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ON VECTOR-VALUED LIPSCHITZ FUNCTION SPACES

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ABSTRACT. This paper is devoted to obtaining sequence space representations of spaces of vector-valued C^k-functions defined on an open subset, Ω, of $\mathbb{R}^n$, whose k^{th} derivatives satisfy a Lipschitz condition on compact subsets of Ω.

1. **Introduction and notation**. Representing a space of functions means finding topological isomorphisms with other better known spaces (usually sequence spaces). In 1960 Ciesielski [4] showed that the spaces $\Lambda^w([0, 1])$ and $\lambda^w([0, 1])$ are isomorphic to l^∞ and c_0 resp.. A few years later Bonic, Frampton and Tromba [3] extended this result to compact subsets, X, of $\mathbb{R}^n$ and represented $\Lambda^w(X, E)$ (resp. $\lambda^w(X, E)$) as $l^\infty(E)$ (resp. $c_0(E)$), E being a Banach space. In a quite different way Wulbert [16] characterised the compact metric spaces, Y, such that $\lambda^w(Y)$ separates points of Y and is isomorphic to c_0. Frampton and Tromba [6] found that $\lambda^{k,w}(L)$ is isomorphic to c_0 and $\Lambda^{k,w}(L)$ to l^∞, L being a compact C^∞-Riemanian n-manifold. A deep study of Lipschitz functions was done by Glaeser [9] including a version of the Whitney extension theorem and the identification between the bidual of $\lambda^{k,w}(Q)$ and $\Lambda^{k,w}(Q)$, Q being a non-trivial compact n-interval.

To get our representations we use representations of $\lambda^{k,w}(Q, E)$ and $\Lambda^{k,w}(Q, E)$ (see definition 1), Frampton and Tromba results, an adequate vector-valued version of the Whitney extension theorem and we follow a method due to Valdivia [15].

Definitions and simple properties of vector-valued C^k-functions can be found in ([8]). For the general theory of locally convex spaces we refer to ([10]) or ([13]).

In what follows we denote by E a separated locally convex space (l.c.s.) and by cs(E) the system of all continuous seminorms defined on it. The symbol $\simeq$ means topological isomorphism. We denote by the greek letters α and β n-indices whose component sum is less or equal than k, $k \in \mathbb{N}$ fixed.

If $M \subset \mathbb{R}^n$ is compact, we say, as usual, that $f: M \to E$ is a C^k-function if there exists a C^k extension, $\hat{f}$, of f to an open set including M. Q is a compact n-interval with non-empty interior. If $M = Q$ the definition of a C^k-function is equivalent to saying that f is a C^k-function on $\overset{\circ}{Q}$ and its derivatives have continuous extension to Q (see [7] for instance). If $q \in$ cs(E), we set $\|f\|_q^M := \Sigma_\alpha \sup\{q[D^\alpha f(x)] : x \in M\}$.

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Let w be a real number, $0 < w < 1$.

DEFINITION 1. *We say that* $f: M \to E$ *belongs to* $\Lambda^{k,w}(M, E)$ *if it is* C^k *on* M *and for every* $q \in \mathrm{cs}(E)$

(D.1) $$\|f\|_q^{M,w} := \max\left\{\sup\left\{\frac{q[D^\alpha f(x) - D^\alpha f(y)]}{d(x, y)^w} : \begin{matrix} x, y \in M \\ x \neq y \end{matrix}\right\} : |\alpha| = k\right\} < +\infty.$$

(D.2) If, in addition, $\displaystyle\lim_{d(x,y)\to 0} \frac{D^\alpha f(x) - D^\alpha f(y)}{d(x,y)^w} = 0 \qquad |\alpha| = k$, then f is said to

belong to $\lambda^{k,w}(M, E)$. ($d(\cdot, \cdot)$ is the usual metric on $\mathbb{R}^n$).

The topology on $\Lambda^{k,w}(M, E)$ is given by the family of seminorms $\{\|\cdot\|_q^M + \|\cdot\|_q^{M,w} : q \in \mathrm{cs}(E)\}$ and $\lambda^{k,w}(M, E)$ is endowed with the induced topology. (When $E = \mathbb{R}$ or $\mathbb{C}$, we omit q).

The following spaces are constructed because they are useful in finding our representations

$$\mathcal{D}^{k,w}(M, E) := \{f \in C^k(\mathbb{R}^n, E) : \operatorname{supp} f \subset M \text{ and } f_{|_M} \in \Lambda^{k,w}(M, E)\}$$

$$\lambda_c^{k,w}(M, E) := \{f \in C^k(\mathbb{R}^n, E) : \operatorname{supp} f \subset M \text{ and } f_{|_M} \in \lambda^{k,w}(M, E)\}$$

We get from these definitions that every f belonging to $\mathcal{D}^{k,w}(M, E)$ (resp. to $\lambda_c^{k,w}(M, E)$) is a function on $\mathbb{R}^n$ satisfying condition D.1 (resp. D.2) with $\mathbb{R}^n$ instead M. From now on we identify every function, f, of $\mathcal{D}^{k,w}(M, E)$ with its restriction, $f_{|_M}$, to M and consequently we consider $\mathcal{D}^{k,w}(M, E)$ and $\lambda_c^{k,w}(M, E)$ as topological subspaces of $\Lambda^{k,w}(M, E)$.

2. **Previous results**. For our purposes the following vector-valued version of Glaeser's extension theorem is enough.

THEOREM 1. *Let* A, B *be two* n*-intervals,* $A \subset \overset{\circ}{B}$. *There is a continuous linear mapping* $T: \Lambda^{k,w}(A, E) \to \mathcal{D}^{k,w}(B, E)$ *such that* $Tf_{|_A} = f$ *for every* $f \in \Lambda^{k,w}(A, E)$, *i.e.,* T *is a linear extension operator. Furthermore if* $f \in \lambda^{k,w}(A, E)$, *then* $Tf \in \lambda_c^{k,w}(B, E)$.

Applying this theorem we obtain the first part of the next proposition. the second part can be shown using Valdivia's techniques to be found in ([15]), p. 380 (14)).

PROPOSITION 1. *a)* $\Lambda^{k,w}(Q, E)$ *(resp.* $\lambda^{k,w}(Q, E)$*) is isomorphic to a complemented subspace of* $\mathcal{D}^{k,w}(Q, E)$ *(resp.* $\lambda_c^{k,w}(Q, E)$*). b)* $\mathcal{D}^{k,w}(Q, E)$ *(resp.* $\lambda_c^{k,w}(Q, E)$*) is isomorphic to a complemented subspace of* $\Lambda^{k,w}(Q, E)$ *(resp.* $\lambda^{k,w}(Q, E)$*).*

Using ϵ-products of L. Schwartz we have (see also [11] 3.4 Th.)

PROPOSITION 2. *a) If* E *is semi-Montel, then* $\Lambda^{k,w}(Q, E) \simeq \Lambda^{k,w}(Q) \in E$. *b) If* E *is quasi-complete, then* $\lambda^{k,w}(Q, E) \simeq \lambda^{k,w}(Q) \in E$.

To obtain isomorphisms in the following theorem and in the next section we introduce the following property: "A l.c.s. G has the Pełczynski complementation property (P.c.p.) if given any l.c.s. F isomorphic to a complemented subspace of G and

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containing a complemented subspace isomorphic to G, then $G \simeq F$". If J is an infinite set, we denote G^J and $G^{(J)}$ the product and the locally convex direct sum of card(J) copies of G resp.. Both spaces have P.c.p. (see [15] p. 332). $l^\infty(G)$ and $c_0(G)$ have P.c.p. too ([1]).

We recall the representation results of Frampton and Tromba, i.e., $\Lambda^{k,w}(Q) \simeq l^\infty$ and $\lambda^{k,w}(Q) \simeq c_0$. Then applying propositions 1 and 2 and the P.c.p. we obtain

THEOREM 2. *a) If E is semi-Montel, then $\Lambda^{k,w}(Q, E) \simeq \mathfrak{D}^{k,w}(Q, E) \simeq l^\infty(E)$. b) If E is quasi-complete, then $\lambda^{k,w}(Q, E) \simeq \lambda_c^{k,w}(Q, E) \simeq c_0(E)$.*

Unfortunately we are not able to remove in general the condition assumed on E in a), however the result can be strengthened by taking E to be quasi-complete when we restrict ourselves to 1-intervals. (see [7]).

REMARK. It is a well known fact that $\lambda^{k,w}(Q)'' \simeq \Lambda^{k,w}(Q)$ (see [9] for instance). This is not always true in the vector-valued case as is easily seen taking $E = c_0$, then $\lambda^{k,w}(Q, c_0) \simeq c_0(c_0) \simeq c_0$ so $\lambda^{k,w}(Q, c_0)'' \simeq l^\infty$ but $\Lambda^{k,w}(Q, c_0) \simeq l^\infty(c_0)$ is not isomorphic to l^∞ because it contains c_0 as complemented subspace.

3. **The main results**. From now on Ω denotes a non-void open subset of $\mathbb{R}^n$.

DEFINITION 2. *We say that $f : \Omega \to E$ belongs to $C^{k,w}(\Omega, E)$ (resp. $Z^{k,w}(\Omega, E)$) if it is a C^k-function and for any compact set, M, included in Ω, $f_{|M} \in \Lambda^{k,w}(M, E)$ (resp. $f_{|M} \in \lambda^{k,w}(M, E)$).*

The topology of $C^{k,w}(\Omega, E)$ is defined by the family of seminorms $\{\| \; \|_q^M + \| \; \|_q^{M,w} : q \in \mathrm{cs}(E)$, M compact set in $\Omega\}$. $Z^{k,w}(\Omega, E)$ has the induced topology (see also [12] p. 154).

We set $\mathfrak{D}^{k,w}(\Omega, E) := \{f \in C^{k,w}(\mathbb{R}^n, E)$: supp f is a compact subset of $\Omega\}$. Then $\mathfrak{D}^{k,w}(\Omega, E) = \bigcup\{\mathfrak{D}^{k,w}(M, E) : M$ compact set in $\Omega\}$ and therefore we provide $\mathfrak{D}^{k,w}(\Omega, E)$ with the inductive limit topology of the family $\{\mathfrak{D}^{k,w}(M, E) : M$ compact set in $\Omega\}$.

In an analogous way, we set $Z_c^{k,w}(\Omega, E) := \{f \in Z^{k,w}(\mathbb{R}^n, E)$: supp f is a compact subset of $\Omega\}$ and we define on $Z_c^{k,w}(\Omega, E)$ the inductive limit topology of $\{\lambda_c^{k,w}(M, E) : M$ compact set in $\Omega\}$.

Now we give representations of these spaces. We let I^n denote the unit cube in $\mathbb{R}^n$, $I^n := [0, 1]^n$.

Let $\{Q_m : m \in \mathbb{N}\}$ be a locally finite family of n-intervals in Ω such that $\{\mathring{Q}_m : m \in \mathbb{N}\}$ is a covering of Ω. Let $\{g_m : m \in \mathbb{N}\}$ be a C^∞ partition of unity subordinated to this covering. For the next lemma we write $\|g_m\|^{k+1}$ for the norm of g_m in $C^{k+1}(Q_m)$.

LEMMA. *Fix $m \in \mathbb{N}$. If $g \in C^{k,w}(\Omega, E)$ (resp. $Z^{k,w}(\Omega, E)$), then $g \cdot g_m \in \mathfrak{D}^{k,w}(Q_m, E)$ (resp. $\lambda_c^{k,w}(Q_m, E)$). Furthermore there exists a positive constant, C_m, not depending on g such that*

$$\|g \cdot g_m\|_q^{Q_m, w} \leqslant C_m (\|g\|_q^{Q_m} + \|g\|_q^{Q_m, w}) \quad q \in \mathrm{cs}(E).$$

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PROOF. Define $g \cdot g_m$ as 0 out of Ω. Take α, $q \in \mathrm{cs}(E)$ and $x, y \in Q_m$ $x \neq y$, then

$$q[D^\alpha(g\cdot g_m)(x) - D^\alpha(g\cdot g_m)(y)] \leq \sum_{\beta\leq\alpha} q[D^\beta g(x)\cdot D^{\alpha-\beta}g_m(x)$$
$$- D^\beta g(y) D^{\alpha-\beta}g_m(y)] \leq \sum_{\beta\leq\alpha} q[D^\beta g(x) D^{\alpha-\beta}g_m(x)$$
$$- D^\beta g(x) D^{\alpha-\beta} g_m(y)] + q[D^\beta g(x) D^{\alpha-\beta}g_m(y) - D^\beta g(y) D^{\alpha-\beta}g_m(y)]$$
$$\leq \sum_{\beta\leq\alpha} q[D^\beta g(x)]\cdot |D^{\alpha-\beta}g_m(x) - D^{\alpha-\beta}g_m(y)| + q[D^\beta g(x)$$
$$- D^\beta g(y)]\cdot |D^{\alpha-\beta}g_m(y)|.$$

If we set $U: = \{u \in E : q(u) < 1\}$, then for any $v \in U^0$ and any β, $|\beta| < k$, we apply the mean value theorem to the functions $\mathrm{Re}[v \circ D^\beta g]$ and $\mathrm{Im}[v \circ D^\beta g]$ to obtain $|v[D^\beta g(x) - D^\beta g(y)]| \leq 2\sqrt{n}\cdot\|g\|_q^{Q_m}\cdot d(x,y)$ and so $q[D^\beta g(x) - D^\beta g(y)] \leq 2\sqrt{n}\cdot\|g\|_q^{Q_m}\cdot d(x,y)$. In the case that $|\beta| = k$, we obtain directly $q[D^\beta g(x) - D^\beta g(y)] \leq \|g\|_q^{Q_m,w}\cdot d(x,y)^w$. Now we apply again the mean value theorem to g_m for finding analogous inequalities. Collecting all them we get

$$\frac{q[D^\alpha g\cdot g_m(x) - D^\alpha g\cdot g_m(y)]}{d(x,y)^w} \leq \sum_{|\beta|<k} \|g\|_q^{Q_m}\cdot\|g_m\|^{k+1}\cdot 2\sqrt{n}\cdot d(x,y)^{1-w} + \sum_{|\beta|=k}\|g_m\|$$
$$\cdot\frac{q[D^\beta g(x) - D^\beta g(y)]}{d(x,y)^w} \leq (\|g\|_q^{Q_m} + \|g\|_q^{Q_m,w})\|g_m\|^{k+1}\Big(\sum_\beta 2\sqrt{n}\cdot\delta(Q_m) + 1\Big)$$

from where the lemma follows. □

PROPOSITION 3. *$C^{k,w}(\Omega, E)$ (resp. $Z^{k,w}(\Omega,E)$) is isomorphic to a complemented subspace of $\Lambda^{k,w}(I^n,E)^{\mathbb{N}}$ (resp. $\lambda^{k,w}(I^n,E)^{\mathbb{N}}$.*

PROOF. Construct $Y: C^{k,w}(\Omega,E) \to \Pi_{m=1}^\infty \mathcal{D}^{k,w}(Q_m,E)$ according to $Y(g): = (g\cdot g_m)_m$ Y is well defined, linear and continuous. So happens also with its restriction $Y: Z^{k,w}(\Omega,E) \to \Pi_{m=1}^\infty \lambda_c^{k,w}(Q_m,E)$.

Let $X: \Pi_{m=1}^\infty \mathcal{D}^{k,w}(\mathring{Q}_m, E) \to C^{k,w}(\Omega,E)$ be the mapping defined by $X[(f_m)] := \Sigma_m f_m$. Since $\{Q_m\}$ is locally finite it is easy to see that X is well defined linear and continuous. The same is true for its restriction $X: \Pi_{m=1}^\infty \lambda_c^{k,w}(Q_m,E) \to Z^{k,w}(\Omega,E)$.

Since $(X\circ Y)(g)(x) = \Sigma_m (g\cdot g_m)(x) = g(x)$ for each $x \in \Omega$ we have that Y is an isomorphism from $C^{k,w}(\Omega,E)$ (resp. $Z^{k,w}(\Omega,E)$) onto its image. But we also have that $Y\cdot X$ is a continuous projection onto $Y(C^{k,w}(\Omega,E))$ (resp. $Y(Z^{k,w}(\Omega,E))$. Now the proposition follows from proposition 1.b). □

Let $\{P_m : m \in \mathbb{N}\}$ be a sequence of n-intervals in Ω with non-empty interior, pairwise disjoint. For every m we find another n-interval, O_m, such that $\emptyset \neq \mathring{Q}_m \subset O_m \subset P_m$ and according to theorem 1 there exists a linear extension operator $T_m: \Lambda^{k,w}(O_m,E) \to \mathcal{D}^{k,w}(P_m,E)$ $(T_m: \lambda^{k,w}(O_m,E) \to \lambda_c^{k,w}(P_m,E))$.

PROPOSITION 4. *There is a complemented subspace of* $C^{k,w}(\Omega, E)$ *(resp.* $Z^{k,w}(\Omega, E)$*) isomorphic to* $\Lambda^{k,w}(I^n, E)^{\mathbb{N}}$ *(resp.* $\lambda^{k,w}(I^n, E)^{\mathbb{N}}$*).*

PROOF. Define $S: \Pi_{m=1}^{\infty} \Lambda^{k,w}(O_m, E) \to C^{k,w}(\Omega, E)$ by $S[(f_m)(x) := \Sigma_m T_m f_m(x)$ for each $x \in \Omega$, S is well defined, linear and continuous. (Clearly $S: \Pi_{m=1}^{\infty} \lambda^{k,w}(O_m, E) \to Z^{k,w}(\Omega, E)$). Now taking $R: C^{k,w}(\Omega, E) \to \Pi_{m=1}^{\infty} \Lambda^{k,w}(O_m, E)$ as $R(g) := (g_{|O_m})_m$, R is also linear and continuous. $(R: Z^{k,w}(\Omega, E) \to \Pi_{m=1}^{\infty} \lambda^{k,w}(O_m, E))$. Moreover $(R \circ S)[(f_m)] = \{\Sigma_i T_i f_{i|O_m}\}_m = (f_m)$ so $R \circ S = \mathrm{Id}$ and then proceeding as at the end of the later proposition, the proof finishes. □

THEOREM 3.

$$C^{k,w}(\Omega, E) \simeq \Lambda^{k,w}(I^n, E)^{\mathbb{N}} \simeq \mathcal{D}^{k,w}(I^n, E)^{\mathbb{N}}$$

$$Z^{k,w}(\Omega, E) \simeq \lambda^{k,w}(I^n, E)^{\mathbb{N}} \simeq \lambda_c^{k,w}(I^n, E)^{\mathbb{N}}.$$

Furthermore if E is semi-Montel, $C^{k,w}(\Omega, E) \simeq l^{\infty}(E)^{\mathbb{N}}$ *and if E is quasi-complete,* $Z^{k,w}(\Omega, E) \simeq c_o(E)^{\nu}$.

PROOF. It is enough to apply the former results having in mind the P.c.p. □

The next theorem, stated without proof, is proven in an analogous way to the former one, simply substituting infinite products by locally convex direct sums and taking adequate restrictions of the mapping X, Y, R and S.

THEOREM 4.

$$\mathcal{D}^{k,w}(\Omega, E) \simeq \Lambda^{k,w}(I^n, E)^{(\mathbb{N})} \simeq \mathcal{D}^{k,w}(I^n, E)^{(\mathbb{N})}$$

$$Z_c^{k,w}(\Omega, E) \simeq \lambda^{k,w}(I^n, E)^{(\mathbb{N})} \simeq \lambda_c^{k,w}(I^n, E)^{(\mathbb{N})}.$$

If E is semi-Montel, $\mathcal{D}^{k,w}(\Omega, E) \simeq l^{\infty}(E)^{(\mathbb{N})}$ *and if E is quasi-complete,* $Z_c^{k,w}(\Omega, E) \simeq c_o(E)^{(\mathbb{N})}$.

In the scalar case we obtain the following isomorphisms:

$$C^{k,w}(\Omega) \simeq l^{\infty\mathbb{N}},\ Z^{k,w}(\Omega) \simeq c_o^{\mathbb{N}},\ \mathcal{D}^{k,w}(\Omega) \simeq l^{\infty(\mathbb{N})} \quad \text{and} \quad Z_c^{k,w}(\Omega) \simeq c_o^{(\mathbb{N})}.$$

Consequently the isomorphisms between the strong bidual of $Z^{k,w}(\Omega)$ (resp. $Z_c^{k,w}(\Omega)$) and $C^{k,w}(\Omega)$ (resp. $\mathcal{D}^{k,w}(\Omega)$) still hold.

We recall the following abstract theorem dealing with the stability of barrelledness properties of ϵ-products: "If E is a quasi-complete l.c.s. then $l^{\infty}\epsilon E$ is barrelled iff $c_o \epsilon E$ is barrelled iff E is barrelled and its strong dual E_b' has the property (B) of Pietsch ([14] p. 30)" ([2] and [5]).

As a consequence we obtain

THEOREM 5. *Let E be a l.c.s. such that* E_b' *has the property (B) of Pietsch. If E is Montel, then* $\Lambda^{k,w}(Q, E)$, $\mathcal{D}^{k,w}(Q, E)$, $C^{k,w}(\Omega, E)$ and $\mathcal{D}^{k,w}(\Omega, E)$ *are barrelled. If E is quasi-complete and barrelled,* $\lambda^{k,w}(Q, E)$, $\lambda_c^{k,w}(Q, E)$, $Z^{k,w}(\Omega, E)$ *and* $Z_c^{k,w}(\Omega, E)$ *are barrelled.*

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