equations have been solved up to a certain stage. Proceeding one step further, we are faced with the task of solving the following system of equations:

$$\underset{2k-2}{\Phi_{00}} + 2\underset{2k-2}{\Lambda_{00}} = 0 \tag{5.1a}$$

$$\underset{2k-1}{\Phi_{0m}} + 2\underset{2k-1}{\Lambda_{0m}} = 0 \tag{5.1b}$$

$$\underset{2k}{\Phi_{mn}} + 2\underset{2k}{\Lambda_{mn}} = 0 \tag{5.1c}$$

(cf. [2, (8.1)]). Because of the Bianchi identities this system is generally not solvable. In order to solve (5.1c) we have to add dipoles to the *known* solution $\underset{2k-2}{\gamma_{00}}$. For, then we can obtain that the surface integrals

$$\underset{2k-1}{\overset{i}{C_0}} = \frac{1}{4\pi}\int_i \underset{2k-1}{\Lambda_{0n}}\, n_n\, dS \tag{5.2}$$

$$\underset{2k}{\overset{i}{C_m}} = \frac{1}{4\pi}\int_i \underset{2k}{\Lambda_{mn}} n_n\, dS \tag{5.3}$$

vanish, which is the condition of integrability. Indeed, by inspecting (5.3) we note that adding dipoles $\underset{2k-2}{\overset{i}{S_m}}\psi_{,m}$ to $\underset{2k-2}{\gamma_{00}}$ changes these surface integrals into $\underset{2k}{\overset{i}{C^*_m}}$ with

$$\underset{2k}{\overset{i}{C^*_m}} = \underset{2k}{\overset{i}{C_m}} + \underset{2k-2}{\overset{i}{S_{m,\,00}}} \tag{5.4}$$

which can be made zero by choosing

$$\underset{2k-2}{\overset{i}{S_{m,\,00}}} = -\underset{2k}{\overset{i}{C_m}}. \tag{5.5}$$

However, the contribution to the surface integrals obtained by adding dipoles to $\underset{2k-2}{\gamma_{00}}$ is very similar to the one obtained by performing a co-ordinate transformation at the $(2k-4)$th stage of the approximation procedure. We have seen that a co-ordinate transformation

$$x^m = x^{m*} + \lambda^{2k-4} \underset{(2k-4)}{T^m}(x^*) \tag{5.6}$$

causes a change in the surface integrals. This change is, if we assume that the space derivatives of T vanish,

$$\underset{2k}{\overset{i}{C^*_m}} = \underset{2k}{\overset{i}{C_m}}(\eta^*, \zeta^*) + 4\overset{i}{m}\, \underset{2k-4}{\overset{i}{T^m}}{}_{,\,00}(x^* = \eta^*) \tag{5.7}$$

Thus, to enforce the integrability of (5.1c) we can either add dipoles or change the co-ordinate system according to (5.6). Furthermore, we see that $\underset{2k-4}{S_{m,\,00}}$ in $\underset{2k-2}{\gamma_{00}}$ has the same effect upon the equations of motion as $4m\underset{2k-4}{T^m}$ in the $(2k-4)$th step of the approximation.

We may emphasize once more that the form in which the equations of motion finally appear does not influence any of the well known results of general relativity theory. It is only a matter of representation whether these relativistic effects are explicitly contained in the equations of motion or in the metric field.

Let us illustrate this by a specific example. Robertson [7] has integrated the differential relativistic equations of motion of the sixth order for the two-body problem. His result was that one obtains the same effect as when considering the motion of a small body in the Schwarzschild field of a large one by applying the geodesic principle; i.e., the orbit of a double star in general relativity theory

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differs in its secular behaviour from the classical orbit only in an advance of perihelion equal to that which an infinitesimal planet, describing the same relative orbit, would undergo in the field of a star whose mass is the sum of those of the two components of the double star. Hence it is intuitively seen that introducing a co-ordinate system rotating at the right speed will reduce the non-Newtonian orbit to a Newtonian one. This co-ordinate transformation needs only to take place in the immediate neighbourhood of the trajectory, a statement which is in agreement with the fact that we had to know T^m only near the world lines of the particles. A detailed investigation (in [8]) shows indeed that it is possible to find such a co-ordinate system and also that the metric in that system contains components $g_{0m} \neq 0$ which confirms that light rays no longer have a simple trajectory in that system.

Thus we obtain either simple (Newtonian) equations of motion and a complicated metric field, or a simple field (of Newtonian character near the singularities) but non-Newtonian equations of motion.

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