A FEW INFINITE INTEGRALS INVOLVING E-FUNCTIONS

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1. The object of this paper is to evaluate a few infinite integrals involving E-functions by applying the Parseval-Goldstein [**1**] theorem of Operational Calculus; that, if

$$\phi(p) \doteqdot h(x)$$

and
$$\psi(p) \doteqdot g(x),$$

then
$$\int_0^\infty \phi(x) g(x) x^{-1}\, dx = \int_0^\infty \psi(x) h(x) x^{-1}\, dx, \qquad (1)$$

when the integrals are convergent.

The notation $\phi(p) \doteqdot h(x)$ or $h(x) \doteqdot \phi(p)$ means that

$$\phi(p) = p \int_0^\infty e^{-px} h(x)\, dx.$$

We shall require the following results [**2**, 98], [**4**, 134].

$$\int_0^\infty e^{-zx} x^{\gamma-1} (1+x)^{\alpha+\beta-\delta} {}_2F_1(\alpha, \beta;\ \delta;\ -x)\, dx$$
$$= \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} z^{-\gamma} E(\delta-\alpha, \delta-\beta, \gamma : \delta : z), \quad R(\gamma)>0,\ R(z)>0. \qquad (2)$$

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E(\gamma, \delta : : \lambda)\, E(p;\ \alpha_r : q;\ \rho_s : z/\lambda^m)\, d\lambda$$
$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{k-\frac{1}{2}} \Gamma(\gamma)\Gamma(\delta)\, E(p+2m;\ \alpha_r : q+m;\ \rho_s : z/m^m), \qquad (3)$$

where $\alpha_{p+\nu+1} = (\gamma+k+\nu)/m$, $\alpha_{p+m+\nu+1} = (\delta+k+\nu)/m$, $\rho_{q+\nu+1} = (\gamma+\delta+k+\nu)/m$, $\nu = 0, 1, 2, \ldots, m-1$; $R(k+\gamma)>0$, $R(k+\delta)>0$.

2. (i) Take
$$\phi(p) = \frac{\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)} p(p+z)^{-\gamma} E(\alpha, \beta, \gamma : \delta : p+z)$$
$$\doteqdot e^{-zx} x^{\gamma-1} {}_2F_1(\alpha, \beta;\ \delta;\ -x)$$
$$= h(x), \quad R(\gamma)>0,$$

and
$$g(x) = \frac{x^{\delta-\alpha-\beta-\nu-1}}{\Gamma(\delta-\alpha-\beta-\nu)} {}_1F_1(\delta-\alpha-\beta;\ \delta-\alpha-\beta-\nu;\ -x)$$
$$\doteqdot p^{\nu+1}(1+p)^{\alpha+\beta-\delta} = \psi(p), \quad R(\delta-\alpha-\beta-\nu)>0.$$

Using these relations in (1) and evaluating the integral on the right with the help of (2), we get

$$\int_0^\infty x^{\delta-\alpha-\beta-\nu-1} (x+z)^{-\gamma} E(\alpha, \beta, \gamma : \delta : x+z) {}_1F_1(\delta-\alpha-\beta;\ \delta-\alpha-\beta-\nu;\ -x)\, dx$$
$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\delta-\alpha-\beta-\nu)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} z^{-\gamma-\nu} E(\delta-\alpha, \delta-\beta, \gamma+\nu : \delta : z), \qquad (4)$$

valid, by analytic continuation, for $R(\gamma+\nu)>0$, $R(\delta-\alpha-\beta-\nu)>0$, $|\arg z|<\pi$, $z\neq 0$.

In particular, when $\nu=0$, we have

$$\int_0^\infty e^{-x}x^{\delta-\alpha-\beta-1}(x+z)^{-\gamma}E(\alpha,\beta,\gamma:\delta:x+z)\,dx$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}z^{-\gamma}E(\delta-\alpha,\delta-\beta,\gamma:\delta:z), \qquad \ldots\ldots(5)$$

where $R(\delta-\alpha-\beta)>0$, $|\arg z|<\pi$, $z\neq 0$.

(ii) Now take

$$\psi(p)=p^{\gamma-\nu+1}{}_2F_1(\delta-\alpha,\delta-\beta;\ \delta;\ -p)$$
$$=\Gamma(\delta)p^{\gamma-\nu-\delta+1}\sum_{\alpha,\beta}\left\{\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\delta-\beta)}p^{\alpha}{}_2F_1\left(\delta-\alpha,1-\alpha;\ 1+\beta-\alpha;\ -\frac{1}{p}\right)\right\}$$
$$\risingdotseq \Gamma(\delta)x^{\nu+\delta-\gamma-1}$$
$$\times\sum_{\alpha,\beta}\left\{\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\delta-\beta)\Gamma(\delta+\nu-\gamma-\alpha)}x^{-\alpha}{}_2F_2(\delta-\alpha,1-\alpha;\ 1+\beta-\alpha,\delta+\nu-\gamma-\alpha;\ -x)\right\}$$
$$=g(x),\quad R(\delta+\nu-\gamma-\alpha)>0,\quad R(\delta+\nu-\gamma-\beta)>0,$$

and

$$h(x)=e^{-zx}x^{\nu-1}(1+x)^{\delta-\alpha-\beta}$$
$$\risingdotseq \frac{p(p+z)^{-\nu}}{\Gamma(\alpha+\beta-\delta)}E(\nu,\alpha+\beta-\delta::p+z)$$
$$=\phi(p),\quad R(\nu)>0.$$

Applying (1) and using (2) we get

$$\int_0^\infty x^{\delta+\nu-\gamma-1}(x+z)^{-\nu}E(\nu,\alpha+\beta-\delta::x+z)$$
$$\times\left\{\sum_{\alpha,\beta}\frac{\Gamma(\alpha-\beta)\Gamma(\beta)x^{-\alpha}}{\Gamma(\delta-\beta)\Gamma(\delta+\nu-\gamma-\alpha)}{}_2F_2(\delta-\alpha,1-\alpha;\ 1+\beta-\alpha,\delta+\nu-\gamma-\alpha;\ -x)\right\}dx$$
$$=\Gamma(\alpha+\beta-\delta)z^{-\gamma}E(\alpha,\beta,\gamma:\delta:z), \qquad \ldots\ldots(6)$$

valid, by analytic continuation, for

$R(\delta+\nu-\gamma-\alpha)>0$, $R(\delta+\nu-\gamma-\beta)>0$, $R(\gamma)>0$, $R(\gamma-\delta)>-1$, $|\arg z|<\pi$, $z\neq 0$.

When $\delta\to\alpha+\beta$, this gives

$$\int_0^\infty x^{\alpha+\beta+\nu-\gamma-1}(x+z)^{-\nu}$$
$$\times\left\{\sum_{\alpha,\beta}\frac{\Gamma(\alpha-\beta)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta+\nu-\gamma)}x^{-\alpha}{}_2F_2(\beta,1-\alpha;\ 1+\beta-\alpha,\nu-\gamma+\beta;\ -x)\right\}dx$$
$$=\{\Gamma(\nu)\}^{-1}z^{-\gamma}E(\alpha,\beta,\gamma:\alpha+\beta:z), \qquad \ldots\ldots(7)$$

$R(\alpha+\nu-\gamma)>0$, $R(\beta+\nu-\gamma)>0$, $R(\gamma)>0$, $R(\gamma-\alpha-\beta)>-1$, $|\arg z|<\pi$, $z\neq 0$.

When $\nu=\gamma$, (6) yields a result obtained by me previously [**5**, 174].

(iii) From the integral [**4**, 131]

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha-1}E(l;\ \alpha_r:n;\ \rho_s:z/\lambda^m)\,d\lambda=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\alpha-\frac{1}{2}}E(l+m;\ \alpha_r:n;\ \rho_s:z/m^m), \quad \ldots\ldots(8)$$

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$R(\alpha)>0$, $\alpha_{l+k+1}=(\alpha+k)/m$, $k=0, 1, 2, \ldots, m-1$, we find that

$$x^{\alpha-1}E(l\,;\ \alpha_r:n\,;\ \rho_s:1/x^m) \doteq (2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\alpha-\frac{1}{2}}p^{1-\alpha}E\{l+m\,;\ \alpha_r:n\,;\ \rho_s:(p/m)^m\}, \qquad (9)$$

$R(\alpha)>0$, $\alpha_{l+k+1}=(\alpha+k)/m$, $k=0, 1, 2, \ldots, m-1$.

We take

$$h(x)=e^{-zx}x^{\alpha-1}E(l\,;\ \alpha_r:n+m\,;\ \rho_s:1/x^m)$$
$$\doteq (2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\alpha-\frac{1}{2}}p(p+z)^{-\alpha}E[l\,;\ \alpha_r:n\,;\ \rho_s:\{(p+z)/m\}^m]$$
$$=\phi(p),\quad R(\alpha)>0,\ \rho_{n+k+1}=(\alpha+k)/m,\ k=0, 1, 2, \ldots, m-1,$$

and

$$g(x)=x^{\nu-1}{}_2F_1(\lambda, \mu\,;\ \nu\,;\ -x/z)$$
$$\doteq \frac{\Gamma(\nu)}{\Gamma(\lambda)\,\Gamma(\mu)}p^{1-\nu}E(\lambda, \mu::pz)$$
$$=\psi(p),\quad R(\nu)>0.$$

Applying (1) to the above relations and using (3) we get after a little simplification,

$$\int_0^\infty x^{\nu-1}(x+z)^{-\alpha}{}_2F_1(\lambda, \mu\,;\ \nu\,;\ -x/z)\,E[l\,;\ \alpha_r:n\,;\ \rho_s:\{(x+z)/m\}^m]\,dx$$
$$=\Gamma(\nu)z^{\nu-\alpha}m^{-\nu}E\{l+2m\,;\ \alpha_r:n+2m\,;\ \rho_s:(z/m)^m\}, \qquad (10)$$

where $R(\nu)>0$, $R(\alpha+\lambda-\nu)>0$, $R(\alpha+\mu-\nu)>0$, $|\arg z|<\pi$, $z\neq 0$, $\alpha_{l+k+1}=(\alpha+\lambda-\nu+k)/m$, $\alpha_{l+m+k+1}=(\alpha+\mu-\nu+k)/m$, $\rho_{n+k+1}=(\alpha+k)/m$, $\rho_{n+m+k+1}=(\alpha+\lambda+\mu-\nu+k)/m$, $k=0, 1, 2, \ldots, m-1$.

When $\mu=\nu$, we have

$$\int_0^\infty x^{\nu-1}(x+z)^{-\alpha-\lambda}E[l\,;\ \alpha_r:n\,;\ \rho_s:\{(x+z)/m\}^m]\,dx$$
$$=\Gamma(\nu)z^{\nu-\alpha-\lambda}m^{-\nu}E\{l+m\,;\ \alpha_r:n+m\,;\ \rho_s:(z/m)^m\}, \qquad (11)$$

$R(\nu)>0$, $R(\alpha+\lambda-\nu)>0$, $|\arg z|<\pi$, $z\neq 0$, $\alpha_{l+k+1}=(\alpha+\lambda-\nu+k)/m$, $\rho_{n+k+1}=(\alpha+\lambda+k)/m$, $k=0, 1, 2, \ldots, m-1$.

This result is equivalent to one given by MacRobert [3, 190].

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