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On possible limit functions on a Fatou component in non-autonomous iteration

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Abstract. The possibilities for limit functions on a Fatou component for the iteration of a single polynomial or rational function are well understood and quite restricted. In non-autonomous iteration, where one considers compositions of arbitrary polynomials with suitably bounded degrees and coefficients, one should observe a far greater range of behavior. We show this is indeed the case and we exhibit a bounded sequence of quadratic polynomials which has a bounded Fatou component on which one obtains as limit functions every member of the classical Schlicht family of normalized univalent functions on the unit disc. The proof is based on quasiconformal surgery and the use of high iterates of a quadratic polynomial with a Siegel disc which closely approximate the identity on compact subsets. Careful bookkeeping using the hyperbolic metric is required to control the errors in approximating the desired limit functions and ensure that these errors ultimately tend to zero.

Key words: non-autonomous iteration, bounded sequences of polynomials, Schlicht class, limit functions, Siegel disc
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Contents

1. *Introduction*

This work is concerned with non-autonomous iteration of bounded sequences of polynomials, a relatively new area of complex dynamics. In classical complex dynamics, one studies the iteration of a (fixed) rational function on the Riemann sphere. Often in applications of dynamical systems, noise is introduced, and thus it is natural to consider a scheme of iteration where the function at each stage is allowed to vary. Here we study the situation where the functions being applied are polynomials with appropriate bounds on the degrees and coefficients.

Non-autonomous iteration, in our context, was first studied by Fornaess and Sibony [**FS91**] and also by Sester, Sumi, and others who were working in the closely related area of skew-products [**Ses99**, **Sum00**, **Sum01**, **Sum06**, **Sum10**]. Further work was done by Rainer Brück, Stefan Reitz, Matthias Büger [**Brü00**, **Brü01**, **BBR99**, **Büg97**], Michael Benedicks, and the first author [**Com04**, **Com06**, **Com08**, **Com13b**, **MC13**], among others.

One of the main topics of interest in non-autonomous iteration is discovering which results in classical complex dynamics generalize to the non-autonomous setting and which do not. For instance, the first author proved there is a generalization of the Sullivan straightening theorem [**CG93**, **Com12**, **DH85**], while Sullivan's non-wandering theorem [**CG93**, **Sul85**] no longer holds in this context [**Com03**]. One can thus construct polynomial sequences which either provide counterexamples or have interesting properties in their own right.

1.1. *Non-autonomous iteration.*

Following [**Com12**, **FS91**], let $d \geq 2$, $M \geq 0$, $K \geq 1$, and let $\{P_m\}_{m=1}^{\infty}$ be a sequence of polynomials where each $P_m(z) = a_{d_m,m} z^{d_m} +$

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$a_{d_m-1,m}z^{d_m-1}+\cdots\cdots+a_{1,m}z+a_{0,m}$ is a polynomial of degree $2\le d_m\le d$ whose coefficients satisfy

$$1/K\le|a_{d_m,m}|\le K,\quad m\ge 1,\quad |a_{k,m}|\le M,\quad m\ge 1,\quad 0\le k\le d_m-1.$$

Such sequences are called *bounded sequences of polynomials* or simply *bounded sequences*. For a constant $C\ge 1$, we will say that a bounded sequence is *C-bounded* if all of the coefficients in the sequence are bounded above in absolute value by C while the leading coefficients are also bounded below in absolute value by $1/C$.

For each $1\le m$, let Q_m be the composition $P_m\circ\cdots\cdots\circ P_2\circ P_1$ and, for each $0\le m\le n$, let $Q_{m,n}$ be the composition $P_n\circ\cdots\cdots\circ P_{m+2}\circ P_{m+1}$ (where we set $Q_{m,m}=\mathrm{Id}$ for each $m\ge 0$). Let the degrees of these compositions be D_m and $D_{m,n}$, respectively, so that $D_m=\prod_{i=1}^m d_i$, $D_{m,n}=\prod_{i=m+1}^n d_i$.

For each $m\ge 0$, define the *mth iterated Fatou set* or simply the *Fatou set* at time m, $\mathcal{F}_m$, by

$$\mathcal{F}_m=\{z\in\hat{\mathbb{C}}:\{Q_{m,n}\}_{n=m}^{\infty}\text{ is a normal family on some neighborhood of }z\}$$

where we take our neighborhoods with respect to the spherical topology on $\hat{\mathbb{C}}$. Components of $\mathcal{F}_m$ are referred to as *Fatou components at time m* and we also define the *mth iterated Julia set* or simply the *Julia set* at time m, $\mathcal{J}_m$, to be the complement $\hat{\mathbb{C}}\setminus\mathcal{F}_m$.

It is easy to show that these iterated Fatou and Julia sets are completely invariant in the following sense.

THEOREM 1.1. *For any $0\le m\le n$, $Q_{m,n}(\mathcal{J}_m)=\mathcal{J}_n$ and $Q_{m,n}(\mathcal{F}_m)=\mathcal{F}_n$, with components of $\mathcal{F}_m$ being mapped surjectively onto those of $\mathcal{F}_n$ by $Q_{m,n}$.*

It is easy to see that, given bounds d, K, M as above, we can find some radius R depending only on d, K, M so that, for any sequence $\{P_m\}_{m=1}^{\infty}$ with these bounds and any $m\ge 0$,

$$|Q_{m,n}(z)|\to\infty\quad\text{as } n\to\infty,\quad |z|>R,$$

which shows in particular that, as for classical polynomial Julia sets, there will be a *basin of infinity at time m*, denoted $\mathcal{A}_{\infty,m}$ on which all points escape locally uniformly to infinity under iteration. Such a radius will be called an *escape radius* for the bounds d, K, M. Note that the maximum principle shows that, just as in the classical case (see [**CG93**]), there can be only one component on which ∞ is a limit function and so the sets $\mathcal{A}_{\infty,m}$ are completely invariant in the sense given in Theorem 1.1.

The complement of $\mathcal{A}_{\infty,m}$ is called the *filled Julia set* at time m for the sequence $\{P_m\}_{m=1}^{\infty}$ and is denoted by $\mathcal{K}_m$. The same argument using Montel's theorem as in the classical case then shows that $\partial\mathcal{K}_m=\mathcal{J}_m$ (see [**CG93**]). When $m=0$, we will refer to the Fatou set, Julia set, filled Julia set, and basin of infinity for a bounded polynomial sequence $\{P_m\}_{m=1}^{\infty}$ as simply the Fatou set, Julia set, filled Julia set, and basin of infinity (respectively) for $\{P_m\}_{m=1}^{\infty}$ and denote them by $\mathcal{F}$, $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{A}_\infty$.

In view of the existence of the escape radius above, we have the following obvious result which we will use in proving our main result (see Theorem 1.3 below).

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PROPOSITION 1.2. *If V is an open connected set for which there exists a subsequence $\{m_k\}_{k=1}^{\infty}$ such that the sequence of forward images $\{Q_{m_k}(V)\}_{k=1}^{\infty}$ is uniformly bounded, then V is contained in a bounded Fatou component for $\{P_m\}_{m=1}^{\infty}$.*

1.2. *The Schlicht class.* The *Schlicht* class of functions, commonly denoted by $\mathcal{S}$, is the set of univalent functions defined on the unit disk such that, for all $f \in \mathcal{S}$, we have $f(0) = 0$ and $f'(0) = 1$. This is a classical class of functions for which many results are known. A common and useful technique is to use scaling or conformal mapping to apply results for $\mathcal{S}$ to arbitrary univalent functions defined on arbitrary domains (see for example Theorem 1.4).

1.2.1. *Statement of the main theorem.* We now give the statement of the main result of this paper.

THEOREM 1.3. *There exists a bounded sequence of quadratic polynomials $\{P_m\}_{m=1}^{\infty}$ and a bounded Fatou component $V \subset \mathbb{D}$ for this sequence containing 0 such that for all $f \in \mathcal{S}$, there exists a subsequence of iterates $\{Q_{m_k}\}_{k=1}^{\infty}$ which converges locally uniformly to f on V.*

The strength of this statement is that *every* member of $\mathcal{S}$ is a limit function on the *same* Fatou component for a *single* polynomial sequence.

The proof relies on a scaled version of the polynomial $P_\lambda(z) = \lambda z(1 - z)$, where $\lambda = e^{2\pi i(\sqrt{5}-1)/2}$. As P_λ is conjugate to an irrational rotation on its Siegel disk about 0, which we denote by U_λ, we may find a subsequence of iterates which converges uniformly to the identity on compact subsets of U_λ. We will rescale P_λ so that the filled Julia set for the scaled version P of P_λ is contained in a small Euclidean disc about 0. This is done so that, for any $f \in \mathcal{S}$, we can use the distortion theorems to control $|f'|$ on a relatively large hyperbolic disk inside U, the scaled version of the Siegel disc U_λ (see Figure 1).

The initial inspiration for this proof came from the concept of Löwner chains (see e.g. [**CDMG10**, **Dur83**]), particularly the idea that a univalent function can be expressed as a composition of many functions which are close to the identity. Given our remarks above about iterates of P_λ which converge to the identity locally uniformly on U_λ, this encouraged us to think we might be able to approximate these univalent functions which are close to the identity in some way with polynomials, and then compose these polynomials to get an approximation of the desired univalent function on some suitable subset of U_λ, a principle which we like to summarize as 'Do almost nothing and you can do almost anything'. As a matter of fact, there is now only one point in our proof where we make use of a Löwner chain, although it is not necessary to know this: the interested reader can find this in the 'up' section in the proof of Phase II (Lemma 5.17).

The proof of Theorem 1.3 will follow from an inductive argument, and each step in the induction will be broken up into two phases.

- Phase I: Construct a bounded polynomial composition which approximates a suitable net of functions from $\mathcal{S}$ on a subset of the unit disk.
- Phase II: Construct a bounded polynomial composition which corrects the error of the previous Phase I composition to arbitrary accuracy on a slightly smaller subset.

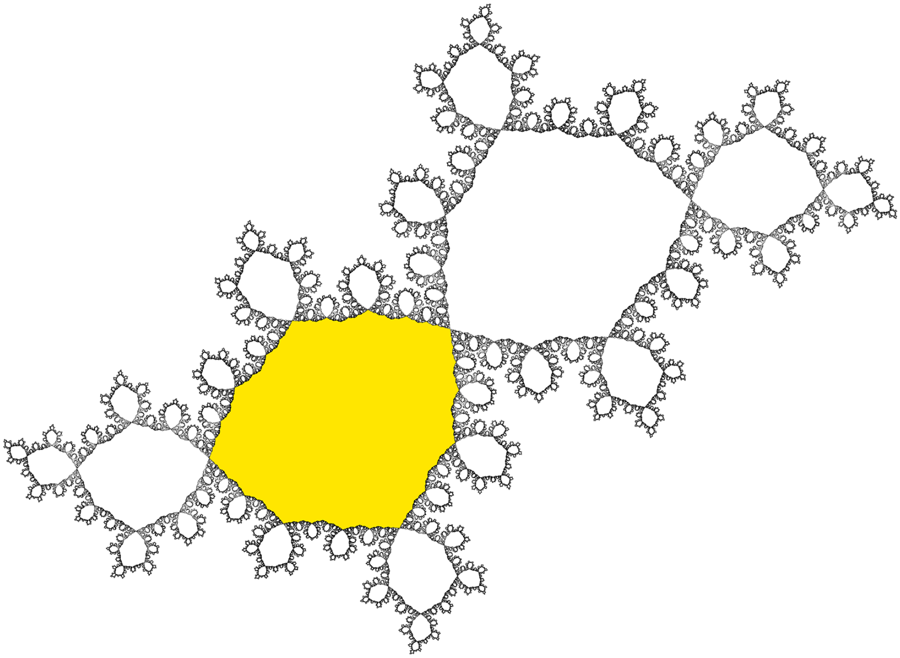

FIGURE 1. The filled Julia set $\mathcal{K}_\lambda$ for P_λ with Siegel disc highlighted.

A key idea in the proof is the fact that, since $\mathcal{S}$ is normal, it has a countable dense subset and we can approximate all of $\mathcal{S}$ to any desired accuracy on any compact subset of $\mathbb{D}$ by choosing a suitable finite net of functions (see Lemma 2.1). Great care is needed to control the error in the approximations and to ensure that the domain loss that necessarily occurs in each Phase II correction eventually stabilizes, so that we are left with a non-empty region upon which the desired approximations hold. To be a little more specific, the induction hypothesis will consist of nine parts. The first three are bookkeeping estimates about the radii of the hyperbolic discs in U on which our approximations hold which ensure that these radii do not get too small. The fourth hypothesis states that our polynomial sequence will be bounded while the fifth allows us to compose inverse branches which is necessary, since the Phase II correction to the error needs to 'undo' the error accumulated thus far and so it is the inverse of this error which needs to be approximated. The sixth hypothesis says that the forward compositions will be univalent on a disc which is not too small and which eventually becomes part of the Fatou component V of the statement of Theorem 1.3. The seventh hypothesis concerns the accuracy of the Phase II correction of the error while the eighth hypothesis is a bound on the size of the error after Phase I which needs to be corrected by the next Phase II. The ninth and final hypothesis is about how Phase I gives accurate approximations to all the functions in a suitably chosen net for $\mathcal{S}$ which is obtained using Lemma 2.1.

To create our polynomial approximations, we use what we call the Polynomial Implementation Lemma (Lemma 3.9). Suppose we want to approximate a given univalent function f with a polynomial composition. Let $\mathcal{K}$ be the filled Julia set for P and let γ,

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Γ be two Jordan curves enclosing $\mathcal{K}$ with γ lying inside Γ. In addition, we require that f be defined inside and on a neighborhood of γ, and that $f(\gamma)$ lie inside Γ. We construct a homeomorphism of the sphere as follows: define it to be f inside γ, the identity outside Γ, and extend by interpolation to the region between γ and Γ. The homeomorphism can be made quasiconformal, with non-zero dilatation (possibly) only on the region between γ and Γ. If we then pull back with a high iterate of P which is close to the identity, the support of the dilatation becomes small, which will eventually allow us to conclude that, when we straighten, we get a polynomial composition that approximates f closely on a large compact subset of U. In Phase I (Lemma 4.8), we then apply this process repeatedly to create a polynomial composition which approximates a finite set of functions from $\mathcal{S}$.

In Phase II (Lemma 5.17), we wish to correct the error from the Phase I composition. This error is defined on a subset of the Siegel disk, but to apply the Polynomial Implementation Lemma to create a composition which corrects the error, we need the error to be defined on a region which contains $\mathcal{K}$.

To get around this, we conjugate so that the conjugated error is defined on a region which contains $\mathcal{K}$. This introduces a further problem, namely that we must now cancel the conjugacy with polynomial compositions. A key element of the proof is viewing the expanding part of the conjugacy as a dilation in the correct conformal coordinates. An inevitable loss of domain occurs in using these conformal coordinates, but we are, in the end, able to create a Phase II composition which corrects the error of the Phase I approximation on a (slightly smaller) compact subset of U. What allows us to control the loss of domain is first that, while some loss of domain is unavoidable, the accuracy of the Phase II correction is completely at our disposal. Second, one can show that the loss of domain will tend to zero as the size of the error to be corrected tends to zero (Lemma 5.15 and also Claims 5.18, 5.19 in the proof of Phase II—Lemma 5.17). This eventually allows us to control the loss of domain. We then implement a fairly lengthy inductive argument to prove the theorem, getting better approximations to more functions in the Schlicht class with each stage in the induction, while ensuring that the region upon which the approximation holds does not shrink to nothing.

Theorem 1.3 can be generalized somewhat to suitable normal families on arbitrary open sets.

THEOREM 1.4. *Let $\Omega \subset \mathbb{C}$ be open and let $\mathcal{N}$ be a locally bounded normal family of univalent functions on Ω all of whose limit functions are non-constant. Let $z_0 \in \Omega$. Then there exists a bounded sequence $\{\tilde{P}_m\}_{m=1}^{\infty}$ of quadratic polynomials and a bounded Fatou component $W \subset \Omega$ for this sequence containing z_0 such that for all $f \in \mathcal{N}$, there exists a subsequence of iterates $\{\tilde{Q}_{m_k}\}_{k=1}^{\infty}$ which converges locally uniformly to f on W.*

1.3. *Related results.* In our proof, we will make extensive use of the hyperbolic metric. This has two main advantages—conformal invariance and the fact that hyperbolic Riemann surfaces are infinitely large when measured using their hyperbolic metrics which allows one to neatly characterize relatively compact subsets using the external hyperbolic radius (see Definition 2.2 below on the hyperbolic metric). An alternative approach is to try to do everything using the Euclidean metric. This requires, among other things that, in the analogue of the 'up' portion of the proof of our Phase II (Lemma 5.17), we must ensure

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that the image of the Siegel disc under a dilation about the fixed point by a factor which is just larger than 1 will cover the Siegel disc—in other words, we need a Siegel disc which is star-shaped (about the fixed point). Fortunately, there is a result in the paper of Avila, Buff, and Chéritat [**ABC04**, Main Theorem] which guarantees the existence of such Siegel discs. This led the authors to extensively investigate using this approach to prove a version of Theorem 1.3 but, in practice, although this can probably be made to work, we found this to be at least as complicated as the proof outlined in the current manuscript.

Results on approximating a large class of analytic germs of diffeomorphisms were proved in the paper of Loray [**Lor06**], particularly Théorème 3.2.3 in his work, where he uses a pseudo-group induced by a non-solvable subgroup of diffeomorphisms to approximate all germs of conformal maps which send one prescribed point to another with only very mild restrictions. Although we cannot rule out the possibility that these results could be used to obtain a version of our Theorems 1.3 and 1.4, this would be far from immediate. For example, pseudo-groups are closed under taking inverses (see [**Lor06**, Définition 3.4.1]). In our context, we can at best only approximate inverses, e.g. the suitable inverse branch of P_λ on U_λ which fixes 0. Moreover, one would need to be able to compose many such approximations while still ensuring that the resulting composition would be close to the ideal version, as well as being defined on a set which was not too small. Thus, one would unavoidably require a complex bookkeeping scheme for tracking the sizes of errors and domains, which is a large part of what we need to concern ourselves with below.

Also worth mentioning is the work done by a number of authors in the area of polynomial skew-products. A seminal paper was the work of Astorg *et al* [**ABD+16**] who used an ingenious idea of Lyubich based on Lavaurs mappings to construct a polynomial skew-product with a wandering domain. It is worth noting that the more rigid nature of polynomial skew-products combined with the fact that the Fatou set is considered as a subset of $\mathbb{C}^2$ make it more difficult to construct a wandering domain than in the context of non-autonomous iteration where one has greater freedom in choosing the members of one's sequence of polynomials. Further examples of wandering domains were obtained in [**AT**, **ATP23**] for a different class of skew-products than originally considered in [**ABD+16**]. Other classification results were obtained where the possibility of wandering domains was ruled out if the skew-product satisfied certain additional conditions on the dynamics. See for example the work of Ji [**Ji20**, **Ji23**], the work of Ji and Shen [**JS**], as well as Lilov [**Lil04**], Peters and Raissy [**PR19**], Peters and Smit [**PS18**], and finally Peters and Vivas [**PV16**].

Finally, in [**GT10**], Gelfriech and Turaev show that an area-preserving two-dimensional map with an elliptic periodic point can be renormalized so that the renormalized iterates are dense in the set of all real-analytic symplectic maps of a two-dimensional disk. However, this is clearly not as close to what we do as the two other cases mentioned above.

2. *Background*

We will now discuss some background which will be instrumental in proving Theorem 1.3. Some of the more standard results we need can be found in the appendices—see Appendix A.1.

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2.1. *The hyperbolic metric.* We first establish some notation for hyperbolic discs. Let R be a hyperbolic Riemann surface and let $\Delta_R(z, r)$ be the (open) hyperbolic disc in R centered at z of radius r. If the domain is obvious in context, we may simply denote this disc as $\Delta(z, r)$. Lastly, let $d\rho_R$ represent the hyperbolic length element for R. If D is a domain in $\mathbb{C}$ and $z \in D$, let $\delta_D(z)$ denote the Euclidean distance to ∂D. We will be using the hyperbolic metric to measure both the accuracy of our approximations and the loss of domain that occurs in each Phase II composition. One immediate application of Lemma A.4 in the appendices is the following which will be essential to us later in the proof of the induction (Lemma 6.2) leading up to the main result (Theorem 1.3).

LEMMA 2.1. *Let $K \subset \mathbb{D}$ be relatively compact and let $\varepsilon > 0$. We can then find a finite set $\{f_i\}_{i=1}^N \subset \mathcal{S}$ such that, given $f \in \mathcal{S}$, there exists (at least one) $1 \le k \le N$ such that*

$$\sup_{z \in K} \rho_{\mathbb{D}}(f(z), f_k(z)) < \varepsilon.$$

Proof. This follows immediately from the normality of $\mathcal{S}$ (Corollary A.3), combined with [**Con78**, Proposition VII.1.16]. □

A set $\{f_i\}_{i=1}^N \in \mathcal{S}$ as above will be called an *ε-net* for $\mathcal{S}$ on K or simply an *ε-net* if the set K is clear from the context.

Next, we will need a notion of internal and external hyperbolic radii, which is one of the crucial bookkeeping tools we will be using, especially for controlling loss of domain in Phase II.

Definition 2.2. Suppose V is a hyperbolic Riemann surface, $v \in V$, and X is a non-empty subset of V. Define the *external hyperbolic radius of X in V about v*, denoted $R^{\text{ext}}_{(V,v)}X$, by

$$R^{\text{ext}}_{(V,v)}X = \sup_{z \in X} \rho_V(v, z).$$

If $v \in X$, we further define the *internal hyperbolic radius of X in V about v*, denoted $R^{\text{int}}_{(V,v)}X$, by

$$R^{\text{int}}_{(V,v)}X = \inf_{z \in V \setminus X} \rho_V(v, z).$$

If $v \in X$ and it happens that $R^{\text{int}}_{(V,v)}X = R^{\text{ext}}_{(V,v)}X$, we will call their common value the *hyperbolic radius of X in V about v*, and denote it by $R_{(V,v)}X$.

We remark that, for any $v \in V$, if $X = V$, then $R^{\text{int}}_{(V,v)}X = R^{\text{ext}}_{(V,v)}X = \infty$. Also, if $v \in X$ and $X \subsetneq V$, then $R^{\text{int}}_{(V,v)}X < \infty$. Indeed, let $w \in V \setminus X$. Then,

$$\begin{aligned} R^{\text{int}}_{(V,v)}X &= \inf_{z \in V \setminus X} \rho_V(v, z) \\ &\le \rho_V(v, w) \\ &< \infty. \end{aligned}$$

We also remark that the internal and external hyperbolic radii are increasing with respect to set-theoretic inclusion in the obvious way. Namely, if $\emptyset \ne X \subset Y$ are subsets of V, then $R^{\text{ext}}_{(V,v)}X \le R^{\text{ext}}_{(V,v)}Y$, while if $v \in X$, we also have $R^{\text{int}}_{(V,v)}X \le R^{\text{int}}_{(V,v)}Y$.

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The names 'internal hyperbolic radius' and 'external hyperbolic radius' are justified in view of the following lemma which is how they are often used in practice.

LEMMA 2.3. *Let $V \subsetneq \mathbb{C}$ be a simply connected domain, $v \in V$, and X be a non-empty subset of V. We then have the following:*

(1) *if $0 < R^{\text{ext}}_{(V,v)} X < \infty$, then $X \subset \overline{\Delta}_V(v, R^{\text{ext}}_{(V,v)} X)$;*

(2) *if $v \in X$ and $0 < R^{\text{int}}_{(V,v)} X < \infty$, then $R^{\text{int}}_{(V,v)} X = \sup\{r : \Delta_V(v,r) \subset X\}$ so that, in particular, $\Delta_V(v, R^{\text{int}}_{(V,v)} X) \subset X$;*

(3) *if $v \in X$, then $R^{\text{int}}_{(V,v)} X \le R^{\text{ext}}_{(V,v)} X$.*

Proof. Item (1) follows immediately from the above definition for external hyperbolic radius. For item (2), if we temporarily let $R := \sup\{r : \Delta_V(v,r) \subset X\}$, then from the definition of internal hyperbolic radius, it follows easily that $V \setminus X \subset V \setminus \Delta_V(v, R^{\text{int}}_{(V,v)} X)$ so that $\Delta_V(v, R^{\text{int}}_{(V,v)} X) \subset X$ whence we have that $R^{\text{int}}_{(V,v)} X \le R$. Note that since $R^{\text{int}}_{(V,v)} X > 0$, this means that $R > 0$ and the set of which we take the supremum to find R must be non-empty. However, if we let $z \in V \setminus X$ (note that the requirement that $R^{\text{int}}_{(V,v)} X < \infty$ ensures that we can always find such a point), then we must have that $\rho_V(v, x) \ge R$, and on taking an infimum over all such x, we have $R^{\text{int}}_{(V,v)} X \ge R > 0$ from which we obtain item (2). Item (3) then follows from items (1) and (2) (the result being trivial in the cases where the external hyperbolic radius is infinite or the internal hyperbolic radius is zero) which completes the proof. □

We remark that item (2) above illustrates how the internal hyperbolic radius $R^{\text{int}}_{(V,v)} X$ is effectively the radius of the largest disc about v which lies inside X. The reason that we took $R^{\text{int}}_{(V,v)} X = \inf_{z \in V \setminus X} \rho_V(v, z)$ as our definition and not the alternative $\sup\{r : \Delta_V(v,r) \subset X\}$ is that this version still works, even if $\inf_{z \in V \setminus X} \rho_V(v, z)$ is zero or infinite. This lemma leads to the following handy corollary.

COROLLARY 2.4. *Suppose $V \subsetneq \mathbb{C}$ is a simply connected domain, $v \in V$, and that X, Y are subsets of V, with $v \in Y$.*

(1) *If $R^{\text{ext}}_{(V,v)} X \le R^{\text{int}}_{(V,v)} Y$, then $\overline{X} \subset \overline{Y}$.*

(2) *If $R^{\text{ext}}_{(V,v)} X < R^{\text{int}}_{(V,v)} Y$, then $\overline{X} \subsetneq Y$.*

We also have the following equivalent formulation for the internal and external hyperbolic radii which is often very useful in practice.

LEMMA 2.5. *Let $V \subsetneq \mathbb{C}$ be a simply connected domain, let $v \in V$, and let X be a non-empty subset of V. We then have the following:*

(1) *if $v \in X$, then $R^{\text{int}}_{(V,v)} X = \inf_{z \in (\partial X) \cap V} \rho_V(v, z)$;*

(2) $R^{\text{ext}}_{(V,v)} X \ge \sup_{z \in (\partial X) \cap V} \rho_V(v, z)$.

If, in addition, $R^{\text{ext}}_{(V,v)} X < \infty$ or $X \subsetneq V$ and $\hat{\mathbb{C}} \setminus X$ is connected, we also have:

(3) $R^{\text{ext}}_{(V,v)} X = \sup_{z \in (\partial X) \cap V} \rho_V(v, z)$.

In particular, the above holds if $X = U \subsetneq V$ is a simply connected domain.

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Note that we can get strict inequality in item (2) above. For example, let $V = \mathbb{D}$, $v = 0$, and let $X = \{z : \frac{2}{3} \le |z| < 1\}$. We leave the elementary details to the interested reader.

Proof. To prove item (1), we first observe that the result is trivial if $R^{\text{int}}_{(V,v)} X = \infty$ which happens if and only if $X = V$. So suppose now that $X \subsetneq V$. Note that, in this case, $\partial X \cap V \ne \emptyset$, since otherwise int X and $V \setminus \overline{X}$ would give a separation of the connected set V.

Now let $z \in \partial X \cap V$ and pick $\varepsilon > 0$. Since $z \in \partial X$, there exists $w \in V \setminus X$ with $\rho_V(z, w) < \varepsilon$. By the triangle inequality $\rho_V(v, w) < \rho_V(v, z) + \varepsilon$ and, on taking the infimum on the left-hand side,

$$R^{\text{int}}_{(V,v)} X \le \rho_V(v, z) + \varepsilon.$$

If we then take the infimum over all $z \in \partial X \cap V$ on the right-hand side and let ε tend to 0, we then obtain that

$$R^{\text{int}}_{(V,v)} X \le \inf_{z \in (\partial X) \cap V} \rho_V(v, z).$$

Now we show $R^{\text{int}}_{(V,v)} X \ge \inf_{z \in (\partial X) \cap V} \rho_V(v, z)$. Take a point $w \in V \setminus X$ and connect v to w with a geodesic segment γ in V. Then γ must meet ∂X since otherwise, int X and $V \setminus \overline{X}$ would give a separation of the connected set $[\gamma]$ (the track of γ). So let $z_0 \in \partial X \cap [\gamma]$. Clearly,

$$\rho_V(v, z_0) \le \rho_V(v, w),$$

so

$$\inf_{z \in (\partial X) \cap V} \rho_V(v, z) \le \rho_V(v, w),$$

and thus,

$$\inf_{z \in (\partial X) \cap V} \rho_V(v, z) \le R^{\text{int}}_{(V,v)} X.$$

This completes the proof of item (1).

To prove item (2), we first consider the case when $\sup_{z \in (\partial X) \cap V} \rho_V(v, z) = \infty$. Note that, since the supremum of the empty set is minus infinity, this in particular implies that $(\partial X) \cap V \ne \emptyset$. Thus, we can find a sequence $\{z_n\} \in (\partial X) \cap V$ such that $\rho_V(v, z_n) \to \infty$. For each z_n, choose $x_n \in X$ such that $\rho_V(z_n, x_n) \le 1$. Then, $\rho_V(v, x_n) \to \infty$ by the reverse triangle inequality, which shows $R^{\text{ext}}_{(V,v)} X = \infty$ so that we have equality.

Now consider the case when $\sup_{z \in (\partial X) \cap V} \rho_V(v, z) < \infty$. The result is trivial if this supremum is minus infinity, so again we can assume that $(\partial X) \cap V \ne \emptyset$. Similarly to above, we can take a sequence $\{z_n\} \in (\partial X) \cap V$ for which $\rho_V(v, z_n) \to \sup_{z \in (\partial X) \cap V} \rho_V(v, z)$. Then take a sequence $\{x_n\} \in X$ such that $\rho_V(x_n, z_n) < 1/n$. By definition of the external hyperbolic radius, we must have

$$\rho_V(v, x_n) \le R^{\text{ext}}_{(V,u)} X,$$

and since $\rho_V(x_n, z_n) < 1/n$, by the reverse triangle inequality, on letting n tend to infinity,

$$\sup_{z \in (\partial X) \cap V} \rho_V(v, z) \le R^{\text{ext}}_{(V,v)} X,$$

which proves item (2) as desired.

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Now we show that, under the additional assumptions that $R^{\text{ext}}_{(V,v)}X < \infty$ or $X \subsetneq V$ and $\hat{\mathbb{C}} \setminus X$ is connected, $\sup_{z\in(\partial X)\cap V} \rho_V(v, z) \geq R^{\text{ext}}_{(V,v)}X$ from which item (3) follows. Assume first that $R^{\text{ext}}_{(V,v)}X < \infty$ and let $\{x_n\} \in X$ be a sequence in X such that $\rho_V(v, x_n) \to R^{\text{ext}}_{(V,v)}X$ as n tends to infinity (recall that we have assumed $X \neq \emptyset$ so that the set over which we are taking our supremum to obtain the external hyperbolic radius is non-empty). Note also that $R^{\text{ext}}_{(V,v)}X = 0$ if and only if $X = \partial X = \{v\}$ in which case, the result is trivial, so we can assume that $\rho_V(v, x_n) > 0$ for each n. For each n, let γ_n be the unique hyperbolic geodesic in V which passes through v and x_n. Then there must be a point z_n (which may possibly be x_n itself) on $\gamma_n \cap \partial X$ which does not lie on the same side of x_n as v since otherwise, the portion of γ_n on the same side of v as x_n and which runs from x_n to ∂V would be separated by the open sets int X and $V \setminus \overline{X}$. However, this is impossible since $x_n \in X$ while $R^{\text{ext}}_{(V,v)}X < \infty$ which forces γ_n to eventually leave X (in both directions). It then follows that for each n, we have that

$$\rho_V(v, z_n) \geq \rho_V(v, x_n)$$

so that

$$\sup_{z\in(\partial X)\cap V} \rho_V(v, z) \geq \rho_V(v, x_n),$$

and the desired conclusion then follows on letting n tend to infinity.

Now suppose that $X \subsetneq V$ and $\hat{\mathbb{C}} \setminus X$ is connected. We observe that, since V is connected, we must have $(\partial X) \cap V \neq \emptyset$ similarly to in the proof of item (1) above, while if $X = U$ is a simply connected domain, then $\hat{\mathbb{C}} \setminus X$ is automatically connected (e.g. [**New51**, VI.4.1] or [**Con78**, Theorem VIII.2.2]).

In view of item (2) above, item (3) holds if $\sup_{z\in(\partial X)\cap V} \rho_V(v, z) = \infty$, so assume from now on that $\sup_{z\in(\partial X)\cap V} \rho_V(v, z) < \infty$ and note that $(\partial X) \cap V \neq \emptyset$ implies that this supremum will be non-negative so that we can set $\rho := \sup_{z\in(\partial X)\cap V} \rho_V(v, z)$. Note that, if $\rho = 0$, then $V \setminus \{v\} \cap \partial X = \emptyset$ and, since $V \setminus \{v\}$ is connected, then either $V \setminus \{v\} \subset \text{int}\, X \subset X$ or $V \setminus \{v\} \subset V \setminus \overline{X} \subset V \setminus X$. In the first case, $X = V \setminus \{v\}$, in which case, one checks easily that item (3) fails. However, we can rule out this case since $\hat{\mathbb{C}} \setminus X = \{v\} \cup \hat{\mathbb{C}} \setminus V$ is disconnected, which violates our hypothesis that $\hat{\mathbb{C}} \setminus X$ be connected. In the second case, we have $X = \{v\}$, in which case one easily checks that item (3) holds. Thus, we can assume from now on that $\rho > 0$.

CLAIM 2.6. $X \subset \overline{\Delta}_V(v, \rho)$.

Proof. Suppose not. Then there exists $x \in X$ such that $\rho_V(v, x) > \rho$. Set $\tilde{\rho} := \rho_V(v, x) > \rho$ and define C to be the hyperbolic circle of radius $\tilde{\rho}$ about v with respect to the hyperbolic metric of V. Then we have $C \cap \partial X = \emptyset$ since, for all $z \in \partial X \cap V$, by definition, we have $\rho_V(v, z) \leq \rho < \tilde{\rho}$. Thus, $x \in \text{int}\, X$.

Now we have $x \in C$. We next argue that each point of C must lie in X. Suppose z is another point on C such that $z \notin X$. Then z would be in $V \setminus X$. As $C \cap \partial X = \emptyset$, we have that $z \in V \setminus \overline{X} = \text{int}\,(V \setminus X)$. However, this is impossible as int X and int $(V \setminus X)$ would then form a separation of the connected set C. Thus, $C \subset X$ and C induces a separation of $\hat{\mathbb{C}} \setminus X$. Indeed, since $\rho < \tilde{\rho}$, ∂X is inside the Jordan curve C and hence there are points

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of $\hat{\mathbb{C}} \setminus X$ inside C. However, $\infty \in \hat{\mathbb{C}} \setminus V \subset \hat{\mathbb{C}} \setminus X$ lies outside C. This contradicts our assumption that $\hat{\mathbb{C}} \setminus X$ is connected. □

Immediately from the above claim, we see that $\rho = \sup_{z \in (\partial X) \cap V} \rho_V(v, z) \geq R^{\text{ext}}_{(V,v)} X$, and thus, in the case where $X \subsetneq V$ and $\hat{\mathbb{C}} \setminus X$ is connected,

$$\sup_{z \in (\partial X) \cap V} \rho_V(v, z) = R^{\text{ext}}_{(V,v)} X,$$

which proves item (3) as desired. □

We will require the following elementary definition from metric spaces.

Definition 2.7. Suppose R is a hyperbolic Riemann surface, and that A and B are non-empty subsets of R. For $z \in R$, we define

$$\rho_R(z, B) = \inf_{w \in B} \rho_R(z, w)$$

and

$$\rho_R(A, B) = \inf_{z \in A} \rho_R(z, B).$$

We say that a subset X of a simply connected domain $V \subsetneq \mathbb{C}$ is *hyperbolically convex* if, for every $z, w \in X$, the geodesic segment $\gamma_{z,w}$ from z to w lies inside X (this is the same as the definition given in [**MM94**, §2]). We then have the following elementary but useful lemma.

LEMMA 2.8. (The hyperbolic convexity lemma) *Let $V \subsetneq \mathbb{C}$ be a simply connected domain. Then any hyperbolic disc $\Delta_V(z, R)$ is hyperbolically convex with respect to the hyperbolic metric of V.*

Proof. Let a, b be two points in $\Delta_V(z, R)$. Using conformal invariance, we can apply a suitably chosen Riemann map from V to the unit disc $\mathbb{D}$ so that, without loss of generality, we can assume that $a = 0$ while b is on the positive real axis whence the shortest geodesic segment from a to b is the line segment $[0, b]$ on the positive real axis. However, the disc $\Delta_{\mathbb{D}}(z, R)$ is a round disc $\mathrm{D}(w, r)$ for some $w \in D$ and $r \in (0, 1)$ which is therefore convex (with respect to the Euclidean metric) and the result follows. □

Ordinary derivatives are useful for estimating how points get moved apart by applying functions when using the Euclidean metric. In our case, we will need a notion of a derivative taken with respect to the hyperbolic metric.

Let R, S be hyperbolic Riemann surfaces with metrics

$$d\rho_R = \sigma_R(z)|dz|,$$
$$d\rho_S = \sigma_S(z)|dz|,$$

respectively, and let ℓ_R, ℓ_S denote the hyperbolic length in R, S, respectively. Let $X \subset R$ and let f be defined and analytic on an open set containing X with $f(X) \subset S$. For $z \in X$, define the *hyperbolic derivative:*

$$f^{\natural}_{R,S}(z) := f'(z)\frac{\sigma_S(f(z))}{\sigma_R(z)}, \tag{2.1}$$

see e.g. the differential operation D_{h1} defined in [**MM94**, §2] and also [**MM99**, §2]. Note that the hyperbolic derivative satisfies the chain rule, that is, if R, S, T are hyperbolic Riemann surfaces with g defined and analytic on an open set containing $X \subset R$, and f defined and analytic on an open set containing $Y \subset S$ with $f(X) \subset Y$, then, on the set X,

$$(f \circ g)^{\natural}_{R,T} = (f^{\natural}_{S,T} \circ g) \cdot g^{\natural}_{R,S}. \tag{2.2}$$

We also have a version of conformal invariance which is essentially [**KL07**, Theorem 7.1.1] or which the interested reader can simply deduce from the formula for the hyperbolic metric using a universal covering map from the disc (see, e.g. [**CG93**, p. 12]), namely:

$$\text{if } f : R \mapsto S \text{ is a covering map, then } |f^{\natural}| = 1 \text{ on } R. \tag{2.3}$$

We observe that the above is basically another way of rephrasing part of the Schwarz lemma for the hyperbolic metric (e.g. [**CG93**, Theorem I.4.1]) where we have an isometry of hyperbolic metrics if and only if the mapping from one Riemann surface to the other lifts to an automorphism of the unit disc. The main utility of the hyperbolic derivative for us will be via the hyperbolic metric version of the standard M-L estimates for line integrals (see Lemma 2.9 below). First, however, we make one more definition.

Let R, S be hyperbolic Riemann surfaces, let X be a non-empty subset of R, and let f be defined and analytic on an open set containing X with $f(X) \subset S$. Define the *hyperbolic Lipschitz bound of f on X* as

$$\|f^{\natural}_{R,S}\|_X := \sup_{z\in X} |f^{\natural}_{R,S}(z)|.$$

We recall that, for any two points z, w in R, the hyperbolic distance $\rho_R(z, w)$ is the same as the length of a shortest geodesic segment in R joining z to w (see e.g. [**KL07**, Theorems 7.1.2 and 7.2.3]).

LEMMA 2.9. (Hyperbolic M-L estimates) *Suppose R, S are hyperbolic Riemann surfaces. Let γ be a piecewise smooth curve in R and let f be holomorphic on an open neighborhood of $[\gamma]$ and map this neighborhood inside S with $|f^{\natural}_{R,S}| \le M$ on $[\gamma]$. Then,*

$$\ell_S(f(\gamma)) \le M\ell_R(\gamma).$$

In particular, if $z, w \in R$ and γ is a shortest hyperbolic geodesic segment connecting z and w, and $|f^{\natural}_{R,S}| \le M$ on $[\gamma]$, then

$$\rho_S(f(z), f(w)) \le M\rho_R(z, w).$$

Proof. For the first part, if $\gamma : [a, b] \to R$, we calculate

$$\begin{aligned}\ell_S(f(\gamma)) &= \int_{f(\gamma)} d\rho_S \\ &= \int_a^b \sigma_S(f(\gamma(t))) \cdot |f'(\gamma(t))| \cdot |\gamma'(t)|\, dt\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b |f^\natural(\gamma(t))| \cdot \sigma_R(\gamma(t)) \cdot |\gamma'(t)|\, dt \\
&= \int_\gamma |f^\natural|\, d\rho_R \\
&\le M \int_\gamma d\rho_R \\
&= M\ell_R(\gamma).
\end{aligned}
$$

The second part then follows immediately from this and the facts that by [KL07, Theorems 7.1.2 and 7.2.3], $\rho_R(z, w)$ is equal to the length of the shortest geodesic segment in R joining z and w, while $f(\gamma)$ is at least as long in S as the distance between $f(z)$ and $f(w)$. $\square$

In this paper, we will be working with hyperbolic derivatives only for mappings which map a subset of U to U, where U is a suitably scaled version of the Siegel disc U_λ introduced in §1 and where we are obviously using the hyperbolic density of U in the definition in equation (2.1). For the sake of readability, from now on, we will suppress the subscripts and simply write $f^\natural$ instead of $f^\natural_{U,U}$ for derivatives taken with respect to the hyperbolic metric of U.

3. *The Polynomial Implementation Lemma*

3.1. *Setup.* Let $\Omega, \Omega' \subset \mathbb{C}$ be bounded Jordan domains with analytic boundary curves γ and Γ, respectively, such that $\overline{\Omega} \subset \Omega'$. By making a translation if necessary, we can assume without loss of generality that $0 \in \Omega$ so that γ then separates 0 from ∞. Suppose f is analytic and injective on a neighborhood of $\overline{\Omega}$ such that $f(\gamma)$ is still inside Γ. Let $A = \Omega' \setminus \overline{\Omega}$ be the conformal annulus bounded by γ and Γ, and let $\tilde{A}$ be the conformal annulus bounded by $f(\gamma)$ and Γ. Define

$$
F(z) = \begin{cases} f(z), & z \in \overline{\Omega}, \\ z, & z \in \hat{\mathbb{C}} \setminus \Omega'. \end{cases}
$$

We wish to extend F to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$. To do this, the main tool we use will be a lemma of Lehto [Leh65] which allows us to define F in the 'missing' region between Ω and $\hat{\mathbb{C}} \setminus \Omega'$. First, however, we need to gather some terminology.

Recall that in [New51], a Jordan curve C in the plane (parameterized on the unit circle $\mathbb{T}$) is said to be *positively oriented* if the algebraic number of times a ray from the bounded complementary domain to the unbounded complementary domain crosses the curve is 1 or, equivalently, the winding number of the curve about points in its bounded complementary region is also 1 (the reader is referred to the discussion in [New51, pp. 188–194]).

Following the proof of Theorem VII.11.1, Newman goes on to define a homeomorphism g defined on $\mathbb{C}$ to be *orientation-preserving* or *sense-preserving* if it preserves the orientation of all simple closed curves. Lehto and Virtanen adopt Newman's definitions in their text on quasiconformal mappings [LV65], and they have a related and more general definition of orientation-preserving maps defined on an arbitrary plane domain G, where g

is said to be *orientation-preserving* on G if the orientation of the boundary of every Jordan domain D with $\overline{D} \subset G$ is preserved [**LV65**, p. 9].

Lehto and Virtanen also introduce the concept of the *orientation of a Jordan curve C with respect to one of its complementary domains G* [**LV65**, p. 8]. Let $C(z) : \mathbb{T} \mapsto C$ be a parameterization of C which defines its orientation and let Φ be a Möbius transformation which maps G to the bounded component of the complement of $\Phi(C)$ such that $0 \in \Phi(G)$. Then, C is said to be *positively oriented with respect to G* if the argument of $\Phi \circ C(t)$ increases by 2π as one traverses $\mathbb{T}$ anticlockwise. Using this, if G is an n-connected domain whose boundary consists of n disjoint Jordan curves (what Lehto and Virtanen in [**LV65**, p. 12] refer to as *free boundary curves*), it is easy to apply the above definition to define the orientation of G with respect to each curve which comprises ∂G.

Recall that in Lehto's paper [**Leh65**], he considers a conformal annulus (ring domain) $D \subset \hat{\mathbb{C}}$ bounded by two Jordan curves C_1 and C_2. If φ is a homeomorphism of $C_1 \cup C_2$ into the plane, then the curves $\varphi(C_1)$, $\varphi(C_2)$ will bound another conformal annulus which we call D'. If, under the mapping φ, the positive orientations of C_1 and C_2 with respect to D correspond to the positive orientations of $\varphi(C_1)$ and $\varphi(C_2)$ with respect to D', then φ is called an *admissible boundary function* for D.

LEMMA 3.1. (Lehto [**Leh65**]) *Let D be a conformal annulus in $\hat{\mathbb{C}}$ bounded by the Jordan curves C_1 and C_2, and let $w_h : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$, $h = 1, 2$ be quasiconformal mappings such that the restrictions of w_h to C_h, $h = 1, 2$, constitute an admissible boundary function for D. Then there exists a quasiconformal mapping w of D such that $w(z) = w_h(z)$ for $z \in [C_h]$, $h = 1, 2$ (where for each h, $[C_h]$ is the track of the curve C_h).*

Applying this result to our situation, we have the following.

LEMMA 3.2. *For Ω, Ω', f, F as above, we can extend the mapping F above to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$.*

Proof. To apply Lehto's lemma above, we need to verify two things: first that f (and the identity) can be extended as quasiconformal mappings from $\hat{\mathbb{C}}$ to itself and second that we have an admissible pair of mappings on $\partial A = \partial(\Omega' \setminus \overline{\Omega})$ according to Lehto's definition given above.

First note that, in view of the argument principle, f, being univalent, is an orientation-preserving mapping on a neighborhood of $\overline{\Omega}$. Using [**LV65**, Satz II.8.1], f (and trivially the identity) can be extended as a quasiconformal mapping of $\mathbb{C}$ to itself. Using either [**New51**, Theorem VII.11.1] or the Orientierungssatz in [**LV65**, p. 9], the above extension can be easily extended to an orientation-preserving homeomorphism of $\hat{\mathbb{C}}$, which is then readily seen to be a quasiconformal mapping of $\hat{\mathbb{C}}$ to itself as follows easily from [**LV65**, Satz I.8.1].

Both f and the identity preserve the positive orientations of γ and Γ, respectively. In addition, since $f(\gamma)$ lies inside Γ, it follows that the orientations of γ and Γ with respect to A are the same as those of $\tilde{A}$. To be precise, let γ be positively oriented with respect to A and let $1/A$, $1/(\tilde{A} - f(0))$ denote the images of A, $\tilde{A}$, respectively, under $1/z$, $1/(z - f(0))$, respectively. Since A lies in the unbounded complementary component of γ,

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it follows from the above definition of the orientation of a boundary curve for a domain that the winding number of $1/\gamma$ about points of $1/A$ is 1 so that the winding number of γ about 0 (which lies inside γ) is -1. By the argument principle, $f(0)$ lies inside $f(\gamma)$ and, since f is orientation-preserving, the winding number of $f(\gamma)$ about $f(0)$ is also -1.

A simple calculation then shows that the winding number of $1/(f(\gamma)-f(0))$ about 0 and thus also about points in $1/(\tilde{A}-f(0))$ is also 1. This shows that f and thus F preserve the positive orientations of γ, $f(\gamma)$ with respect to A and $\tilde{A}$, respectively. Since F is the identity on $[\Gamma]$, it trivially preserves the positive orientation of Γ with respect to A and $\tilde{A}$ (both of which lie inside Γ) and, with this, we have shown the hypotheses of Lemma 3.1 above from [**Leh65**] are met.

Lemma 3.1 now allows us to extend F to a quasiconformal mapping on the conformal annulus $A = \Omega' \setminus \overline{\Omega}$ such that this extension agrees with the original values of F on the boundary and maps A to $\tilde{A}$. We can then use [**LV65**, Satz I.8.3] on the removeability of analytic arcs or, remembering that f is defined on a neighborhood of $\overline{\Omega}$ while the identity is defined on all of $\hat{\mathbb{C}}$, twice invoke Rickman's lemma (e.g. [**DH85**, Lemma 2]) to conclude that the resulting homeomorphism of $\hat{\mathbb{C}}$ is quasiconformal. □

We can summarize the above in the following useful definition.

Definition 3.3. If f, F, γ, Γ, and A are all as above, with F an admissible boundary function for A, we will say that (f, Id) is an admissible pair on (γ, Γ).

Recall we have $P_\lambda = \lambda z(1-z)$, where $\lambda = e^{2\pi i(\sqrt{5}-1)/2}$. Let $\mathcal{K}_\lambda$ be the filled Julia set for P_λ and let U_λ be the corresponding Siegel disc containing 0. Let $\kappa \geq 1$ and set $P = P_\kappa = (1/\kappa)P_\lambda(\kappa z) = \lambda z - \lambda\kappa z^2$. Then, if $\mathcal{K}$ is the filled Julia set for P, we have $\mathcal{K} \subset D(0, (2/\kappa))$. Let U be the Siegel disk for P and note that $U = \{z \in \mathbb{C} : z = w/\kappa \text{ for some } w \in U_\lambda\}$. Now choose the Jordan domains Ω, Ω' above such that $\mathcal{K} \subset \Omega \subset \overline{\Omega} \subset \Omega' \subset \mathrm{D}(0, (2/\kappa))$, where from above $2/\kappa \leq 2$ is an escape radius for P.

Let (f, Id) be an admissible pair where f, F, γ, Γ, and A are all as above. In view of Lemma 3.2, F can be extended to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ and we let μ_F denote the complex dilatation of F. Next let $N \in \mathbb{N}$ and, for each $0 \leq m \leq N$, set $\mu_m^N := (P^{N-m})^*\mu_F$ that is $\mu_m^N(z) = \mu_{F\circ P^{N-m}}(z)$ (here and in what follows, we draw the reader's attention to the fact that the superscript N is an index rather than an iterate or a power). Let $\varphi_N^N := F$ and, for $0 \leq m \leq N-1$, let φ_m^N be the unique normalized solution of the Beltrami equation for μ_m^N which satisfies $\varphi_m^N(z) = z + \mathcal{O}(1/|z|)$ near ∞ (see e.g. [**CG93**, Theorem I.7.4]). For $1 \leq m \leq N$, let

$$\tilde{P}_m^N(z) = \varphi_m^N \circ P \circ (\varphi_{m-1}^N)^{-1}(z).$$

Then for each m, $\tilde{P}_m^N$ is an analytic degree 2 branched cover of $\mathbb{C}$ which has a double pole at ∞ and no other poles. Thus, $\tilde{P}_m^N$ is a quadratic polynomial and the fact that each φ_m^N is tangent to the identity at ∞ ensures that the leading coefficient of $\tilde{P}_m^N$ is $-\lambda\kappa$ and thus has absolute value κ. Let $\alpha_m^N := \varphi_m^N(0)$. Since the dilatation of φ_m^N is zero on $\hat{\mathbb{C}} \setminus \overline{\mathrm{D}}(0, (2/\kappa))$, we know φ_m^N is univalent on this set. Thus, $1/\varphi_m^N(1/z)$ is univalent on $\mathrm{D}(0, (\kappa/2))$ and is tangent to the identity at 0. It follows from the Koebe one-quarter theorem (Theorem A.1) and the injectivity of φ_m^N that $|\alpha_m^N| \leq 4(2/\kappa) = 8/\kappa$.

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Define $\psi_m^N(z) := \varphi_m^N(z) - \alpha_m^N$. Then for each $0 \leq m \leq N$, if we define

$$P_m^N(z) = \psi_m^N \circ P \circ (\psi_{m-1}^N)^{-1}(z), \tag{3.1}$$

we have that P_m^N is a quadratic polynomial whose leading coefficient is again $-\lambda\kappa$ and thus has absolute value κ. Moreover, P_m^N fixes 0 as it is $\tilde{P}_m^N$ composed with suitably chosen (uniformly bounded) translations. We now turn to calculating bounds on the coefficients of each P_m^N.

LEMMA 3.4. *Any sequence formed from the polynomials $P_m^N(z)$ for $0 \leq m \leq N$ as above is a $(17+\kappa)$-bounded sequence of polynomials.*

Proof. By the construction in equation (3.1) above, the leading coefficient has absolute value κ while the constant term is zero. Now, for $|z|$ sufficiently large,

$$\begin{aligned} P_m^N(z) = \lambda\bigg(z + \alpha_{m-1}^N + \mathcal{O}\bigg(\frac{1}{|z|}\bigg)\bigg)\bigg(1 - \kappa z - \kappa\alpha_{m-1}^N + \mathcal{O}\bigg(\frac{1}{|z|}\bigg)\bigg) \\ - \alpha_m^N + \mathcal{O}\bigg(\frac{1}{|P \circ (\psi_{m-1}^N)^{-1}(z))|}\bigg), \end{aligned}$$

and one sees easily that the $\mathcal{O}(1/|P \circ (\psi_{m-1}^N)^{-1}(z))|)$ term is actually $\mathcal{O}(1/|z|^2)$. Therefore, the coefficient of the linear term is $\lambda - 2\lambda\kappa\alpha_{m-1}^N$, and thus is bounded in modulus by $1 + 2 \cdot 1 \cdot \kappa \cdot (8/\kappa) = 17$. Lastly, since $\kappa \geq 1$, $1/(17+\kappa) < \kappa < 17+\kappa$ and so we have indeed constructed a $17+\kappa$-bounded sequence of polynomials (as defined near the start of §1.1), proving the lemma as desired. □

LEMMA 3.5. *Both ψ_0^N and $(\psi_0^N)^{-1}$ converge uniformly to the identity on $\mathbb{C}$ (with respect to the Euclidean metric).*

Proof. Recall that Γ is the boundary of Ω' and that we chose $\mathcal{K} \subset \Omega \subset \overline{\Omega} \subset \Omega' \subset \mathrm{D}(0, (2/\kappa))$. Let $G(z)$ be the Green's function for P and set $h := \sup_{z\in\Gamma} G(z)$. Then, for each $0 \leq m < N$, supp $\mu_m^N \subset \{z : 0 < G(z) \leq h \cdot 2^{m-N}\}$ and so supp $\mu_0^N \subset \{z : 0 < G(z) \leq h \cdot 2^{-N}\}$. Thus, $\mu_0^N \to 0$ everywhere as $N \to \infty$. By [**CG93**, Theorem I.7.5] (see also [**Ahl66**, Lemma 1]), we have that φ_0^N and $(\varphi_0^N)^{-1}$ both converge uniformly to the identity on $\mathbb{C}$ (recall that the unique solution for $\mu \equiv 0$ is the identity in view of the uniqueness part of the measurable Riemann mapping theorem for solving the Beltrami equation e.g. [**CG93**, Theorem I.7.4]). Finally, $\alpha_0^N = \varphi_0^N(0) \to 0$ as $N \to \infty$, and since $\psi_0^N = \varphi_0^N(z) - \alpha_0^N$, the result follows. □

The support of each μ_0^N is contained in the basin of infinity for P, A_∞. Since we had $2^{-N} \inf_{z\in\gamma} G(z) > 0$, ψ_0^N is analytic on a neighborhood of $\overline{U}$. Then if we define $U^N := \psi_0^N(U)$, we have that $(\psi_0^N)^{-1}$ is analytic on a neighborhood of $\overline{U^N}$. We now prove two fairly straightforward technical lemmas (see Figure 2).

LEMMA 3.6. $(U^N, 0) \to (U, 0)$ *in the Carathéodory topology.*

Proof. Define $\psi^{-1} : \mathbb{D} \to U$ to be the unique inverse Riemann map from $\mathbb{D}$ to U satisfying $\psi^{-1}(0) = 0$, $(\psi^{-1})'(0) > 0$. By Lemma 3.5, $\psi_0^N \circ \psi^{-1}$ converges locally

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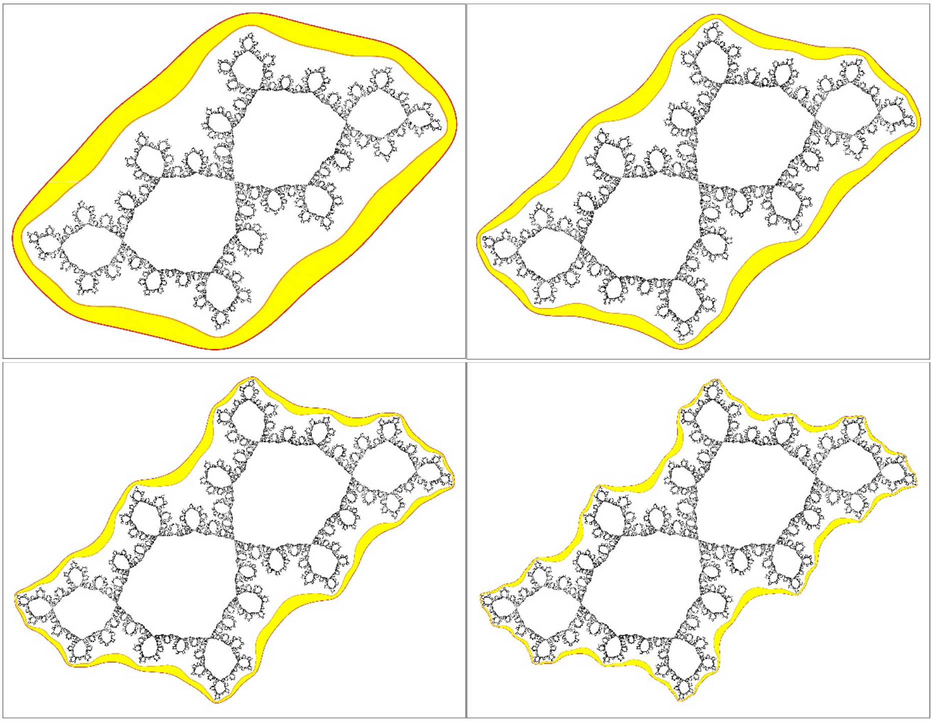

FIGURE 2. Supports of dilatations converging to zero almost everywhere.

uniformly to $\mathrm{Id} \circ \psi^{-1}$ on $\mathbb{D}$. The result then follows from Theorem A.9 in view of the fact that by the above result, $(\psi_0^N)'(0) \to 1$ as $N \to \infty$ (so that the argument of $(\psi_0^N)'(0)$ converges to 0 as $N \to \infty$). $\square$

LEMMA 3.7. *For any $\varepsilon > 0$ and any relatively compact subset A of U, there exists an N_0 such that*

$$|(\psi_0^N)^\natural(z) - 1| < \varepsilon,$$
$$|((\psi_0^N)^{-1})^\natural(z) - 1| < \varepsilon$$

for all z in A, $N \geq N_0$.

Proof. Let $d\rho_U = \sigma_U(z)|dz|$, where the hyperbolic density σ_U is continuous on U (e.g. [**KL07**, Theorem 7.2.2]) and bounded away from 0 on any relatively compact subset of U. For each N, ψ_0^N is analytic on a neighborhood of $\overline{U}$, while by Lemma 3.6 and part (2) of the definition of convergence in the Carathéodory topology (Definition A.7), $(\psi_0^N)^{-1}$ is analytic on any relatively compact subset of U for N sufficiently large, so that by Lemma 3.5, both $(\psi_0^N)'$ and $((\psi_0^N)^{-1})'$ converge uniformly to 1 on A. Since A is a relatively compact subset of U, there exists $\delta_0 > 0$ such that U contains a Euclidean $2\delta_0$-neighborhood of A. Let $\tilde{A}$ denote a Euclidean δ_0-neighborhood of A, so that $\tilde{A}$ is still a relatively compact subset of U. By Lemma 3.5 again, we can choose N_0 large enough such that $\psi_0^N(A) \subset \tilde{A}$ for all $N \geq N_0$. Then, since σ_U is continuous on the relatively compact subset A of U, there exists $\sigma > 0$ such that $|\sigma_U| \geq \sigma$ on A. Then for $z \in A$, using the uniform continuity of σ_U on the relatively compact subset $\tilde{A}$ of U,

$$(\psi_0^N)^\natural(z) = \frac{(\psi_0^N)'(z)\sigma_U(\psi_0^N(z))}{\sigma_U(z)}$$

converges uniformly to 1 on A, as desired. The proof for $((\psi_0^N)^{-1})^\natural$ is similar. □

3.2. *Statement and proof of the Polynomial Implementation Lemma.* Recall that we had defined $P_m^N(z) = \psi_m^N \circ P \circ (\psi_{m-1}^N)^{-1}(z)$ so that we have defined P_m^N for $0 \leq m \leq N$. Recall also that we have a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ for which the subsequence $\{P^{\circ n_k}\}_{k=1}^\infty$ converges uniformly to the identity on compact subsets of U (in fact, we can choose $\{n_k\}_{k=1}^\infty$ to be the Fibonacci sequence e.g. [**Mil06**, Problem C-3]).

Define $Q_{n_k}^{n_k}(z) = P_{n_k}^{n_k} \circ P_{n_k-1}^{n_k} \circ \cdots \cdots \circ P_2^{n_k} \circ P_1^{n_k}(z)$ (again we remind the reader that the superscripts n_k here are indices and do not denote powers or iteration) and note that this simplifies so that $Q_{n_k}^{n_k}(z) = \psi_{n_k}^{n_k} \circ P^{\circ n_k} \circ (\psi_0^{n_k})^{-1}(z)$. Essentially the same argument as in the proof of Lemma 3.7 allows us to prove the following.

LEMMA 3.8. *For any $\varepsilon > 0$ and any relatively compact subset A of U, there exists k_0 such that*

$$|(P^{\circ n_k})^\natural(z) - 1| < \varepsilon$$

for all z in A, $k \geq k_0$.

We now state the Polynomial Implementation Lemma. It is by means of this lemma that we create all polynomials constructed in the proofs of Phases I and II. First we make the definition that, for a relatively compact set A of U and $\delta > 0$, the set $\{z \in U : \rho_U(z, A) < \delta\}$ (where $\rho_U(z, A)$ is the hyperbolic distance in U from z to A as specified in Definition 2.7) is called the *δ-neighborhood* of A. Observe that such a neighborhood is again a relatively compact subset of U.

LEMMA 3.9. (The Polynomial Implementation Lemma) *Let P_λ, U_λ, κ, P, U, $\{n_k\}_{k=1}^\infty$, Ω, Ω', γ, Γ, and f be as above where, in addition, we also require $f(0) = 0$. Suppose $A \subset U$ is relatively compact and δ, M are positive such that if $\hat{A}$ is the δ-neighborhood of A with respect to ρ_U as above, then we have $f(\hat{A}) \subset U$ and $\|f^\natural\|_{\hat{A}} \leq M$. Then, for all $\varepsilon > 0$, there exists $k_0 \geq 1$ (determined by κ, the curves γ, Γ, the function f, as well as A, δ, M, and ε) such that for each $k_1 \geq k_0$, there exists a (17+κ)-bounded finite sequence of quadratic polynomials $\{P_m^{n_{k_1}}\}_{m=1}^{n_{k_1}}$ (which also depends on κ, γ, Γ, f, A, δ, M, ε, as well as k_1) such that $Q_{n_{k_1}}^{n_{k_1}}$ is univalent on A and:*

(1) $\rho_U(Q_{n_{k_1}}^{n_{k_1}}(z), f(z)) < \varepsilon$ *for all $z \in A$;*

(2) $\|(Q_{n_{k_1}}^{n_{k_1}})^\natural\|_A \leq M(1 + \varepsilon)$*;*

(3) $Q_{n_{k_1}}^{n_{k_1}}(0) = 0$.

Before embarking on the proof, a couple of remarks: first, this result is set up so that the subsequence of iterates $\{n_k\}_{k=1}^\infty$ used is always the same. Although we do not require this, it is convenient as it allows us to apply the theorem to approximate many functions simultaneously (which may be of use in some future application) while using the same number of polynomials in each of the compositions we obtain. Second, one can view this result as a weak form of our main theorem (Theorem 1.3), in that it allows to to approximate a single element of $\mathcal{S}$ with arbitrary accuracy using a finite composition of quadratic polynomials.

Proof. Let ε, δ be as above and, without loss of generality, take $\varepsilon < \min\{\delta, 1\}$. By Lemma A.4, the Euclidean and hyperbolic metrics are equivalent on compact subsets of U, and we can then use Lemma 3.5 to pick k_0 sufficiently large so that for all $k_1 \geq k_0$,

$$\rho_U((\psi_0^{n_{k_1}})^{-1}(z), z) < \frac{\varepsilon}{3(M+1)}, \quad z \in A. \tag{3.2}$$

This also implies that if we let $\check{A}$ be the $\delta/2$-neighborhood of A in U, then, since $\varepsilon < \delta$,

$$(\psi_0^{n_{k_1}})^{-1}(A) \subset \check{A}. \tag{3.3}$$

Next, by Lemma 3.7, we can make k_0 larger if needed such that for all $k_1 \geq k_0$,

$$|((\psi_0^{n_{k_1}})^{-1})^{\natural}(z) - 1| < \frac{\varepsilon}{3}, \quad z \in A. \tag{3.4}$$

From above, since $\{P^{\circ n_k}\}_{k=1}^{\infty}$ converges locally uniformly to the identity on U (with respect to the Euclidean metric), using Lemma A.4, we can again make k_0 larger if necessary to ensure for all $k_1 \geq k_0$ that

$$\rho_U(P^{\circ n_{k_1}}(z), z) < \frac{\varepsilon}{3(M+1)}, \quad z \in \check{A}. \tag{3.5}$$

This also implies

$$P^{\circ n_{k_1}}(\check{A}) \subset \hat{A}. \tag{3.6}$$

By Lemma 3.8, we can again make k_0 larger if needed such that for all $k_1 \geq k_0$,

$$|(P^{\circ n_{k_1}})^{\natural}(z) - 1| < \frac{\varepsilon}{3}, \quad z \in \check{A}. \tag{3.7}$$

We remark that this is the last of our requirements on k_0 and we are now in a position to establish the dependencies of k_0 on κ, γ, Γ, f, A, δ, M, ε in the statement. To be precise, the requirements on k_0 in equation (3.2) depend on κ, γ, Γ, f, A, M, ε, and δ, while those in equation (3.4) depend on κ, γ, Γ, f, A, ε, and δ, (but not M). Note that the dependency of these two estimates on the curves γ, Γ, or equivalently on the domains Ω, Ω', (which in turn depend on the scaling factor κ) as well as the function f, arises from the quasiconformal interpolation performed with the aid of Lemma 3.2 which is clearly dependent on these curves and this function. Further, the requirements on k_0 in equation (3.5) depend on κ, A, δ, M, and ε (but not γ, Γ, or f) while those in equation (3.7) depend on κ, A, δ, and ε (but not γ, Γ, f, or M). Finally, for the remaining estimates, equation (3.3) is a direct consequence of equation (3.2), while equation (3.6) follows immediately from equation (3.5) so that none of these three introduces any further dependencies.

Now fix $k_1 \geq k_0$ arbitrarily and let the finite sequence $\{P_m^{n_{k_1}}\}_{m=1}^{k_1}$ be constructed according to the sequence $\{n_k\}_{k=1}^{\infty}$ specified at the start of this section and the prescription given in equation (3.1). Note that this sequence is then $(17+\kappa)$-bounded in view of

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Lemma 3.4. By construction, $Q_{n_k}^{n_k}(0) = 0$ for every k so that item (3) in the statement above will be automatically satisfied.

Now equations (3.1), (3.3), and (3.6), the univalence of P on U, and the univalence of f on a neighborhood of $\mathcal{K}$ imply that $Q_{n_{k_1}}^{n_{k_1}}$ is univalent on A. Now let $z \in A$ and, using equation (3.2), consider a geodesic segment γ connecting z to $(\psi_0^{n_{k_1}})^{-1}(z)$ which, since $\varepsilon < \delta$, has length smaller than $\varepsilon/3$. Since $\varepsilon < \min\{\delta, 1\}$, $\varepsilon/3$ is in turn smaller than $\delta/2$ and so, by the definition of $\check{A}$, we have $[\gamma] \subset \check{A}$. This allows us to apply equations (3.2) and (3.7), and the hyperbolic M-L estimates (Lemma 2.9) for $P^{\circ n_{k_1}}$ to conclude that the length of $P^{\circ n_{k_1}}(\gamma)$ is at most $(1 + (\varepsilon/3))\varepsilon/3(M + 1)$, which is smaller than $\delta/2$ since $\varepsilon < \min\{\delta, 1\}$. As $[\gamma] \subset \check{A}$, by equation (3.6), $[P^{\circ n_{k_1}}(\gamma)] \subset \hat{A}$ and we are then able to apply the hyperbolic M-L estimates for f since by hypothesis, we have $|f^\natural(z)| \le M$ on $\hat{A}$.

In a similar manner, if instead we consider a geodesic segment connecting z to $P^{\circ n_{k_1}}(z)$, then, since $\varepsilon < \delta$, by equation (3.5), this segment again has length less than $\delta/2$ and starts at $z \in A$, whence it lies inside $\check{A} \subset \hat{A}$ and we are again able to apply the hyperbolic M-L estimates for f to this segment. Recall that $\psi_{n_{k_1}}^{n_{k_1}} = f$ on U in view of the definition of this function using quasiconformal interpolation in Lemma 3.2 and also the fact that the hyperbolic distance between any two points of U is less than or equal to the hyperbolic length of any curve connecting them. Using the triangle inequality and applying the estimates in equations (3.2)–(3.7) (except equation (3.4)) as well as $|f^\natural(z)| \le M$ on $\hat{A}$ from the statement, for each $z \in A$, since $P^{\circ n_{k_1}} \circ (\psi_0^{n_{k_1}})^{-1}(z) \in \hat{A}$ and $f(\hat{A}) \subset U$ by hypothesis, we then have

$$\begin{aligned}
\rho_U(Q_{n_{k_1}}^{n_{k_1}}(z), f(z)) &= \rho_U(\psi_{n_{k_1}}^{n_{k_1}} \circ P^{\circ n_{k_1}} \circ (\psi_0^{n_{k_1}})^{-1}(z), f(z)) \\
&\le \rho_U(f \circ P^{\circ n_{k_1}} \circ (\psi_0^{n_{k_1}})^{-1}(z), f \circ P^{\circ n_{k_1}}(z)) \\
&\quad + \rho_U(f \circ P^{\circ n_{k_1}}(z), f(z)) \\
&< M\left(1 + \frac{\varepsilon}{3}\right)\left(\frac{\varepsilon}{3(M+1)}\right) + M\left(\frac{\varepsilon}{3(M+1)}\right) \\
&< \varepsilon
\end{aligned}$$

(recall that we assumed $\varepsilon < 1$), which proves item (1). Also, using the chain rule in equation (2.2) for the hyperbolic derivative, the estimate $|f^\natural(z)| \le M$ on $\hat{A}$, and equations (3.3), (3.4), (3.6), and (3.7), for each $z \in A$,

$$\begin{aligned}
|((Q_{n_{k_1}}^{n_{k_1}})^\natural)(z)| &= |f^\natural(P^{\circ n_{k_1}} \circ (\psi_0^{n_{k_1}})^{-1}(z)) \cdot (P^{\circ n_{k_1}})^\natural((\psi_0^{n_{k_1}})^{-1}(z)) \cdot ((\psi_0^{n_{k_1}})^{-1})^\natural(z)| \\
&\le M\left(1 + \frac{\varepsilon}{3}\right)\left(1 + \frac{\varepsilon}{3}\right) \\
&< M(1 + \varepsilon),
\end{aligned}$$

again using $\varepsilon < 1$ at the end, which proves item (2) as desired. □

4. *Phase I*

4.1. *Setup.* We begin by finding a suitable disk on which $f \circ g^{-1}$ is defined for arbitrary $f, g \in \mathcal{S}$.

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LEMMA 4.1. *If $f, g \in \mathcal{S}$, then $f \circ g^{-1}$ is defined on* $\mathrm{D}(0, (1/12))$ *and*

$$(f \circ g^{-1})(\mathrm{D}(0, (1/12))) \subset \mathrm{D}(0, \tfrac{1}{3}).$$

Proof. Let $f, g \in \mathcal{S}$. By the Koebe one-quarter theorem (Theorem A.1) we have $\mathrm{D}(0, \frac{1}{4}) \subset g(\mathbb{D})$ so g^{-1} is defined on $\mathrm{D}(0, \frac{1}{4})$. Then if $h(w) := 4g^{-1}(w/4)$ for $w \in \mathbb{D}$, we have that $h \in \mathcal{S}$ and $g^{-1}(z) = \frac{1}{4}h(4z)$ for $z \in \mathrm{D}(0, \frac{1}{4})$, where $z = w/4$. Thus, if $|z| \le 1/12$, we have $|w| \le \frac{1}{3}$ and by the distortion theorems (Theorem A.2), we have that $|h(w)| \le \frac{3}{4}$ and $|g^{-1}(z)| \le (3/16) < 1$ so that, in particular, $f \circ g^{-1}(z)$ exists. Then, using the distortion theorems again, if $z \in \mathrm{D}(0, (1/12))$, we have that $(f \circ g^{-1})(z) \le (48/169) < \frac{1}{3}$. Thus, $f \circ g^{-1}$ is defined on $\mathrm{D}(0, (1/12))$ for all $f, g \in \mathcal{S}$ and maps $\mathrm{D}(0, (1/12))$ into $\mathrm{D}(0, \frac{1}{3})$ as required. $\square$

In the proof of Phase I, we will scale the filled Julia set for the polynomial $P_\lambda(z) = \lambda z(1-z)$, where $\lambda = e^{2\pi i((\sqrt{5}-1)/2)}$ so that the filled Julia set is a subset of $\mathrm{D}(0, (1/12))$. We are then able to apply $f \circ g^{-1}$ for $f, g \in \mathcal{S}$, which are then defined on this filled Julia set. We wish to find a suitable subdomain of this scaled filled Julia set so that we may control the size of the hyperbolic derivative $(f \circ g^{-1})^\natural$ on that subdomain. There are two possible strategies for doing this: one can either consider a small hyperbolic disk in the Siegel disc, or one can scale P_λ so that the scaled filled Julia set lies inside a small Euclidean disc about 0. We found the second option more convenient, as it allows us to consider an arbitrarily large hyperbolic disk inside the scaled Siegel disc on which $|(f \circ g^{-1})'|$ is tame and $|(f \circ g^{-1})^\natural|$ is thus easier to control. Lemmas 4.2–4.7 deal with finding a suitable scaling which allows us to obtain good estimates for $|(f \circ g)^\natural|$.

LEMMA 4.2. *There exists $K_1 > 0$ such that for all $f, g \in \mathcal{S}$, if $|z| \le 1/24$, then*

$$|(f \circ g^{-1})(z) - z| \le K_1 |z|^2.$$

Proof. Let $f, g \in \mathcal{S}$. By Lemma 4.1, the function $f \circ g^{-1}$ is defined on $\mathrm{D}(0, (1/12))$. Let $w \in \mathbb{D}$, $z = (1/12)w$, so that $z \in \mathrm{D}(0, (1/12))$, and define $h(w) = 12(f \circ g^{-1})(w/12)$ so that $h \in \mathcal{S}$. Then, letting $w + \sum_{n=2}^{\infty} a_n w^n$ denote the Taylor series about 0 for h and setting $K_0 = e\sum_{n=2}^{\infty} n^3(1/2^{n-2})$, if $|w| \le \frac{1}{2}$, we have

$$\begin{aligned}
|h'(w) - 1| &= \left| w \sum_{n=2}^{\infty} n a_n w^{n-2} \right| \\
&\le |w| \sum_{n=2}^{\infty} n|a_n||w|^{n-2} \\
&\le |w| e \sum_{n=2}^{\infty} n^3 \frac{1}{2^{n-2}} \\
&= K_0 |w|,
\end{aligned}$$

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where we used that $|a_n| \le en^2$ as $h \in \mathcal{S}$ (see e.g. [CG93, Theorem I.1.8]). Let $\gamma = [0, w]$ be the radial line segment from 0 to w. Then, if $|w| \le \frac{1}{2}$,

$$\begin{aligned}|h(w) - w| &= \left| \int_\gamma [h'(\zeta) - 1]\, d\zeta \right| \\ &\le K_0|w| \int_\gamma |\, d\zeta| \\ &= K_0|w|^2.\end{aligned}$$

Then, if $|z| \le 1/24$ (so that $|w| \le \frac{1}{2}$), a straightforward calculation shows

$$|(f \circ g^{-1})(z) - z| \le 12K_0|z|^2,$$

from which the lemma follows on setting $K_1 = 12K_0$. □

Recall $P_\lambda = \lambda z(1 - z)$ and the corresponding Siegel disc U_λ. Now fix $R > 0$ arbitrarily and let $\tilde{U}_R$ denote $\Delta_{U_\lambda}(0, R)$, the hyperbolic disc of radius R about 0 in U_λ. Let $\psi_\lambda : U_\lambda \to \mathbb{D}$ be the unique Riemann map satisfying $\psi_\lambda(0) = 0$, $\psi_\lambda'(0) > 0$. Let $\tilde{r}_0 = \tilde{r}_0(R) := d(\partial\tilde{U}_R, \partial U_\lambda)$, the Euclidean distance from $\partial\tilde{U}_R$ to ∂U_λ. Similarly to in §3, for $\kappa > 0$ arbitrary, set $P := (1/\kappa)P_\lambda(\kappa z)$ and note that P obviously depends on κ. Then, if $\mathcal{K} = \mathcal{K}(\kappa)$ is the filled Julia set for P, we have $\mathcal{K} \subset \mathrm{D}(0, (2/\kappa))$. Let $U = \{z : \kappa z \in U_\lambda\}$ be the corresponding Siegel disc for P and set $U_R = \Delta_U(0, R)$. Define $\psi(z) := \psi_\lambda(\kappa z)$ and observe that ψ is the unique Riemann map from U to $\mathbb{D}$ satisfying $\psi(0) = 0$, $\psi'(0) > 0$. Lastly, define $r_0 = r_0(\kappa, R) := d(\partial U_R, \partial U)$ and note $r_0 = \tilde{r}_0/\kappa$. Observe that $\tilde{r}_0$ and r_0 are decreasing in R while we must have $\tilde{r}_0 \le 2$. In what follows, let P_λ, U_λ, ψ_λ, P, U, ψ, $\tilde{r}_0$, and r_0 be fixed. For the moment, we let $\kappa > 0$ be arbitrary. We will, however, be fixing a lower bound on κ in the lemmas which follow.

LEMMA 4.3. (Local distortion) *For all κ, $R_0 > 0$, there exists $C_0 = C_0(R_0)$ depending on R_0 (in particular, C_0 is independent of κ) which is increasing, real-valued, and (thus) bounded on any bounded subset of* $[0, \infty)$ *such that, if U_{R_0} and $r_0 = r_0(\kappa, R_0) = d(\partial U_{R_0}, \partial U)$ are as above and $z_0 \in \overline{U}_{R_0}$, $z \in U$ with $|z - z_0| \le s < r_0$, we have:*

(1) $|\psi(z) - \psi(z_0)| \le C_0(s/r_0)/(1 - (s/r_0))^2$;

(2) $(1 - (s/r_0))/(1 + (s/r_0))^3 \le |\psi'(z)/\psi'(z_0)| \le (1 + (s/r_0))/(1 - (s/r_0))^3$.

Proof. Set $C_0 = C_0(R_0) = 2\max_{z\in\overline{\tilde{U}}_{R_0}} |\psi_\lambda'(z)| = (2/\kappa)\max_{z\in\overline{U}_{R_0}} |\psi'(z)|$. Then $C_0(R_0)$ does not depend on κ and is clearly increasing in R_0, and therefore bounded on any bounded subinterval of $[0, \infty)$. For $z \in \mathrm{D}(z_0, r_0)$, set $\zeta = (z - z_0)/r_0$ and note that if we define $\varphi(\zeta) := (\psi(r_0\zeta + z_0) - \psi(z_0))/r_0\psi'(z_0)$, we have that $\varphi \in \mathcal{S}$. Applying the distortion theorems (Theorem A.2) to φ, we see

$$\begin{aligned}|\varphi(\zeta)| &\le \frac{|\zeta|}{(1 - |\zeta|)^2} \\ &\le \frac{s/r_0}{(1 - (s/r_0))^2},\end{aligned}$$

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from which we can conclude (using $r_0 = \tilde{r}_0/\kappa$ and $\tilde{r}_0 \leq 2$)

$$|\psi(z) - \psi(z_0)| \leq \frac{s/r_0}{(1-(s/r_0))^2} \cdot C_0,$$

which proves item (1). For (2), we again apply the distortion theorems to φ and observe

$$\frac{1-(s/r_0)}{(1+(s/r_0))^3} \leq \frac{1-|\zeta|}{(1+|\zeta|)^3} \leq |\varphi'(\zeta)| \leq \frac{1+|\zeta|}{(1-|\zeta|)^3} \leq \frac{1+(s/r_0)}{(1-(s/r_0))^3},$$

from which item (2) follows as $\varphi'(\zeta) = \psi'(z)/\psi'(z_0)$. □

LEMMA 4.4. *For any $R_0 > 0$ and $\eta > 0$, there exists $\kappa_0 = \kappa_0(R_0, \eta) \geq 48$ such that, for all $\kappa \geq \kappa_0$, $f, g \in \mathcal{S}$ and $z \in U$,*

$$|(f \circ g^{-1})(z) - z| \leq \eta r_0,$$

where $r_0 = r_0(\kappa, R_0) = d(\partial U_{R_0}, \partial U)$ is as above. In particular, this holds for $z \in \overline{U}_{R_0}$.

Proof. Fix $\kappa_0 \geq 48$. By Lemma 4.2, we have, on $U \subset \mathrm{D}(0, (2/\kappa)) \subset \mathrm{D}(0, (1/24))$, that $|(f \circ g^{-1})(z) - z| < K_1|z|^2$ for some $K_1 > 0$ (note that $f \circ g^{-1}$ is defined on U by Lemma 4.1). So $|(f \circ g^{-1})(z) - z| < 4K_1/\kappa^2$ since $|z| < 2/\kappa$. Then make κ_0 larger if necessary to ensure that $4K_1/\kappa^2 \leq \eta r_0 = \eta\tilde{r}_0/\kappa$ for all $\kappa \geq \kappa_0$ (where we recall that $\tilde{r}_0 = \tilde{r}_0(R_0) = d(\partial\tilde{U}_{R_0}, \partial U_\lambda) = \kappa r_0$). In fact, $\kappa_0 = \max\{48, (4K_1/\eta\tilde{r}_0)\}$ will suffice and since $\tilde{r}_0$ depends only on R_0, we have the correct dependencies for κ_0 and the proof is complete. □

Lemmas 4.2–4.4 are technical lemmas that assist in proving the following result which will be essential for controlling the hyperbolic derivative of ψ.

LEMMA 4.5. *Given $R_0 > 0$, there exists $\kappa_0 = \kappa_0(R_0) \geq 48$ such that for all $\kappa \geq \kappa_0$, $f, g \in \mathcal{S}$, and $z \in \overline{U}_{R_0}$, $(f \circ g^{-1})(z) \in U$ and:*
(1) $(1 - |\psi(z)|^2)/(1 - |\psi((f \circ g^{-1})(z))|^2) \leq 10/9$;
(2) $|\psi'((f \circ g^{-1})(z))|/|\psi'(z)| \leq \frac{9}{8}$.

Proof. For $R > 0$, set $c_R := e^R - 1/e^R + 1$. Then, if we fix $z_0 \in \overline{U}_{R_0}$, we have that $|\psi(z_0)| \leq c_{R_0}$ (recall that $\rho_{\mathbb{D}}(0, z) = \log((1+|z|)/(1-|z|))$ for $z \in \mathbb{D}$). Thus, $c_{R_0} < 1$ and

$$1 - |\psi(z_0)|^2 \geq 1 - c_{R_0}^2 > 0. \tag{4.1}$$

As in the proof of Lemma 4.3, set $C_0 = C_0(R_0) = 2\max_{z \in \overline{\tilde{U}}_{R_0}} |\tilde{\psi}_\lambda'(z)|$. Let $0 < \eta_1 = \eta_1(R_0) < \frac{1}{2}$ be such that

$$\frac{C_0\eta_1}{(1-\eta_1)^2} \leq \frac{1}{2}(\log 10 - \log 9)(1 - c_{R_0+\log 3}^2) \tag{4.2}$$

and note that η_1 depends only on R_0. Using Lemma 4.4, we can pick $\kappa_1 = \kappa_1(R_0, \eta_1) = \kappa_1(R_0) > 0$ such that, if $\kappa \geq \kappa_1$, then $|(f \circ g^{-1})(z) - z| < \eta_1 r_0$ on $U \supset \overline{U}_{R_0}$ (recall the definitions of $\tilde{r}_0 = \tilde{r}_0(R_0)$ and $r_0 = r_0(\kappa, R_0)$ given before Lemma 4.3).

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Now set $s := |(f \circ g^{-1})(z_0) - z_0|$. We have $|(f \circ g^{-1})(z_0) - z_0| = s < \eta_1 r_0 < r_0/2$ as $\eta_1 < \frac{1}{2}$. Then, recalling the definition of $r_0 = d(\partial U_{R_0}, \partial U)$, we have $(f \circ g^{-1})(z_0) \in \mathrm{D}(z_0, (r_0/2)) \subset \mathrm{D}(z_0, r_0) \subset U$ as in the statement so that, in particular, $\psi(f \circ g^{-1})(z_0)$ is well defined. Again using $\rho_{\mathbb{D}}(0, z) = \log((1 + |z|)/(1 - |z|))$ for $z \in \mathbb{D}$ combined with the Schwarz lemma for the hyperbolic metric, we must have that $(f \circ g^{-1})(z_0) \in \overline{\Delta}_U(z_0, \log 3)$. By the triangle inequality for the hyperbolic metric, $(f \circ g^{-1})(z_0) \in \overline{\Delta}_U(0, R_0 + \log 3) = \overline{U}_{R_0+\log 3}$ so that $|\psi(f \circ g^{-1})(z_0)| \le c_{R_0+\log 3}$. Then, similarly to equation (4.1),

$$1 - |\psi((f \circ g^{-1})(z_0))|^2 \ge 1 - c^2_{R_0+\log 3} > 0. \tag{4.3}$$

We may then apply item (1) of Lemma 4.3 and equation (4.2) to see that

$$\begin{aligned}|\psi(z_0) - \psi((f \circ g^{-1})(z_0))| &\le \frac{C_0(s/r_0)}{(1 - (s/r_0))^2}\\ &\le \frac{C_0\eta_1}{(1 - \eta_1)^2}\\ &\le \frac{1}{2}(\log 10 - \log 9)(1 - c^2_{R_0+\log 3}).\end{aligned}$$

Thus, using the triangle inequality (and the fact that ψ is a Riemann mapping to the unit disc which has radius 1), we see that

$$|(1 - |\psi(z_0)|^2) - (1 - |\psi((f \circ g^{-1})(z_0))|^2)| < (\log 10 - \log 9)(1 - c^2_{R_0+\log 3}).$$

Making use of equations (4.1) and (4.3), noting that C_R is an increasing function of R, and applying the mean value theorem to the logarithm function on the interval $[1 - c^2_{R_0+\log 3}, \infty)$, we have

$$|\log(1 - |\psi(z_0)|^2) - \log(1 - |\psi((f \circ g^{-1})(z_0))|^2)| < \log 10 - \log 9$$

from which item (1) follows easily. For item (2), let $0 < \eta_2 < 1$ (e.g. $\eta_2 = 1/35$) be such that

$$\frac{1 + \eta_2}{(1 - \eta_2)^3} < \frac{9}{8}.$$

By Lemma 4.4, using the same $R_0 > 0$ as above, we can pick $\kappa_2 = \kappa_2(R_0, \eta_2) = \kappa_2(R_0) > 48$ such that for all $\kappa \ge \kappa_2$, if $z \in U \supset \overline{U}_{R_0}$,

$$|(f \circ g^{-1})(z) - z| < \eta_2 r_0.$$

Using the same $z_0 \in \overline{U}_{R_0}$ as above, in a similar way to how we used item (1) of Lemma 4.3 above, we can apply item (2) of the same result to see that

$$\frac{|\psi'((f \circ g^{-1})(z_0))|}{|\psi'(z_0)|} \le \frac{9}{8},$$

as desired. The result follows if we set $\kappa_0 = \kappa_0(R_0) = \max\{\kappa_1(R_0), \kappa_2(R_0)\}$. □

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LEMMA 4.6. *For all $\kappa \geq \kappa_0 := 576$, for any $f, g \in \mathcal{S}$ and $z \in \overline{U}$,*

$$|(f \circ g^{-1})'(z)| \leq \frac{6}{5}.$$

Proof. As in the proof of Lemma 4.2, define $h(w) = 12(f \circ g^{-1})(w/12)$. Note that h is defined on all of $\mathbb{D}$ by Lemma 4.1 and that $h \in \mathcal{S}$. Let $z = w/12$. Using the distortion theorems (Theorem A.2), we have that, for $z \in \mathrm{D}(0, (1/12))$,

$$|(f \circ g^{-1})'(z)| \leq \frac{1 + |12z|}{(1 - |12z|)^3}. \tag{4.4}$$

If $\kappa \geq \kappa_0$, we have that $\mathrm{D}(0, (2/\kappa)) \subset \mathrm{D}(0, (2/\kappa_0)) = \mathrm{D}(0, (1/288))$. Let $z \in \overline{U}$ and, since $\overline{U} \subset \mathcal{K} \subset \mathrm{D}(0, (2/\kappa)) \subset \mathrm{D}(0, (1/288))$, we have $|z| < 1/288$ for $\kappa \geq \kappa_0$. Thus, the right-hand side of equation (4.4) is less than $25 \cdot 24^2/23^3$, which in turn is less than $\frac{6}{5}$ for all $\kappa \geq \kappa_0$ as desired. □

As all the previous lemmas hold for all κ sufficiently large, applying them in tandem in the next result is valid. In general, each lemma may require a different choice of κ_0, but we may choose the maximum so that all results hold simultaneously. The purpose of Lemmas 4.5 and 4.6 is to prove the following.

LEMMA 4.7. *Given $R_0 > 0$, there exists $\kappa_0 = \kappa_0(R_0) \geq 576$ such that, for all $\kappa \geq \kappa_0$, for any $f, g \in \mathcal{S}$, and $z \in \overline{U}_{R_0}$, $(f \circ g^{-1})(z) \in U$ and*

$$|(f \circ g^{-1})^{\natural}(z)| \leq \frac{3}{2}.$$

Proof. Applying Lemmas 4.5 and 4.6 to the definition in equation (2.1) of the hyperbolic derivative taken with respect to the hyperbolic metric of U, and letting κ_0 be the maximum of the two lower bounds on κ in these lemmas, we have that there exists a $\kappa_0 \geq 576$ depending on R_0 such that, for all $\kappa \geq \kappa_0$, and $z \in \overline{U}_{R_0}$, $(f \circ g^{-1})(z) \in U$ and

$$\begin{aligned}
|(f \circ g^{-1})^{\natural}(z)| &= \frac{1 - |\psi(z)|^2}{1 - |\psi((f \circ g^{-1})(z))|^2} \cdot \frac{2 \cdot |\psi'((f \circ g^{-1})(z))|}{2 \cdot |\psi'(z)|} \cdot |(f \circ g^{-1})'(z)| \\
&\leq \frac{10}{9} \cdot \frac{9}{8} \cdot \frac{6}{5} \\
&= \frac{3}{2},
\end{aligned}$$

as desired. □

4.2. *Statement and proof of Phase I.*

LEMMA 4.8. (Phase I) *Let P_λ, U_λ, κ, P, and U be as above. Let $R_0 > 0$ be given, and let $\tilde{U}_{R_0}$ and U_{R_0} also be as above. Then, there exists $\kappa_0 = \kappa_0(R_0) \geq 576$ such that, for all $\kappa \geq \kappa_0$, $\varepsilon > 0$, and $N \in \mathbb{N}$, if $\{f_i\}_{i=0}^{N+1}$ is a collection of mappings with $f_i \in \mathcal{S}$ for $i = 0, 1, 2, \ldots\ldots, N+1$ with $f_0 = f_{N+1} = \mathrm{Id}$, there exists an integer M_N and a $(17+\kappa)$-bounded finite sequence $\{P_m\}_{m=1}^{(N+1)M_N}$ of quadratic polynomials both of which depend on R_0, κ, N, the functions $\{f_i\}_{i=0}^{N+1}$, and ε such that, for each $1 \leq i \leq N+1$:*

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(1) $Q_{iM_N}(0) = 0$;
(2) Q_{iM_N} *is univalent on* U_{2R_0};
(3) $\rho_U(f_i(z), Q_{iM_N}(z)) < \varepsilon$ *on* U_{2R_0};
(4) $\|Q^{\natural}_{iM_N}\|_{U_{R_0}} \le 7$.

Before proving this result, we remark first that the initial function $f_0 = \mathrm{Id}$ in the sequence $\{f_i\}_{i=0}^{N+1}$ does not actually get approximated. The reason we included this function was purely for convenience as this allowed us to describe all the functions being approximated in the proof using the Polynomial Implementation Lemma (Lemma 3.9) as $f_{i+1} \circ f_i^{-1}, 0 \le i \le N$.

Second, we can view this result as a weak form of our main theorem in that it allows to approximate finitely many elements of $\mathcal{S}$ with arbitrary accuracy using a finite composition of quadratic polynomials. Phase I is thus intermediate in strength between the Polynomial Implementation Lemma (Lemma 3.9) and our main result (Theorem 1.3).

Proof. Step 1: Setup. Without loss of generality, make ε smaller if necessary to ensure $\varepsilon < R_0$. Let $\kappa_0 = \kappa_0(R_0) \ge 576$ be as in the statement of Lemma 4.7 so that the conclusions of this lemma as well as those of Lemmas 4.5 and 4.6 also hold. Then for all $\kappa \ge \kappa_0$, we have $U \subset \mathcal{K} \subset \mathrm{D}(0, (2/\kappa)) \subset \mathrm{D}(0, (1/288)) \subset \mathrm{D}(0, (1/12))$. Note that the last inclusion implies that, if $f, g \in \mathcal{S}$, then $f \circ g^{-1}$ is defined on U in view of Lemma 4.1.

Step 2: Application of the Polynomial Implementation Lemma. First apply Lemma 4.7 with $5R_0 + 1$ replacing R_0 so that, for all $\kappa \ge \kappa_0$, if $f, g \in \mathcal{S}$, we have $(f \circ g^{-1})(U_{5R_0+1}) \subset U$ and

$$\|(f \circ g^{-1})^{\natural}\|_{U_{5R_0}} \le \|(f \circ g^{-1})^{\natural}\|_{U_{5R_0+1}} \le \frac{3}{2}. \tag{4.5}$$

Note that, by Lemmas 2.8 and 2.9, since U_{2R_0} is then hyperbolically convex while $(f \circ g^{-1})(0) = 0$, this implies

$$(f \circ g^{-1})(U_{2R_0}) \subset U_{3R_0}. \tag{4.6}$$

We observe that since $\mathrm{Id} \in \mathcal{S}$, in particular, we have $f(U_{2R_0}) \subset U_{3R_0}$ for all $f \in \mathcal{S}$.

Fix $\kappa \ge \kappa_0$ and for each $0 \le i \le N$, using equation (4.5), apply the Polynomial Implementation Lemma (Lemma 3.9), with $\Omega = \mathrm{D}(0, (1/24))$, $\Omega' = \mathrm{D}(0, \frac{1}{2})$, $\gamma = \mathrm{C}(0, (1/24))$, $\Gamma = \mathrm{C}(0, \frac{1}{2})$ (where both of these circles are positively oriented with respect to the round annulus of which they form the boundary), $f = f_{i+1} \circ f_i^{-1}$, $A = U_{5R_0}$, $\delta = 1$ (and hence, $\hat{A} = U_{5R_0+1}$), $M = \frac{3}{2}$, and ε replaced with $\varepsilon/3^N$. Note that $f(0) = 0$, $(f \circ g^{-1})(U_{5R_0+1}) \subset U$ (as noted above), and that, in view of Lemma 4.1, f is analytic and injective on a neighborhood of $\overline{\Omega}$ and maps γ inside $\mathrm{D}(0, \frac{1}{3})$ which lies inside Γ, so that (f, Id) is indeed an admissible pair on (γ, Γ) in the sense given in Definition 3.3 in §3 on the Polynomial Implementation Lemma, which then allows us to obtain a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ using Lemma 3.2.

Let M_N be the maximum of the integers n_{k_0} in the statement of Lemma 3.9 for each of the $N + 1$ applications of this lemma above. Note that each k_0 depends on κ, the curves $\gamma = \mathrm{C}(0, (1/24))$, $\Gamma = \mathrm{C}(0, \frac{1}{2})$, and the individual function $f = f_{i+1} \circ f_i^{-1}$ being approximated, as well as A, δ, the upper bound M on the hyperbolic derivative (which, in

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our case, by equation (4.5) is $\frac{3}{2}$ for every function we are approximating) and finally ε. Thus, M_N, in addition to N, then also depends on R_0, κ, the finite sequence of functions $\{f_i\}_{i=0}^{N+1}$, and, finally (recalling that here we have $\gamma = \mathrm{C}(0, (1/24))$, $\Gamma = \mathrm{C}(0, \frac{1}{2})$ and $A = U_{5R_0}$, $\delta = 1$, $M = \frac{3}{2}$), ε. From these $N + 1$ applications, we also then obtain (after a suitable and obvious labeling) a finite $(17 + \kappa)$-bounded sequence $\{P_m\}_{m=1}^{(N+1)M_N}$ such that each $Q_{iM_N,(i+1)M_N}$ is univalent on U_{5R_0}, and we have, for each $0 \le i \le N$ and each $z \in U_{5R_0}$,

$$\rho_U(Q_{iM_N,(i+1)M_N}(z), f_{i+1} \circ f_i^{-1}(z)) < \frac{\varepsilon}{3^N}. \tag{4.7}$$

It also follows from Lemma 3.9 that each $Q_{iM_N,(i+1)M_N}$ depends on N, R_0, κ, the functions f_i, f_{i+1}, and ε so that we obtain the correct dependencies for M_N and $\{P_m\}_{m=1}^{(N+1)M_N}$ in the statement. Lastly, by item (3) of Lemma 3.9, $Q_{iM_N,(i+1)M_N}(0) = 0$, for each i, proving item (1) in the statement above.

Step 3: Estimates on the compositions $\{Q_{iM_N}\}_{i=1}^{N+1}$. We use the following claim to prove items (2) and (3) in the statement (note that we do not require part (ii) of the claim below for this, but we will need it in proving item (4) later).

CLAIM 4.9. *For each $1 \le j \le N + 1$, we have that Q_{jM_N} is univalent on U_{2R_0} and, for each $z \in U_{2R_0}$,*

$$\text{(i) } \rho_U(Q_{jM_N}(z), f_j(z)) < \frac{\varepsilon}{3^{N+1-j}},$$

$$\text{(ii) } \rho_U(Q_{jM_N}(z), 0) < 4R_0.$$

Note that the error in this polynomial approximation for $j = N + 1$ is the largest, as this error combines errors from the largest number of prior mappings.

Proof. We prove the claim by induction on j. Let $z \in U_{2R_0}$. For the base case, we have that univalence and part (i) in the claim follow immediately from our applications of the Polynomial Implementation Lemma and in particular from equation (4.7) (with $j = i + 1 = 1$ so that $i = 0$) since $f_0 = \mathrm{Id}$. For part (ii), using part (i) (or equation (4.7)) and equation (4.6), compute

$$\begin{aligned}
\rho_U(Q_{M_N}(z), 0) &\le \rho_U(Q_{M_N}(z), f_1(z)) + \rho_U(f_1(z), 0) \\
&< \frac{\varepsilon}{3^N} + 3R_0 \\
&< 4R_0,
\end{aligned}$$

which completes the proof of the base case since we had assumed $\varepsilon < R_0$. Now suppose the claim holds for some $1 \le j < N + 1$. Then,

$$\begin{aligned}
\rho_U(Q_{(j+1)M_N}(z), f_{j+1}(z)) \le{}& \rho_U(Q_{jM_N,(j+1)M_N} \circ Q_{jM_N}(z), (f_{j+1} \circ f_j^{-1}) \circ Q_{jM_N}(z)) \\
&+ \rho_U((f_{j+1} \circ f_j^{-1}) \circ Q_{jM_N}(z), (f_{j+1} \circ f_j^{-1}) \circ f_j(z)).
\end{aligned}$$

Now $Q_{jM_N}(z) \in U_{4R_0} \subset U_{5R_0}$ by the induction hypothesis, so equation (4.7) implies that the first term on the right-hand side in the inequality above is less than $\varepsilon/3^N$. Again by the induction hypothesis, $Q_{jM_N}(z) \in U_{4R_0} \subset U_{5R_0}$, while we also have $f_j(z) \in U_{3R_0} \subset$

U_{5R_0} by equation (4.6). Thus, equation (4.5), the hyperbolic convexity of U_{5R_0} from Lemmas 2.8, 2.9, and the induction hypothesis imply that the second term in the inequality is less than $\frac{3}{2} \cdot \varepsilon/3^{N+1-j}$. Thus, we have $\rho_U(Q_{(j+1)M_N}(z), f_{j+1}(z)) < \varepsilon/3^{N+1-(j+1)}$, proving the first part of the claim.

Also, using what we just proved, equation (4.6), and our assumption that $\varepsilon < R_0$,

$$\begin{aligned}\rho_U(Q_{(j+1)M_N}(z), 0) &\leq \rho_U(Q_{(j+1)M_N}(z), f_{j+1}(z)) + \rho_U(f_{j+1}(z), 0)\\ &< \frac{\varepsilon}{3^{N+1-(j+1)}} + 3R_0\\ &< 4R_0,\end{aligned}$$

which proves part (ii) in the claim. Univalence of $Q_{(j+1)M_N}$ follows by hypothesis as $Q_{jM_N}(U_{2R_0}) \subset U_{4R_0}$, while $Q_{(j+1)M_n, jM_N}$ is univalent on $A = U_{5R_0} \supset U_{4R_0}$ by the Polynomial Implementation Lemma as stated immediately before equation (4.7). This completes the proof of the claim, from which items (2) and (3) in the statement of Phase I follow easily. □

Step 4: Proof of item (4) in the statement. To finish the proof, we need to give a bound on the size of the hyperbolic derivatives of the compositions Q_{iM_N}, $1 \leq i \leq N+1$. It will be of essential importance to us later that this bound not depend on the number of functions being approximated, the reason being that, in the inductive construction in Lemma 6.2, the error from the prior application of Phase II (Lemma 5.17) needs to pass through all these compositions while remaining small. This means that the estimate on the size of the hyperbolic derivative in item (2) of the statement of Lemma 3.9 is too crude for our purposes and so we have to proceed with greater care.

Let $d\rho_U(z)$ be the hyperbolic length element in U and write $d\rho_U(z) = \sigma_U(z)|dz|$, where the hyperbolic density σ_U (as introduced in the proof of Lemma 3.7) is continuous and positive on U (e.g. [**KL07**, Theorem 7.2.2]) and therefore uniformly continuous on U_{4R_0}, as U_{4R_0} is relatively compact in U. Let $\sigma = \sigma(R_0) > 0$ be the infimum of σ_U on U_{4R_0} so that

$$\sigma_U(z) \geq \sigma, \quad z \in U_{4R_0}. \tag{4.8}$$

Let $z \in U_{2R_0}$ and observe that, since $\kappa \geq \kappa_0 \geq 576$, $U \subset \mathrm{D}(0, (1/288)) \subset \mathbb{D}$. Then item (3) in the statement together with the Schwarz lemma for the hyperbolic metric (e.g. [**CG93**, Theorems 4.1 or 4.2]) give, for $1 \leq i \leq N+1$, $\rho_{\mathbb{D}}(Q_{iM_N}(z), f_i(z)) \leq \rho_U(Q_{iM_N}(z), f_i(z)) < \varepsilon$. If γ is a geodesic segment in $\mathbb{D}$ from $Q_{iM_N}(z)$ to $f_i(z)$, we see that

$$\begin{aligned}\varepsilon &> \rho_U(Q_{iM_N}(z), f_i(z))\\ &\geq \rho_{\mathbb{D}}(Q_{iM_N}(z), f_i(z))\\ &= \int_\gamma d\rho_{\mathbb{D}}\\ &= \int_\gamma \frac{2|dw|}{1-|w|^2}\end{aligned}$$

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$$\begin{aligned}
&\geq \int_{\gamma} 2|dw| \\
&= 2l(\gamma) \\
&\geq 2|Q_{iM_N}(z) - f_i(z)|
\end{aligned}$$

and so, in particular,

$$|Q_{iM_N}(z) - f_i(z)| < \varepsilon. \tag{4.9}$$

Now suppose further that $z \in U_{R_0}$ and set

$$\delta_0 = \delta_0(R_0) = \min_{w \in \partial U_{R_0}} d(w, \partial U_{(3/2)R_0}), \tag{4.10}$$

where $d(\cdot\,,\cdot)$ denotes Euclidean distance. By [New51, Theorem VII.9.1], the winding number of $\partial U_{\frac{3}{2}R_0}$ (suitably oriented) around z is 1. Then, using [Con78, Corollary IV.5.9] together with the standard distortion estimates in Theorem A.2 and equation (4.9) above, we obtain

$$\begin{aligned}
|Q'_{iM_N}(z)| &\leq |f_i'(z)| + |Q'_{iM_N}(z) - f_i'(z)| \\
&= |f_i'(z)| + \left| \frac{1}{2\pi i} \int_{\partial U_{(3/2)R_0}} \frac{Q_{iM_N}(w) - f_i(w)}{(w-z)^2} \, dw \right| \\
&\leq \frac{1 + (1/288)}{(1 - (1/288))^3} + \frac{\varepsilon}{2\pi \delta_0^2} l(\partial U_{(3/2)R_0}),
\end{aligned}$$

where $l(\partial U_{\frac{3}{2}R_0})$ is the Euclidean length of $\partial U_{\frac{3}{2}R_0}$. By making ε smaller if needed, we can thus ensure, for $z \in U_{R_0}$, that

$$|Q'_{iM_N}(z)| \leq \tfrac{3}{2}. \tag{4.11}$$

We can make ε smaller still if needed to guarantee that, if $z, w \in U_{4R_0}$ and $|z - w| < \varepsilon$, then, by uniform continuity of σ_U on U_{4R_0},

$$|\sigma_U(z) - \sigma_U(w)| < \sigma. \tag{4.12}$$

Note that both equations (4.11) and (4.12) required us to make ε smaller, but these requirements depended only on R_0 and, in particular, not on the sequence of polynomials we have constructed. Although this means we may possibly need to run the earlier part of the argument again to find a new integer M_N and then construct a new polynomial sequence $\{P_m \ :\ 1 \leq m \leq (N+1)M_N,\ 0 \leq i \leq N\}$, our requirements on ε above will then automatically be met. Alternatively, these requirements on ε could be made before the sequence is constructed. However, we decided to make them here for the sake of convenience.

If $z \in U_{R_0}$, we then have

$$\begin{aligned}
|Q^{\natural}_{iM_N}(z)| &\leq |f_i^{\natural}(z)| + |Q^{\natural}_{iM_N}(z) - f_i^{\natural}(z)| \\
&= |f_i^{\natural}(z)| + \left| \frac{\sigma_U(Q_{iM_n}(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} f_i'(z) \right|
\end{aligned}$$

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$$\leq |f_i^{\natural}(z)| + \left|\frac{\sigma_U(Q_{iM_n}(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} Q'_{iM_N}(z)\right| + \left|\frac{\sigma_U(f_i(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} f'_i(z)\right|.$$

We need to bound each of the three terms on the right-hand side of the above inequality. Recall that, as $g = \mathrm{Id} \in \mathcal{S}$, we have that $|f_i^{\natural}(z)| \leq \frac{3}{2}$ by equation (4.5). For the second term, by equations (4.6), (4.8), (4.9), (4.11), and (4.12), and part (ii) in Claim 4.9, we have

$$\begin{aligned}\left|\frac{\sigma_U(Q_{iM_n}(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} Q'_{iM_N}(z)\right| &= \frac{1}{|\sigma_U(z)|} \cdot |Q'_{iM_N}(z)| \cdot |\sigma_U(Q_{iM_N}(z)) - \sigma_U(f_i(z))| \\ &\leq \frac{1}{\sigma} \cdot \frac{3}{2} \cdot \sigma = \frac{3}{2}.\end{aligned}$$

For the third and final term, recall that we chose $\kappa_0 = \kappa_0(R_0)$ sufficiently large to ensure that the conclusions of Lemmas 4.5 and 4.6 hold. We can then apply Lemmas 4.5 and 4.6, together with equation (4.11) to obtain that

$$\begin{aligned}\left|\frac{\sigma_U(f_i(z))}{\sigma_U(z)} Q'_{iM_N}(z) - \frac{\sigma_U(f_i(z))}{\sigma_U(z)} f'_i(z)\right| &\leq \left|\frac{\sigma_U(f_i(z))}{\sigma_U(z)}\right| \cdot (|Q'_{iM_N}(z)| + |f'_i(z)|) \\ &\leq \frac{10}{9} \cdot \frac{9}{8}\left(\frac{3}{2} + \frac{6}{5}\right) \\ &< 4.\end{aligned}$$

Thus,

$$|Q^{\natural}_{iM_N}(z)| \leq \frac{3}{2} + \frac{3}{2} + 4 = 7,$$

as desired. □

5. *Phase II*

The approximations in Phase I inevitably involve errors and the correction of these errors is the purpose of Phase II. However, this correction comes at a price in that it is only valid on a domain which is smaller than that on which the error itself is originally defined; in other words there is an unavoidable loss of domain. There are two things here which work in our favor and stop this getting out of control: the first is the Fitting Lemma (Lemma 5.15) which shows us that loss of domain can be controlled and in fact diminishes to zero as the size of the error to be corrected tends to zero, while the second is that the accuracy of the correction can be made arbitrarily small, which allows us to control the errors in subsequent approximations.

We will be interpolating functions between Green's lines of a scaled version of the polynomial $P_\lambda = \lambda z(1-z)$, where $\lambda = e^{2\pi i((\sqrt{5}-1)/2)}$. If we denote the corresponding Green's function by G, we will want to be able to choose h small enough so that the regions

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between the Green's lines $\{z : G(z) = h\}$ and $\{z : G(z) = 2h\}$ are small in a sense to be made precise later. This will eventually allow us to control the loss of domain. However, we will want h to be large enough so that, if we distort the inner Green's line $\{z : G(z) = h\}$ slightly (with a suitably conjugated version of that same error function), the distorted region between them will still be a conformal annulus which will then allow us to invoke the Polynomial Implementation Lemma (Lemma 3.9). However, first we must prove several technical lemmas.

5.1. *Setup and the target and fitting lemmas.* We begin this section with continuous versions of Definition A.7 of Carathéodory convergence and of local uniform convergence and continuity on varying domains [**Com13a**, Definition 3.1].

Definition 5.1. Let $\mathcal{W} = \{(W_h, w_h)\}_{h\in I}$ be a sequence of pointed domains indexed by a non-empty set $I \subset \mathbb{R}$. We say that $\mathcal{W}$ *varies continuously in the Carathéodory topology* at $h_0 \in I$ or is *continuous at* h_0 if, for any sequence $\{h_n\}_{n=1}^{\infty}$ in I tending to h_0, $(W_{h_n}, w_{h_n}) \to (W_{h_0}, w_{h_0})$ as $n \to \infty$. If this property holds for all $h \in I$, we say $\mathcal{W}$ *varies continuously in the Carathéodory topology over I.*

For each $h \in I$, let g_h be an analytic function defined on W_h. If $h_0 \in I$ and $\mathcal{W}$ is continuous at h_0 as above, we say g_h converges locally uniformly to g_{h_0} on W_{h_0} if, for every compact subset K of W_{h_0} and every sequence $\{h_n\}_{n=1}^{\infty}$ in I tending to h_0, g_{h_n} converges uniformly to g_{h_0} uniformly on K as $n \to \infty$.

Finally, if we let $\mathcal{G} = \{g_h\}_{h\in I}$ be the corresponding family of functions, we say that $\mathcal{G}$ is *continuous* at $h_0 \in I$ if g_h converges locally uniformly to g_{h_0} on W_{h_0} as above. If this property holds for all $h \in I$, we say $\mathcal{G}$ *is continuous over I.*

Definition 5.2. Let $I \subset \mathbb{R}$ be non-empty and let $\{\gamma_h\}_{h\in I}$ be a family of Jordan curves indexed over *I*. We say that $\{\gamma_h\}_{h\in I}$ is a *continuously varying family of Jordan curves over I* if we can find a continuous function $F : \mathbb{T} \times I \to \mathbb{C}$ which is injective in the first coordinate such that, for each $h \in I$ fixed, $F(z, h)$ is a parameterization of γ_h.

Recall that a Jordan curve γ divides the plane into exactly two complementary components whose common boundary is $[\gamma]$ (e.g. [**Mun00**, Theorem 63.4] or [**New51**, Theorem V.10.2]). It is well known that we can use winding numbers to distinguish between the two complementary components of $[\gamma]$. More precisely, we can parameterize (that is, orient) γ, such that $n(\gamma, z) = 1$ for those points in the bounded complementary component of $\mathbb{C} \setminus [\gamma]$, while $n(\gamma, z) = 0$ for those points in the unbounded complementary component (e.g. [**New51**, Corollary 2 to Theorem VII.8.7 combined with Theorem VII.9.1]).

LEMMA 5.3. *Let $I \subset \mathbb{R}$ be non-empty and $\{\gamma_h\}_{h\in I}$ be a continuously varying family of Jordan curves indexed over I. For each $h \in I$, let W_h be the Jordan domain which is the bounded component of $\hat{\mathbb{C}} \setminus [\gamma_h]$, and let $w : I \to \mathbb{C}$ be continuous with $w(h) \in W_h$ for all h. Then the family $\{(W_h, w(h))\}_{h\in I}$ varies continuously in the Carathéodory topology over I.*

Proof. The continuity of w implies item (1) of Carathéodory convergence in the sense of Definitions A.7, 5.1. For item (2), fix $h_0 \in I$, let $K \subset W_{h_0}$ be compact, and let $z \in K$. Set

$\delta := \mathrm{d}(K, \partial W_{h_0})$. By the uniform continuity of F on compact subsets of $\mathbb{T} \times I$, we can find $\eta > 0$ such that, for each $h \in I$ with $|h - h_0| < \eta$,

$$|\gamma_h(t) - \gamma_{h_0}(t)| < \frac{\delta}{2} \quad \text{for all } t \in \mathbb{T},$$

and γ_h is thus homotopic to γ_{h_0} in $\mathbb{C} \setminus K$. We observe that we have not assumed that $I \cap (h_0 - \eta, h_0 + \eta)$ is an interval, so we may not be able to use the parameterization induced by $\gamma_h(z)$ to make the homotopy. However, using the above, it is a routine matter to construct the desired homotopy using convex linear combinations. By the above remark on winding numbers and Cauchy's theorem, one then obtains

$$n(\gamma_h, w) = n(\gamma_{h_0}, w) = 1 \quad \text{for all } w \in K.$$

Thus, if $|h - h_0| < \eta$, then $K \subset W_h$, and item (2) of Carathéodory convergence follows readily from this.

To show item (3) of Carathéodory convergence, let $\{h_n\}$ be any sequence in I which converges to h_0 and suppose N is an open connected set containing $w(h_0)$ such that $N \subset W_{h_n}$ for infinitely many n. Without loss of generality, we may pass to a subsequence to assume that $N \subset W_{h_n}$ for all n. Let $z \in N$ and connect z to $w(h_0)$ by a curve η in N. As $[\eta]$ is compact, there exists $\delta > 0$ such that a Euclidean δ-neighborhood of $[\eta]$ is contained in N and thus avoids γ_{h_n} for all n. By the continuity of F, this neighborhood also avoids γ_{h_0}. Since $w(h_0)$ and z are connected by η which avoids γ_{h_0}, they are in the same region determined by γ_{h_0} so that $n(\gamma_{h_0}, z) = n(\gamma_{h_0}, w(h_0))$. Hoewever, since by hypothesis $w(h_0) \in W_{h_0}$, by [**New51**, Corollary 2 to Theorem VII.8.7 combined with Theorem VII.9.1], $n(\gamma_{h_0}, w(h_0)) = 1$ whence $z \in W_{h_0}$. As z is arbitrary, we have $N \subset W_{h_0}$ and item (3) of Carathéodory convergence, and the result then follows. □

Recall that a Riemann surface is said to be *hyperbolic* if its universal cover is the unit disc $\mathbb{D}$. For a simply connected domain $U \subset \mathbb{C}$, this is equivalent to U being a proper subset of $\mathbb{C}$. The next lemma makes use of the following definition, originally given in [**Com14**] for families of pointed domains of finite connectivity. Recall that for a domain $U \subset \mathbb{C}$, we use the notation $\delta_U(z)$ for the Euclidean distance from a point z in U to the boundary of U.

Definition 5.4. [**Com14**, Definition 6.1] Let $\mathcal{V} = \{(V_\alpha, v_\alpha)\}_{\alpha \in A}$ be a family of hyperbolic simply connected domains and let $\mathcal{U} = \{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ be another family of hyperbolic simply connected domains indexed over the same set A, where $U_\alpha \subset V_\alpha$ for each α. We say that $\mathcal{U}$ is bounded above and below or just bounded in $\mathcal{V}$ with constant $K \geq 1$ if:

(1) U_α is a subset of V_α which lies within hyperbolic distance at most K about v_α in V_α;

(2)

$$\delta_{U_\alpha}(u_\alpha) \geq \frac{1}{K} \delta_{V_\alpha}(u_\alpha).$$

In this case, we write $\mathrm{pt} \sqsubset \mathcal{U} \sqsubset \mathcal{V}$.

The essential point of this definition is that the domains of the family $\mathcal{U}$ are neither too large nor too small in those of the family $\mathcal{V}$. For families of pointed domains of higher

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connectivity, two extra conditions are required relating to certain hyperbolic geodesics of the family $\mathcal{U}$. See [Com14] for details.

LEMMA 5.5. *Let $I \subset \mathbb{R}$ be non-empty, $\mathcal{U} = \{(U_h, v_h)\}_{h\in I}$ be a sequence of pointed Jordan domains, and $\mathcal{V} = \{(V_h, v_h)\}_{h\in I}$ be a sequence of pointed hyperbolic simply connected domains with the same base points, both indexed over I. If* pt $\sqsubset \mathcal{U} \sqsubset \mathcal{V}$, *$\mathcal{V}$ varies continuously in the Carathéodory topology over I, and ∂U_h is a continuously varying family of Jordan curves on I, then $R^{\text{ext}}_{(V_h,v_h)} U_h$ is continuous on I.*

Before embarking on the proof, we observe that, since both families $\mathcal{U}$ and $\mathcal{V}$ have the same basepoints, it follows from Lemma 5.3 and the fact that $\mathcal{V}$ varies continuously in the Carathéodory topology over I that $\mathcal{U}$ also varies continuously in the Carathéodory topology over I. However, we do not need to make use of this in the proof below.

Proof. Using Definition 5.2, as ∂U_h is a continuously varying family of Jordan curves, let $F : \mathbb{T} \times I \to \mathbb{C}$ be a continuous mapping, injective in the first coordinate where, for each h fixed, $F(t, h)$ is a parameterization of ∂U_h. We first need to uniformize the domains V_h by mapping to the unit disc $\mathbb{D}$ where we can compare hyperbolic distances directly. So let φ_h be the unique normalized Riemann map from V_h to $\mathbb{D}$ satisfying $\varphi_h(v_h) = 0$, $\varphi_h'(v_h) > 0$.

By Definitions 2.2, 5.4, since pt $\sqsubset \mathcal{U} \sqsubset \mathcal{V}$, there exists $K \geq 1$ such that $R^{\text{ext}}_{(V_h,v_h)} U_h \leq K$ and thus $\varphi_h(U_h) \subset \Delta_{\mathbb{D}}(0, K) = \mathrm{D}(0, ((e^K - 1)/(e^K + 1)))$. Also, for any $h_0 \in I$, we know from Theorem A.9 that φ_h converges to φ_{h_0} locally uniformly on V_{h_0} as $h \to h_0$ since $(V_h, v_h) \to (V_{h_0}, v_{h_0})$ in the sense of Definition 5.1. Now, set $\tilde{\varphi}(z, h) := \varphi_h(z)$.

CLAIM 5.6. *For all $h_0 \in I$ and $z_0 \in V_{h_0}$, $\tilde{\varphi}(z, h)$ is jointly continuous in z, h on a suitable neighborhood of (z_0, h_0).*

Proof. Let $\varepsilon > 0$. Let $\{h_n\}$ be a sequence in I which converges to h_0 and $\{z_n\}$ be a sequence in V_{h_0} which converges to z_0. Using item (2) of Carathéodory convergence (Definition A.7) and the fact that V_{h_0} is open, we have that $z_n \in V_{h_n}$ for all sufficiently large n. Then, for n sufficiently large so that z_n and h_n are sufficiently close to z_0 and h_0, respectively, since φ_h converges to φ_{h_0} locally uniformly on V_{h_0} and φ_{h_0} is continuous, we have

$$\begin{aligned}
|\tilde{\varphi}(z_n, h_n) - \tilde{\varphi}(z_0, h_0)| &= |\varphi_{h_n}(z_n) - \varphi_{h_0}(z_0)| \\
&\leq |\varphi_{h_n}(z_n) - \varphi_{h_0}(z_n)| + |\varphi_{h_0}(z_n) - \varphi_{h_0}(z_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

which proves the claim. □

Using this claim, if we now define $\psi(t, h) := \tilde{\varphi}(F(t, h), h)$, we have that $\psi(t, h)$ is jointly continuous in t and h on $\mathbb{T} \times I$.

Now let $h_0 \in I$ be arbitrary and let $\{h_n\}$ be any sequence in I which converges to h_0. If we write $R_n = R^{\text{ext}}_{(V_{h_n},v_{h_n})} U_{h_n}$ and $R_0 = R^{\text{ext}}_{(V_{h_0},v_{h_0})} U_{h_0}$, we then wish to show that $R_n \to R_0$ as $n \to \infty$. As pt $\sqsubset \mathcal{U} \sqsubset \mathcal{V}$, we may choose a subsequence $\{R_{n_k}\}$ which converges using

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Definition 5.4 to some finite limit in $[0, K]$. If we can show that the limit is R_0, we will have completed the proof. In view of Lemma 2.5, for each k, we have that R_{n_k} is attained at some $z_{n_k} \in \partial U_{n_k}$, so we may write $R_{n_k} = \rho_{V_{h_{n_k}}}(v_{h_{n_k}}, z_{n_k}) = \rho_{\mathbb{D}}(0, \tilde{\varphi}(z_{n_k}, h_{n_k}))$. Now $z_{n_k} = F(t_{n_k}, h_{n_k})$ for some $t_{n_k} \in \mathbb{T}$, so $R_{n_k} = \rho_{\mathbb{D}}(0, \psi(t_{n_k}, h_{n_k}))$. As $h_{n_k} \to h_0$, applying the compactness of $\mathbb{T}$ and passing to a further subsequence if necessary, we have that $(t_{n_k}, h_{n_k}) \to (t_0, h_0)$ for some $t_0 \in \mathbb{T}$, so that $\psi(t_{n_k}, h_{n_k})$ converges to $\psi(t_0, h_0)$ by the continuity of ψ and $R_{n_k} = \rho_{\mathbb{D}}(0, \psi(t_{n_k}, h_{n_k}))$ then converges to some limit $\tilde{R}_0$. Observe that there is no loss of generality in passing to such a further subsequence. □

CLAIM 5.7. $R^{\text{ext}}_{(V_{h_0}, v_{h_0})} U_{h_0} = R_0 = \tilde{R}_0 = \rho_{\mathbb{D}}(0, \psi(t_0, h_0)) = \rho_{V_{h_0}}(v_{h_0}, F(t_0, h_0))$.

Proof. Suppose not. Since ∂U_h is a continuously varying family of Jordan curves on I, $F(t, h_0) \in \partial U_{h_0}$ for any $t \in \mathbb{T}$. In view of Lemma 2.5, this means that the external hyperbolic radius for U_{h_0} is not attained at $F(t_0, h_0)$ and so there must exist $\tilde{t}_0 \in \mathbb{T}$ such that $\rho_{V_{h_0}}(v_{h_0}, F(t_0, h_0)) = \tilde{R}_0 < R_0 = \rho_{V_{h_0}}(v_{h_0}, F(\tilde{t}_0, h_0))$, that is, $\rho_{\mathbb{D}}(0, \psi(t_0, h_0)) < \rho_{\mathbb{D}}(0, \psi(\tilde{t}_0, h_0))$ whence $|\psi(t_0, h_0)| < |\psi(\tilde{t}_0, h_0)|$. Choose a sequence $\{(\tilde{t}_{n_k}, h_{n_k})\}$ in $\mathbb{T} \times I$ which converges to $(\tilde{t}_0, h_0)$. Then by joint continuity of ψ, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have that $|\psi(\tilde{t}_{n_k}, h_{n_k})| > |\psi(t_{n_k}, h_{n_k})|$, which contradicts the fact that $R_{n_k} = \rho_{\mathbb{D}}(0, \psi(t_{n_k}, h_{n_k}))$, again by Lemma 2.5. This completes the proof of both the claim and the lemma. □

Recall that we had $P_\lambda = \lambda z(1 - z)$, where $\lambda = e^{2\pi i((\sqrt{5}-1)/2)}$. For $\kappa \geq 1$, we then defined $P = (1/\kappa)P_\lambda(\kappa z)$ and let G be the Green's function for this polynomial. For each $h > 0$, set $V_h := \{z \in \mathbb{C} : G(z) < h\}$—see Figure 3 for an illustration showing two of these domains.

LEMMA 5.8. *The family* $\{\partial V_h\}_{h>0}$ *gives a continuously varying family of Jordan curves.*

Proof. Let P be as above, let $\mathcal{K}$ be the filled Julia set for P, and let $\varphi : \hat{\mathbb{C}} \setminus \mathcal{K} \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the associated Böttcher map. Then the map $F : \mathbb{T} \times (0, \infty) \to \mathbb{C}$, $F(e^{i\theta}, h) \mapsto \varphi^{-1}(e^{h+i\theta})$ is the desired mapping which yields a continuously varying family of Jordan curves. □

LEMMA 5.9. $(V_h, 0) \to (U, 0)$ *as* $h \to 0_+$.

Proof. By appealing to Definitions A.7, 5.1, and Theorem A.8, we can make use of the Carathéodory kernel version of Carathéodory convergence to prove this. So let h_n be any sequence of positive numbers such that $h_n \to 0$ as $n \to \infty$. From above, we will then be done if we can show that the Carathéodory kernel of $\{(V_{h_n}, 0)\}_{n=1}^\infty$, as well as that of every subsequence of this sequence of pointed domains, is U.

Let $\{(V_{h_{n_k}}, 0)\}_{k=1}^\infty$ be an arbitrary subsequence of $\{(V_{h_n}, 0)\}_{n=1}^\infty$ (which could possibly be all of $\{(V_{h_n}, 0)\}_{n=1}^\infty$) and let W be the Carathéodory kernel of this subsequence $\{(V_{h_{n_k}}, 0)\}_{k=1}^\infty$. Since $U \subset V_h$ for every $h > 0$, clearly $U \subseteq W$. To show containment in the other direction, let $z \in W$ be arbitrary and construct a path γ from 0 to z in W. By definition of W as the Carathéodory kernel of the domains $\{(V_{h_{n_k}}, 0)\}_{k=1}^\infty$, the track $[\gamma]$ is contained in $V_{h_{n_k}}$ for all k sufficiently large. From this, it follows that the iterates of P

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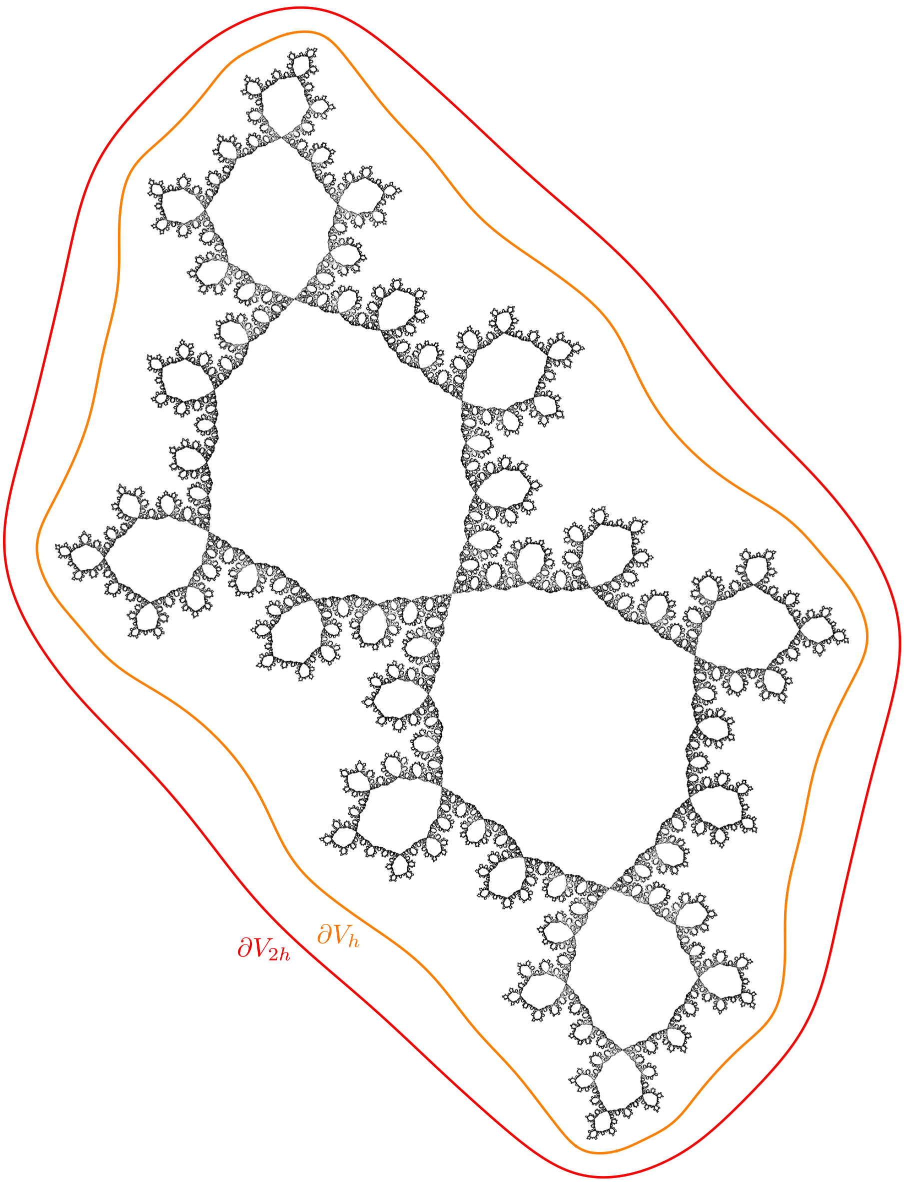

FIGURE 3. The filled Julia set $\mathcal{K}$ for P with the Green's lines $\partial V_h = \{z : G(z) = h\}$ and $\partial V_{2h} = \{z : G(z) = 2h\}$.

are bounded on W which immediately implies that $W \subset \mathcal{K}$. Since W is open, $W \subset \text{int}\,\mathcal{K}$. Moreover, since W is connected, W is then contained in a Fatou component for P and, since $0 \in W$, $W \subseteq U$. Since we have already shown $U \subseteq W$, we have $W = U$, as desired. □

As in the discussion in the proof of Phase I in §4 just before Lemma 4.3, let $\mathcal{K}$ be the filled Julia set for P and let U be the Siegel disc about 0 for P. Again, for $R > 0$, define $U_R := \Delta_U(0, R)$.

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For the remainder of this section, we will be working extensively with these hyperbolic discs U_R of radius R about 0 in U. At this point, we choose $0 < r_0$ and restrict ourselves to $R \geq r_0$ (we will also impose an upper bound on R just before stating the Target Lemma (Lemma 5.13)).

Again, let $\psi : U \to \mathbb{D}$ be the unique normalized Riemann map from U to $\mathbb{D}$ satisfying $\psi(0) = 0$, $\psi'(0) > 0$. For $h > 0$, let $\psi_{2h} : V_{2h} \to \mathbb{D}$ be the unique normalized Riemann map from V_{2h} to $\mathbb{D}$ satisfying $\psi_{2h}(0) = 0$, $\psi'_{2h}(0) > 0$. Set $\tilde{R} = R^{\mathrm{int}}_{(V_{2h},0)} U_R$ and define $\tilde{V}_{2h} = \Delta_{V_{2h}}(0, \tilde{R})$. Let $\varphi_{2h} : \tilde{V}_{2h} \to V_{2h}$ be the unique conformal map from $\tilde{V}_{2h}$ to V_{2h} normalized so that $\varphi_{2h}(0) = 0$ and $\varphi'_{2h}(0) > 0$. An important fact to note is that $\tilde{V}_{2h}$ is round in the conformal coordinates of V_{2h}, that is, $\psi_{2h}(\tilde{V}_{2h})$ is a disc (about 0). This is an essential point we will be making use of later in the 'up' portion of Phase II. We now prove a small lemma concerning this conformal disc $\tilde{V}_{2h}$.

LEMMA 5.10. *For $R \geq r_0$, we have the following:*

(1) *there exists $d_0 > 0$, determined by κ and r_0 such that*
$$d(0, \partial U_R) \geq d_0$$
(where $d(0, \partial U_R)$ denotes the Euclidean distance from 0 to ∂U_R);

(2) *given any finite upper bound $h_0 \in (0, \infty)$, there exists $\rho_0 > 0$, determined by r_0 and h_0, such that, for all $h \in (0, h_0]$, we have that the hyperbolic radius $R_{(V_{2h},0)} \tilde{V}_{2h}$ of $\tilde{V}_{2h}$ in V_{2h} about 0 satisfies*
$$R_{(V_{2h},0)} \tilde{V}_{2h} \geq \rho_0.$$

Proof. Since $R \geq r_0$, we have that ∂U_R is the image under ψ^{-1} of the circle $\mathrm{C}(0, s)$ in $\mathbb{D}$, where $s \geq s_0 := (e^{r_0} - 1)/(e^{r_0} + 1)$. Item (1) then follows on applying the Koebe one-quarter theorem (Theorem A.1).

For item (2), using Lemma 2.5, since $\partial \tilde{V}_{2h}$ is the hyperbolic 'incircle' about 0 of ∂U_R in the hyperbolic metric of V_{2h}, we have that for all $h \in (0, h_0]$, there exists $z_{2h} \in \partial \tilde{V}_{2h} \cap \partial U_R$. By item (1), we have $|z_{2h}| \geq d_0$. However, as the domains $\{V_{2h}\}_{h\in(0,h_0]}$ are increasing in h, there exists D_0 depending only on κ and h_0 such that for all $z \in U$, and for all $h \in (0, h_0]$, we have $\delta_{V_{2h}}(z) \leq D_0$ (where $\delta_{V_{2h}}(z)$ is the Euclidean distance from z to ∂V_{2h}). Letting ρ_{2h} be the hyperbolic radius about 0 of $\tilde{V}_{2h}$ in V_{2h}, we have
$$\rho_{2h} = \int_\gamma d\rho_{V_{2h}(z)},$$
where γ is a geodesic segment in V_{2h} from 0 to z_h. Then, using Lemma A.4, we have
$$\begin{aligned}\rho_{2h} &= \int_\gamma d\rho_{V_{2h}(z)}\\ &\geq \frac{1}{2}\int_\gamma \frac{1}{\delta_{V_{2h}}(z)}|dz|\\ &\geq \frac{1}{2D_0} l(\gamma)\end{aligned}$$

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$$\geq \frac{1}{2D_0}|z_{2h}|$$
$$\geq \frac{d_0}{2D_0},$$

from which the desired lower bound follows by setting $\rho_0 = d_0/2D_0$ (note that in the above, we use l to denote Euclidean arc length). Finally, the fact that ρ_0 does not depend on the scaling factor κ follows immediately by the conformal invariance of the hyperbolic metric of V_{2h} with respect to (Euclidean) scaling. □

Now define $\tilde{V}_h := \varphi_{2h}^{-1}(V_h)$ and recall that $\tilde{V}_{2h} = \varphi_{2h}^{-1}(V_{2h})$. Further, define $\check{R}(h) := R^{\text{ext}}_{(V_{2h},0)} V_h$ and note that the function $\check{R}(h)$ does not depend on the scaling factor κ, while by conformal invariance, we have $\check{R}(h) = R^{\text{ext}}_{(\tilde{V}_{2h},0)} \tilde{V}_h$.

Lemma 5.11. *$\check{R}(h)$ is continuous on* $(0, \infty)$.

Proof. This follows easily from Lemmas 5.3, 5.5, and 5.8. Note that it follows easily from Lemmas A.4 and 5.8 that the family $(V_h, 0)$ is bounded above and below in the family $(V_{2h}, 0)$, where h is allowed to range over any closed bounded subset I of $(0, \infty)$. □

Further, we have the following.

Lemma 5.12. *$\check{R}(h) \to \infty$ as $h \to 0_+$.*

Proof. By Lemma 5.9 and Theorem A.9, ψ_{2h} converges locally uniformly on U to ψ as $h \to 0_+$ (in the sense given in Definition 5.1), where we recall that ψ_{2h} and ψ are the suitably normalized Riemann maps from V_{2h} and U, respectively, to the unit disc (these were introduced in the discussion before Lemma 5.10).

Now let $R > 0$ be large and let $z \in \partial U_R$, where U_R is the hyperbolic disc of radius R about 0 in U introduced above. From the above, we then have that $\rho_{V_{2h}}(0, z) \geq R - 1$ for all h sufficiently small so that by the definition of external hyperbolic radius (Definition 2.2), we must have $R^{\text{ext}}_{(V_{2h},0)} U_R \geq R - 1$. Since $U_R \subset U \subset V_h$, we must have $\check{R}(h) := R^{\text{ext}}_{(V_{2h},0)} V_h \geq R^{\text{ext}}_{(V_{2h},0)} U \geq R^{\text{ext}}_{(V_{2h},0)} U_R \geq R - 1$. The result then follows on letting R tend to infinity. □

At this point, we choose $0 < r_0 < R_0 \leq \pi/2$ and restrict ourselves to $R \in [r_0, R_0]$. The upper bound $\pi/2$ is chosen so that the disc U_R as well as its image under any conformal mapping whose domain of definition contains U is star-shaped (about the image of 0—see Lemma A.6).

Given $\varepsilon_1 > 0$, using the hyperbolic metric of U, construct a $2\varepsilon_1$-open neighborhood of $\partial \tilde{V}_{2h}$ which we will denote by $\hat{N}$. We now fix our upper bound h_0 on the value of the Green's function $G(z)$. Recall the lower bound ρ_0 on the hyperbolic radius about 0 of $\tilde{V}_{2h}$ in V_{2h} as in item (2) of the statement of Lemma 5.10. Recall also the scaling factor κ and that $U \subset \mathrm{D}(0, (2/\kappa))$. We now state and prove one of the most important lemmas we need to prove Phase II (Lemma 5.17).

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LEMMA 5.13. (Target Lemma) *There exist an upper bound* $\tilde{\varepsilon}_1 \in (0, (\rho_0/2))$ *and a continuous function* $T : (0, \tilde{\varepsilon}_1] \to (0, \infty)$*, both of which are determined by* h_0 *and* r_0 *such that, for all* $h \in (0, h_0]$ *and* $R \in [r_0, R_0]$*, we have:*

(1) $R^{\text{int}}_{(\tilde{V}_{2h},0)}(\tilde{V}_{2h} \setminus \hat{N}) \geq T(\varepsilon_1)$ *for all* $\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$*;*

(2) $T(\varepsilon_1) = \frac{1}{2}\log(1/\varepsilon_1) + C_0$ *on* $(0, \tilde{\varepsilon}_1]$*, where* $C_0 = C_0(h_0, r_0)$*, so that, in particular;*

(3) $T(\varepsilon_1) \to \infty$ *as* $\varepsilon_1 \to 0_+$.

Before embarking on the proof, we remark that item (1) in the statement of Lemma 5.13 will help us to interpolate in the 'during' portion of Phase II. Item (3) will be vital for the Fitting Lemma (Lemma 5.15); it allows us to conclude that $h \to 0$ as $\varepsilon_1 \to 0_+$ (see the statement of the Fitting Lemma), which is key to controlling the inevitable loss of domain incurred in correcting the errors in our approximations from Phase I (Lemma 4.8). We observe that, even though we require $R \in [r_0, R_0]$, the upper bound R_0 does not appear in the dependencies for $\tilde{\varepsilon}_1$ and the function T above. The reason for this is that we apply the upper bound $R_0 \leq \pi/2$ in the proof which eliminates the dependence on R_0. Lastly, we observe that, although the domain $\tilde{V}_{2h}$ by definition will depend on R (as will the mapping $\varphi_{2h} : \tilde{V}_{2h} \mapsto V_{2h}$), $\tilde{\varepsilon}_1$ and $T(\varepsilon_1)$ do not depend on R since we are obtaining estimates which work simultaneously for all $R \in [r_0, R_0]$.

Proof. We first deduce the existence of $\tilde{\varepsilon}_1$. Regarding the upper bound $\rho_0/2$ on $\tilde{\varepsilon}_1$ in the statement: we note that, if ε_1 is too large, then we would actually have $\tilde{V}_{2h} \subset \hat{N}$ so that $\tilde{V}_{2h} \setminus \hat{N} = \emptyset$. Recall that, by item (2) of Lemma 5.10, we have that $\rho_0 > 0$ is such that for all $R \in [r_0, R_0]$ and $h \in (0, h_0]$, we have $R_{(V_{2h},0)}\tilde{V}_{2h} \geq \rho_0$. Using the Schwarz lemma for the hyperbolic metric (e.g. [**CG93**, Theorems I.4.1 or I.4.2]), we see that $R^{\text{int}}_{(U,0)}\tilde{V}_{2h} \geq R_{(V_{2h},0)}\tilde{V}_{2h} \geq \rho_0$, so setting $\tilde{\varepsilon}_1 := \rho_0/4 < \rho_0/2$ implies that, if $\varepsilon_1 \leq \tilde{\varepsilon}_1$, then $0 \in \tilde{V}_{2h} \setminus \hat{N}$ and, in particular, $\tilde{V}_{2h} \setminus \hat{N} \neq \emptyset$ (so that the internal hyperbolic radius of this set is well defined in view of Definition 2.2). Note that, in view of Lemma 5.10, since ρ_0 depends on r_0 and h_0, the quantity $\tilde{\varepsilon}_1$ inherits these dependencies.

Recall the lower bound $d_0 = d_0(\kappa, r_0) > 0$ from item (1) of the statement of Lemma 5.10 for which we have $d(0, \partial U_R) \geq d_0$ so that, if $\xi \in \partial U_R$, then $|\xi| \geq d_0$. With the distortion theorems in mind, applied to $\psi_{2h}^{\circ -1}$, we define

$$r_1 := \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1},$$
$$D_1 := \left(\frac{1 + r_1}{1 - r_1}\right)^2 = e^{\pi}.$$

Note that r_1 is chosen so that $\mathrm{D}(0, r_1)$ has hyperbolic radius $\pi/2$ in $\mathbb{D}$, that is, $\mathrm{D}(0, r_1) = \Delta_{\mathbb{D}}(0, (\pi/2))$. By the Schwarz lemma for the hyperbolic metric, since $\tilde{V}_{2h} \subset U_R$, $U \subset V_{2h}$, and $R \leq R_0 \leq \pi/2$, we have

$$R_{(V_{2h},0)}\tilde{V}_{2h} = R^{\text{ext}}_{(V_{2h},0)}\tilde{V}_{2h} \leq R^{\text{ext}}_{(U,0)}\tilde{V}_{2h} \leq R^{\text{ext}}_{(U,0)}U_R = R_{(U,0)}U_R = R \leq \frac{\pi}{2}$$

(recall that $\tilde{V}_{2h}$ and U_R are round in the conformal coordinates of V_{2h}, U, respectively, so that the internal and external hyperbolic radii coincide). By Lemma 2.5 and the definition

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of $\tilde{V}_{2h}$ given before Lemma 5.10, ∂U_R and $\partial \tilde{V}_{2h}$ meet, and it then follows by comparing the maximum and minimum values of $|\psi_{2h}^{-1}|$ given by the distortion theorems (Theorem A.2) that

$$|z| \geq \frac{d_0}{D_1} = d_0 e^{-\pi} \quad \text{if } z \in \partial \tilde{V}_{2h}. \tag{5.1}$$

Now suppose $\zeta_0 \in \partial \tilde{V}_{2h}$ and let $\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$. If $\zeta \in \overline{\Delta}_U(\zeta_0, 2\varepsilon_1)$, we wish to find an upper bound on the Euclidean distance from ζ to ζ_0. Let γ_0 be a geodesic segment in U from ζ_0 to ζ. Then, using Lemma A.4 and the fact that $U \subset \mathrm{D}(0, (2/\kappa))$, we calculate

$$\begin{aligned}
2\varepsilon_1 &\geq \int_{\gamma_0} d\rho_U \\
&\geq \frac{1}{2} \int_{\gamma_0} \frac{|dw|}{\delta_U(w)} \\
&\geq \frac{\kappa}{4} \int_{\gamma_0} |dw| \\
&= \frac{\kappa}{4}\, l(\gamma_0) \\
&\geq \frac{\kappa}{4} |\zeta - \zeta_0|,
\end{aligned}$$

where $l(\gamma_0)$ is (as usual) the Euclidean arc length of γ_0. Thus, $|\zeta - \zeta_0| \leq (8/\kappa)\varepsilon_1$. As ζ_0, ζ were arbitrary, this implies that

$$\overline{\Delta}_U(\zeta_0, 2\varepsilon_1) \subset \overline{\mathrm{D}}\left(\zeta_0, \frac{8}{\kappa}\varepsilon_1\right) \quad \text{for any } \zeta_0 \in \partial \tilde{V}_{2h}. \tag{5.2}$$

Now we aim to specify the value of the function $T(\varepsilon_1)$. Choose a point $z_0 \in \tilde{V}_{2h} \cap \hat{N} = \tilde{V}_{2h} \setminus (\tilde{V}_{2h} \setminus \hat{N})$. Pick $z \in \partial \tilde{V}_{2h}$ which is closest to z_0 in the hyperbolic metric of U (see Figure 4 for an illustration). Then $\rho_U(z_0, z) \leq 2\varepsilon_1$, which by equation (5.2) implies $|z_0 - z| \leq (8/\kappa)\varepsilon_1$. Note that as $|z| \geq d_0/D_1$ by equation (5.1), using the reverse triangle inequality, we have that

$$|z_0| \geq \frac{d_0}{D_1} - \frac{8}{\kappa}\varepsilon_1. \tag{5.3}$$

Note also that, to make sure that $T(\varepsilon_1)$ is defined and positive on $(0, \tilde{\varepsilon}_1]$, it will be essential (because we will be taking the difference of the logs in the two terms in this quantity) that $(d_0/D_1) - (8/\kappa)\varepsilon_1 > 0$, so we may need to make $\tilde{\varepsilon}_1$ smaller if needed so that $\tilde{\varepsilon}_1 < \kappa d_0/8D_1$. Since the constant d_0 is determined by κ and the lower bound r_0 for R, $\tilde{\varepsilon}_1$ will then be determined by these same constants as well as h_0 in view of our earlier discussion on $\tilde{\varepsilon}_1$ above (we will argue later that the dependence on the scaling factor κ can be removed). Now let γ be a geodesic segment in $\tilde{V}_{2h}$ from z_0 to 0. If $w \in [\gamma]$, since $|z_0 - z| \leq (8/\kappa)\varepsilon_1$ from equation (5.2), we have

$$\delta_{\tilde{V}_{2h}}(w) \leq |w - z| \leq |w - z_0| + |z_0 - z| \leq |w - z_0| + \frac{8}{\kappa}\varepsilon_1.$$

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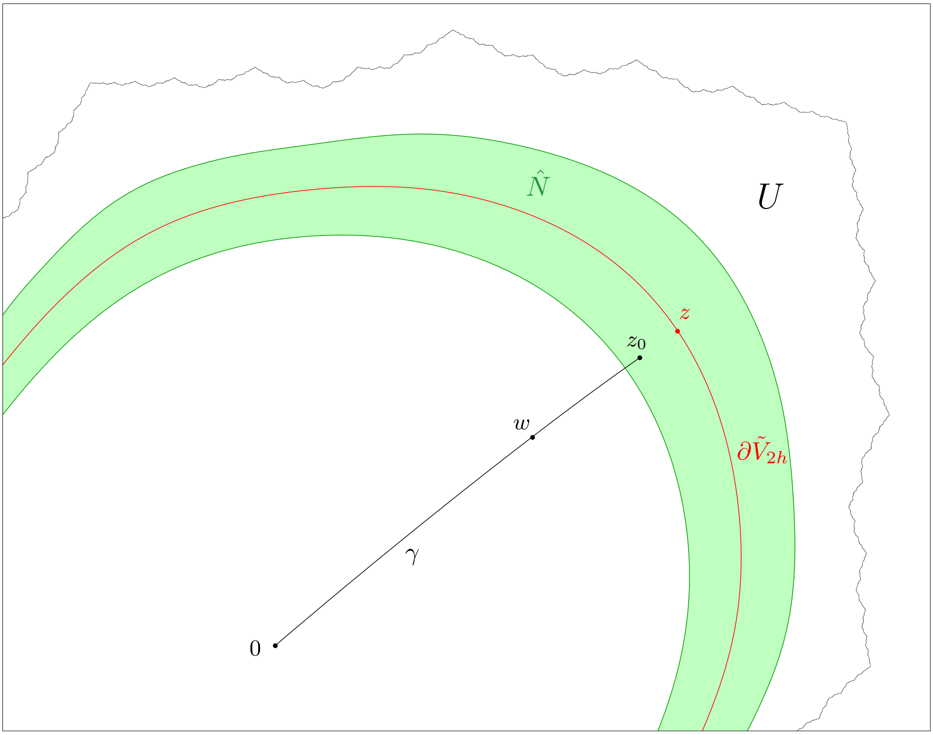

FIGURE 4. Finding a lower bound for $\rho_{\tilde{V}_{2h}}(0, z_0)$.

So, once more using Lemma A.4,

$$\begin{aligned}\rho_{\tilde{V}_{2h}}(0, z_0) &= \int_\gamma d\rho_{\tilde{V}_{2h}(w)} \\ &\geq \frac{1}{2}\int_\gamma \frac{|dw|}{\delta_{\tilde{V}_{2h}}(w)} \\ &\geq \frac{1}{2}\int_\gamma \frac{|dw|}{|w - z_0| + (8/\kappa)\varepsilon_1}.\end{aligned}$$

Now parameterize γ by $w = \gamma(t) = z_0 + r(t)e^{i\theta(t)}$ for $t \in [0, 1]$, and note that, as γ is a geodesic segment in $\tilde{V}_{2h}$ from z_0 to 0, $r(1)e^{i\theta(1)} = -z_0$. Since γ is not self-intersecting, we have $r(t) > 0$ for all $t \in (0, 1]$. Then, using equation (5.3),

$$\begin{aligned}\frac{1}{2}\int_\gamma \frac{|dw|}{|w - z_0| + (8/\kappa)\varepsilon_1} &= \frac{1}{2}\int_0^1 \frac{|r'(t)e^{i\theta(t)} + i\theta'(t)r(t)e^{i\theta(t)}|}{r(t) + (8/\kappa)\varepsilon_1}\, dt \\ &\geq \frac{1}{2}\int_0^1 \frac{|r'(t)|}{r(t) + (8/\kappa)\varepsilon_1}\, dt, \\ &\geq \frac{1}{2}\bigg|\int_0^1 \frac{r'(t)}{r(t) + (8/\kappa)\varepsilon_1}\, dt\bigg|\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}\int_0^{|z_0|} \frac{1}{u+(8/\kappa)\varepsilon_1}\,du \\
&\geq \frac{1}{2}\int_0^{(d_0/D_1)-(8/\kappa)\varepsilon_1} \frac{1}{u+(8/\kappa)\varepsilon_1}\,du \\
&= \frac{1}{2}\int_{(8/\kappa)\varepsilon_1}^{d_0/D_1} \frac{1}{x}\,dx \\
&= \frac{1}{2}\bigg(\log\bigg(\frac{d_0}{D_1}\bigg) - \log\bigg(\frac{8}{\kappa}\varepsilon_1\bigg)\bigg) \\
&= \frac{\log d_0 - \pi - \log\varepsilon_1 - \log 8 + \log\kappa}{2} \\
&= \frac{1}{2}\log\frac{1}{\varepsilon_1} + \frac{\log d_0 - \pi - \log 8 + \log\kappa}{2}.
\end{aligned}
$$

Taking an infimum over all $z_0 \in \tilde{V}_{2h} \cap \hat{N}$ and applying the definition (Definition 2.2) of internal hyperbolic radius (which in particular does not require that $\tilde{V}_{2h} \setminus \hat{N}$ be connected), and setting $T(\varepsilon_1) = \frac{1}{2}\log(1/\varepsilon_1) + \frac{1}{2}(\log d_0 - \pi - \log 8 + \log\kappa)$, then $T(\varepsilon_1)$ is continuous, strictly positive (in view of the definition of $\tilde{\varepsilon}_1$ given in the discussion after equation (5.3)), and gives the desired lower bound on $R^{\mathrm{int}}_{(\tilde{V}_{2h},0)}(\tilde{V}_{2h} \setminus \hat{N})$. Explicitly, the function $T(\varepsilon_1)$ above is determined by κ, r_0, and h_0 (this last being due to the requirement that $\varepsilon_1 \leq \tilde{\varepsilon}_1$). However, similarly to the end of the proof of Lemma 5.10, we may eliminate the dependence on κ (for both $\tilde{\varepsilon}_1$ and T) given the conformal invariance of the hyperbolic metric of $\tilde{V}_{2h}$ with respect to the scaling factor κ. □

Before turning to the Fitting Lemma (Lemma 5.15), we prove a small lemma from real analysis.

LEMMA 5.14. *Let $b > 0$ and let $\varphi : (0, b] \to [0, \infty)$ be a continuous function such that $\varphi(x) \to \infty$ as $x \to 0_+$. Then, for all $y \geq \min\{\varphi(x) : x \in (0, b]\}$, if we set*

$$x(y) := \min\{x : \varphi(x) = y\},$$

we have $x(y) \to 0$ as $y \to \infty$.

Proof. We note first that, since φ is continuous while $\varphi(x) \to \infty$ as $x \to 0_+$, φ attains its minimum on $(0, b]$. Also, in view of the intermediate value theorem, for each $y \geq \min\{\varphi(x) : x \in (0, b]\}$, the set $\{x : \varphi(x) = y\}$ is non-empty. Because $\varphi(x) \to \infty$ as $x \to 0_+$, the infimum of this set is strictly positive, and, as φ is continuous, we must have that $\varphi(x(y)) = y$ so that this infimum is attained and is in fact a minimum. Suppose now the conclusion is false and that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \to \infty$, but $x(y_n) \not\to 0$ as $n \to \infty$. Set $x_n := x(y_n)$. Since $x_n \not\to 0$, we can take a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to a limit $x_0 > 0$. This then leads to a contradiction to the continuity of φ at x_0. □

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Recall the quantity $\check{R}(h) := R^{\mathrm{ext}}_{(V_{2h},0)} V_h = R^{\mathrm{ext}}_{(\tilde{V}_{2h},0)} \tilde{V}_h$ which was introduced before Lemma 5.11 and the $2\varepsilon_1$-open neighborhood $\hat{N}$ of $\partial \tilde{V}_{2h}$ which was introduced before the statement of Lemma 5.13. We now state and prove the Fitting Lemma.

LEMMA 5.15. (The Fitting Lemma) *There exists* $\tilde{\varepsilon}_1 > 0$ *and a function* $h : (0, \tilde{\varepsilon}_1] \mapsto (0, \infty)$ *both of which are determined by* h_0, r_0 *for which the following hold:*

(1) $\overline{\tilde{V}}_{h(\varepsilon_1)} \subset \tilde{V}_{2h(\varepsilon_1)} \setminus \hat{N}$ *for each* $0 < \varepsilon_1 \le \tilde{\varepsilon}_1$;

(2) $h(\varepsilon_1) \to 0$ *as* $\varepsilon_1 \to 0_+$.

Proof. We first apply the Target Lemma (Lemma 5.13) to find $\tilde{\varepsilon}_1 > 0$ and a function $T : (0, \tilde{\varepsilon}_1] \mapsto (0, \infty)$ as above, both of which are determined by h_0, r_0. We now show how to use the function T to define an appropriate value h of the Green's function for which item (1) above holds, which will then allow us to do the interpolation in the 'during' part of the proof of Phase II (Lemma 5.17). Our first step is to fix a (possibly) smaller value of $\tilde{\varepsilon}_1$ which still has the same dependencies as in the statement of Lemma 5.13. Since by item (3) in the statement of Lemma 5.13 $T(\varepsilon_1) \to \infty$ as $\varepsilon_1 \to 0_+$, we can make $\tilde{\varepsilon}_1$ smaller if needed so as to ensure that

$$\min_{0<\varepsilon_1\le\tilde{\varepsilon}_1} T(\varepsilon_1) \ge \min_{0<h\le h_0} \check{R}(h). \tag{5.4}$$

Note that $T(\varepsilon_1)$ and $\check{R}(h)$ attain their minimum values above in view of the fact that $T(\varepsilon_1)$ is continuous on $(0, \tilde{\varepsilon}_1]$ by Lemma 5.13 and $\check{R}(h)$ is continuous on $(0, h_0]$ by Lemma 5.11, while $T(\varepsilon_1) \to \infty$ as $\varepsilon_1 \to 0_+$ and $\check{R}(h) \to \infty$ as $h \to 0_+$ by item (3) in the statement of Lemmas 5.13 and 5.12, respectively. We now define the function h of the variable ε_1 on the interval $(0, \tilde{\varepsilon}_1]$ by setting, for each $0 < \varepsilon_1 \le \tilde{\varepsilon}_1$,

$$h(\varepsilon_1) := \min\{h \in (0, h_0] \ : \ \check{R}(h) = T(\varepsilon_1)\}. \tag{5.5}$$

Note that in view of equation (5.4), again since $\check{R}$ is continuous and $\check{R}(h) \to \infty$ as $h \to 0_+$, using the intermediate value theorem, the set of which we are taking the minimum above will be non-empty, and so this function is well defined. It also follows that the set $\{h \in (0, h_0] \ : \ \check{R}(h) = T(\varepsilon_1)\}$ has a positive infimum which, by the continuity of $\check{R}(h)$, is attained and is thus in fact a minimum, and moreover,

$$\check{R}(h(\varepsilon_1)) = T(\varepsilon_1). \tag{5.6}$$

The right-hand side of equation (5.4) depends only on h_0. Hence, the upper bound $\tilde{\varepsilon}_1$ and the function T from the statement of Lemma 5.13 will still only depend on h_0 and r_0, and the dependencies in the statement of that lemma thus remain unaltered. Lastly, we observe that this function h above is then determined by h_0 and r_0 in view of equation (5.5).

By item (1) of the Target Lemma (Lemma 5.13), for each $0 < \varepsilon_1 \le \tilde{\varepsilon}_1$,

$$R^{\mathrm{int}}_{(\tilde{V}_{2h},0)}(\tilde{V}_{2h} \setminus \hat{N}) \ge T(\varepsilon_1). \tag{5.7}$$

However, by equation (5.6), in view of the definition of $\check{R}(h)$ given before Lemma 5.11,

$$R^{\mathrm{ext}}_{(\tilde{V}_{2h(\varepsilon_1)},0)} \tilde{V}_{h(\varepsilon_1)} = T(\varepsilon_1). \tag{5.8}$$

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Thus, using item (1) of Corollary 2.4, if we set $X = \tilde{V}_{h(\varepsilon_1)}$, $Y = \tilde{V}_{2h(\varepsilon_1)} \setminus \hat{N}$, we have $\overline{\tilde{V}}_{h(\varepsilon_1)} \subset \tilde{V}_{2h(\varepsilon_1)} \setminus \hat{N}$ (this latter set clearly being closed) and so we obtain item (1).

Again by item (3) of the statement of Lemma 5.13, $T(\varepsilon_1) \to \infty$ as $\varepsilon_1 \to 0_+$. Lemma 5.12, together with the fact that $\check{R}$ is continuous in view of Lemma 5.11 then ensure that the hypotheses of Lemma 5.14 are met. Equation (5.5) and Lemma 5.14 then imply that $h(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0_+$ as desired, which proves item (2). □

As we remarked earlier, the Fitting Lemma will be essential for proving Phase II. Basically, item (1) of the statement says that, for each $0 < \varepsilon_1 \le \tilde{\varepsilon}_1$, the domain $\tilde{V}_{h(\varepsilon_1)}$ 'fits' inside $\tilde{V}_{2h(\varepsilon_1)} \setminus \hat{N}$, which will allow us to apply the Polynomial Implementation Lemma, but we will need to correct the error from the Phase I immediately prior to this. However, item (2) of the statement says that $h(\varepsilon_1) \to 0_+$ as $\varepsilon_1 \to 0_+$ which, as we will see, is the key to controlling the loss of domain incurred by the correction of the error from the Phase I immediately prior to this and which is the purpose of Phase II.

Observe that getting $\tilde{V}_h$ to fit inside $\tilde{V}_{2h} \setminus \hat{N}$ as above is easier if the value of h is *large*, while ensuring the loss of domain is small requires a value of h which is *small*. Indeed it is the tension between these competing requirements for h which makes proving Phase II so delicate and why the Target and Fitting Lemmas are so essential. Before we move on to the statement and proof of Phase II, we state one last technical lemma that will be of use to us later.

LEMMA 5.16. *Let $D \subset \mathbb{C}$ be a bounded simply connected domain and let $z_0 \in D$. Then for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that if X is any set containing z_0 and contained in D such that $R^{\mathrm{int}}_{(D,z_0)} X > R_\varepsilon$, then $d(\partial X, \partial D) < \varepsilon$.*

Proof. Define $D_\varepsilon = \{z \in D : d(z, \partial D) \ge \varepsilon/2\}$. Since D is bounded, D_ε is a compact subset of D and we can find $R_\varepsilon > 0$ such that $D_\varepsilon \subset \Delta_D(z_0, R_\varepsilon)$. Then, if X is any set containing z_0 and contained in D such that $R^{\mathrm{int}}_{(D,z_0)} X \ge R_\varepsilon$, by the definition of internal hyperbolic radius (Definition 2.2), for every $z \in D \setminus X$, we have $\rho_D(z_0, z) \ge R_\varepsilon$. It then follows that for every $z \in \partial X \cap D = \partial(D \setminus X) \cap D$, we also have $\rho_D(z_0, z) \ge R_\varepsilon$ from which it follows that $z \notin D_\varepsilon$. Since $\partial X = (\partial X \cap D) \cup (\partial X \cap \partial D)$, it follows that $\partial X \subset \{z \in \overline{D} : d(z, \partial D) < \varepsilon/2\}$, and from the compactness of the bounded set $\partial X \subset \overline{D}$, we get $d(\partial X, \partial D) \le \varepsilon/2 < \varepsilon$, as desired. □

5.2. *Statement and proof of Phase II.* Recall the scaling factor $\kappa \ge 1$ and upper bound h_0 on the value of the Green's function from the statement of Lemma 5.10. Recall also the bounds $0 < r_0 < R_0 \le \pi/2$ for R and that the upper bound of $\pi/2$ was chosen in the discussion before the Target Lemma (Lemma 5.13) so that U_R as well as its image under any conformal mapping whose domain of definition contains U is star-shaped (about the appropriate image of 0).

LEMMA 5.17. (Phase II) *Let κ, h_0, r_0, and R_0 be fixed as above. Then there exist an upper bound $\tilde{\varepsilon}_1 > 0$ and a function $\delta : (0, \tilde{\varepsilon}_1] \to (0, (r_0/4))$, with $\delta(x) \to 0$ as $x \to 0_+$, both of which are determined by h_0, r_0, and R_0 such that, for all $\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$, there exists an upper bound $\tilde{\varepsilon}_2 > 0$, determined by ε_1, h_0, and r_0, R_0, such that, for all $\varepsilon_2 \in (0, \tilde{\varepsilon}_2]$,* *all $R \in [r_0, R_0]$, and all functions $\mathcal{E}$ univalent on U_R with $\mathcal{E}(0) = 0$ and $\rho_U(\mathcal{E}(z), z) < \varepsilon_1$ for $z \in U_R$, there exists a $(17 + \kappa)$-bounded composition $\mathbf{Q}$ of finitely many quadratic polynomials which depend on κ, ε_1, ε_2, R, h_0, r_0, R_0, and $\mathcal{E}$ such that:*

(i) $\mathbf{Q}$ *is univalent on a neighborhood of* $\overline{U}_{R-\delta(\varepsilon_1)}$*;*

(ii) *for all $z \in \overline{U}_{R-\delta(\varepsilon_1)}$, we have*

$$\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) < \varepsilon_2;$$

(iii) $\mathbf{Q}(0) = 0$.

Because we will be using the Polynomial Implementation Lemma repeatedly to construct our polynomial composition, we need to interpolate functions outside of $\mathcal{K}$, the filled Julia set for P. Indeed, as we saw in the Polynomial Implementation Lemma (Lemma 3.9), the solutions to the Beltrami equation converge to the identity precisely because the supports of the Beltrami data become small in measure. However, $\mathcal{E}$ is only defined on a subset of U and hence we will need to map a suitable subset of U on which $\mathcal{E}$ is defined to a domain which contains $\mathcal{K}$, and correct the conjugated error using the Polynomial Implementation Lemma. The trick to doing this is that we choose our subset of U such that the mapping to blow this subset up to U can be expressed as a high iterate of a map which is defined on the whole of the Green's domain V_h, not just on this subset. This will allow us to interpolate outside $\mathcal{K}$. Further, we will then use the Polynomial Implementation Lemma twice more to 'undo' the conjugating map and its inverse.

The two key considerations in the proof are controlling loss of domain (which is measured by the function δ in the statement above), and showing that the error in our polynomial approximation to the function $\mathcal{E}$ (measured by the quantity ε_2 above) is mild and, in particular, can be made as small as desired. In controlling the loss of domain, one main difficulty will arise in converting between the hyperbolic metrics of different domains, U and V_{2h}, and we will deal with this by means of the convergence of the pointed domains $(V_{2h}, 0)$ to $(U, 0)$ in the Carathéodory topology as h tends to zero. One last thing is worth mentioning: since this result involves many functions and quantities which depend on one another, the interested reader is encouraged to make use of the dependency tables in the appendices to help keep track of them.

Proof. Ideal loss of domain: The techniques for controlling loss of domain will be the Fitting Lemma, and again the fact that $(V_{2h}, 0) \to (U, 0)$ in the Carathéodory topology (Lemma 5.9) as h tends to zero combined with the Target Lemma. As stated above, we will apply the Polynomial Implementation Lemma to our conjugated version of $\mathcal{E}$, which will be $\varphi_{2h} \circ \mathcal{E} \circ \varphi_{2h}^{-1}$ in what we call the 'During' portion of the error calculations. To approximate $\mathcal{E}$ itself rather than this conjugated version, we then wish to 'cancel' the conjugacy, so 'During' is bookended by 'Up' and 'Down' portions, in which we apply the Polynomial Implementation Lemma to get polynomial compositions which are arbitrarily close to φ_{2h} and φ_{2h}^{-1}, respectively, on suitable domains.

We begin the proof of Phase II by considering 'Ideal Loss of Domain'. In creating polynomial approximations using Phase I (Lemma 4.8), errors will be created which will

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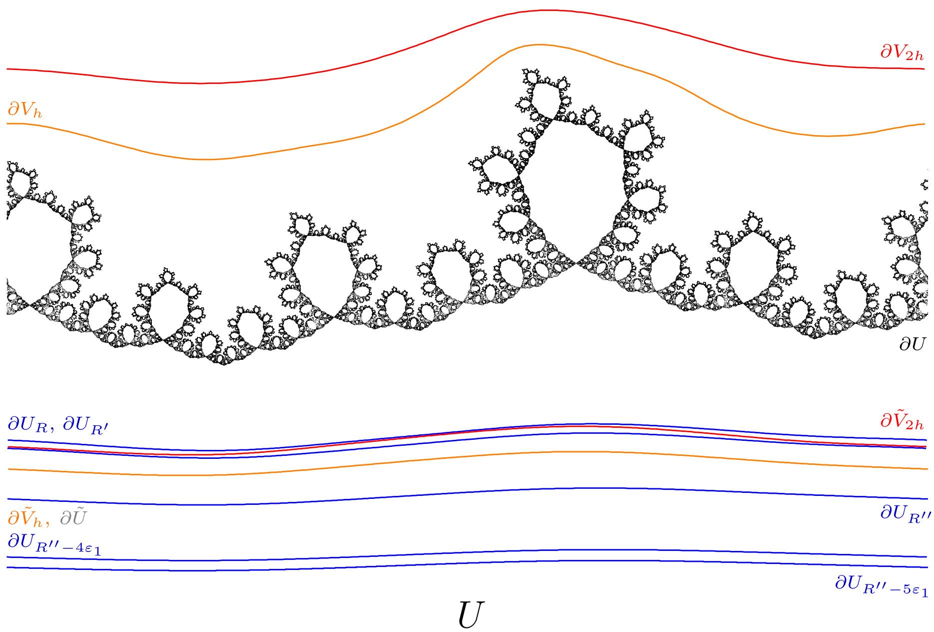

FIGURE 5. The setup for Phase II in rotated logarithmic coordinates.

have an impact on the loss of domain that occurs. We first describe the loss of domain that is forced on us before this error is taken into account. During what follows, the reader might find it helpful to consult Figure 5, where most of the relevant domains are shown in rotated logarithmic coordinates where the up direction corresponds to increasing distance from the fixed point for P at 0.

We first turn our attention to controlling loss of domain. Let $R \in [r_0, R_0]$ be arbitrary as in the statement. Note that because we need to consider uniform convergence in R to define the function δ which measures loss of domain in the statement above, we consider R for now as varying over the whole of the interval $[r_0, R_0]$. However, later we will fix an (arbitrary) value of R from this interval at the start of the 'up' portion of the proof). Recall the discussion before the statement of Lemma 5.10, where we let $\psi : U \to \mathbb{D}$ be the unique normalized Riemann map from U to $\mathbb{D}$ satisfying $\psi(0) = 0$, $\psi'(0) > 0$. Recalling the upper bound h_0 for the value of the Green's function G for P, for $h \in (0, h_0]$ arbitrary, let $\psi_{2h} : V_{2h} \to \mathbb{D}$ be the unique normalized Riemann map from V_{2h} to $\mathbb{D}$ satisfying $\psi_{2h}(0) = 0$, $\psi'_{2h}(0) > 0$. Recall also that we had $\tilde{R} = R^{\mathrm{int}}_{(V_{2h},0)} U_R$, $\tilde{V}_{2h} = \Delta_{V_{2h}}(0, \tilde{R})$ and $\varphi_{2h} : \tilde{V}_{2h} \to V_{2h}$, which was the unique conformal map from $\tilde{V}_{2h}$ to V_{2h} normalized so that $\varphi_{2h}(0) = 0$ and $\varphi'_{2h}(0) > 0$. Now define $R' = R^{\mathrm{int}}_{(U,0)} \tilde{V}_{2h}$ and note that the value of this quantity is completely determined by those of R and h. We prove the following claim.

CLAIM 5.18. *$R - R' \to 0$ uniformly on $[r_0, R_0]$ as $h \to 0_+$.*

Proof. By Lemma 5.9, $(V_{2h}, 0) \to (U, 0)$ in the Carathéodory topology as $h \to 0_+$ and thus ψ_{2h} converges locally uniformly to ψ on U in view of Theorem A.9.

Let $\{h_n\}_{n=1}^{\infty}$ be an arbitrary sequence of positive numbers such that $h_n \to 0$ as $n \to \infty$. By the definitions of $\tilde{V}_{2h}$ and R', and Lemma 2.5, there exists $w_{h_n,1} \in \partial \tilde{V}_{2h_n} \cap \partial U_R$ and $w_{h_n,2} \in \partial \tilde{V}_{2h_n} \cap \partial U_{R'}$. Let $0 < s, s_n', s_n'' < 1$ be such that $\psi(\partial U_R) = \mathrm{C}(0, s)$, $\psi(\partial U_{R'}) = \mathrm{C}(0, s_n')$, and $\psi_{2h_n}(\partial \tilde{V}_{2h_n}) = \mathrm{C}(0, s_n'')$.

Let $\varepsilon_0 > 0$. By the local uniform convergence of ψ_{2h_n} to ψ on U, there exists n_0, such that for all $n \geq n_0$,

$$|\psi_{2h_n}(z) - \psi(z)| < \frac{\varepsilon_0}{2}, \quad z \in \overline{U}_{R_0}.$$

Thus, for any $n \geq n_0$ and any $R \in [r_0, R_0]$,

$$|s - s_n''| = \Big||\psi(w_{h_n,1})| - |\psi_{2h_n}(w_{h_n,1})|\Big| \leq |\psi(w_{h_n,1}) - \psi_{2h_n}(w_{h_n,1})| < \frac{\varepsilon_0}{2},$$
$$|s_n'' - s_n'| = ||\psi_{2h_n}(w_{h_n,2})| - |\psi(w_{h_n,2})|| \leq |\psi_{2h_n}(w_{h_n,2}) - \psi(w_{h_n,2})| < \frac{\varepsilon_0}{2}$$

whence

$$|s - s_n'| < \varepsilon_0.$$

Since the sequence $\{h_n\}_{n=1}^{\infty}$ was arbitrary, the desired uniform convergence then follows on applying the conformal invariance of the hyperbolic metric under ψ^{-1}. □

Now define the *Internal Siegel disc*, $\tilde{U} := \varphi_{2h}^{-1}(U)$, and set $R'' = R^{\mathrm{int}}_{(U,0)}\tilde{U}$, noting again that the value of this quantity is completely determined by those of R and h. Next, we show the following claim.

CLAIM 5.19. *$R - R'' \to 0$ uniformly on $[r_0, R_0]$ as $h \to 0_+$.*

Proof. First, we show $R^{\mathrm{int}}_{(V_{2h},0)}U \to \infty$ as $h \to 0_+$ (note that this convergence will be trivially uniform with respect to R on $[r_0, R_0]$ as there is no dependence on R). Fix $R_1 > 0$ and set $X := U_{R_1}$, $Y := U_{R_1+1}$ so that $\psi(X) = \Delta_{\mathbb{D}}(0, R_1)$, $\psi(Y) = \Delta_{\mathbb{D}}(0, R_1 + 1)$. As $\overline{\Delta}_{\mathbb{D}}(0, R_1) \subset \Delta_{\mathbb{D}}(0, R_1 + 1)$, if we let $\eta = d(\partial\Delta_{\mathbb{D}}(0, R_1), \partial\Delta_{\mathbb{D}}(0, R_1 + 1))$, then $\eta > 0$. Now let $z \in \partial Y$ and $w \in \Delta_{\mathbb{D}}(0, R_1)$. We have that $(V_{2h}, 0) \to (U, 0)$ as $h \to 0_+$ in view of Lemma 5.9, so, by Theorem A.9, we again have that ψ_{2h} converges to ψ uniformly on compact subsets of U in the sense given in Definition 5.1. Then, for all h sufficiently small, we have

$$\begin{aligned} |(\psi_{2h}(z) - w) - (\psi(z) - w)| &= |\psi_{2h}(z) - \psi(z)| \\ &< \eta \\ &\leq |\psi(z) - w|. \end{aligned}$$

Thus, by Rouché's theorem, since the convergence is uniform and $w \in \Delta_{\mathbb{D}}(0, R_1)$ was arbitrary, $\Delta_{\mathbb{D}}(0, R_1) \subset \psi_{2h}(Y)$. Then $\psi_{2h}^{-1}(\Delta_{\mathbb{D}}(0, R_1)) \subset Y$, so $R^{\mathrm{int}}_{(V_{2h},0)}Y \geq R_1$. We also have that $Y \subset U$ so $R^{\mathrm{int}}_{(V_{2h},0)}U \geq R^{\mathrm{int}}_{(V_{2h},0)}Y$, and thus $R^{\mathrm{int}}_{(V_{2h},0)}U \geq R_1$. Since R_1 was arbitrary, we do indeed have that $R^{\mathrm{int}}_{(V_{2h},0)}U \to \infty$ as $h \to 0_+$.

For a constant $c > 0$ and a set $X \subset \mathbb{C}$, define the scaled set $cX := \{z \in \mathbb{C} : z = cw$ for some $w \in X\}$. Let $0 < r_{2h} < 1$ be such that $\psi_{2h}(\tilde{V}_{2h}) = \mathrm{D}(0, r_{2h})$. The quantity r_{2h} then depends on r_0, R_0, R, and h (and thus ultimately on ε_1, R, h_0, r_0, and R_0 once

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we make our determination of the function $h = h(\varepsilon_1)$ immediately before equation (5.11) below) and clearly $1/r_{2h}\psi_{2h}(\tilde{V}_{2h}) = \mathbb{D}$.

By conformal invariance,

$$R^{\text{int}}_{(\frac{1}{r_{2h}}\psi_{2h}(\tilde{V}_{2h}),0)}\bigg(\frac{1}{r_{2h}}\psi_{2h}(\tilde{U})\bigg) = R^{\text{int}}_{(\psi_{2h}(\tilde{V}_{2h}),0)}(\psi_{2h}(\tilde{U})) = R^{\text{int}}_{(\tilde{V}_{2h},0)}\tilde{U} = R^{\text{int}}_{(V_{2h},0)}U.$$

As $R^{\text{int}}_{(V_{2h},0)}U \to \infty$ as $h \to 0_+$ from above, it follows that, uniformly on $[r_0, R_0]$,

$$R^{\text{int}}_{(\frac{1}{r_{2h}}\psi_{2h}(\tilde{V}_{2h}),0)}\bigg(\frac{1}{r_{2h}}\psi_{2h}(\tilde{U})\bigg) = R^{\text{int}}_{(V_{2h},0)}U \to \infty \quad \text{as } h \to 0_+.$$

We can then apply Lemma 5.16 to conclude using $1/r_{2h}\psi_{2h}(\tilde{V}_{2h}) = \mathbb{D}$ that on letting $h \to 0_+$, we have

$$d\bigg(\partial\bigg(\frac{1}{r_{2h}}\psi_{2h}(\tilde{U})\bigg), \partial\bigg(\frac{1}{r_{2h}}\psi_{2h}(\tilde{V}_{2h})\bigg)\bigg) = d\bigg(\partial\bigg(\frac{1}{r_{2h}}\psi_{2h}(\tilde{U})\bigg), \partial\mathbb{D}\bigg) \to 0,$$

where again the convergence is uniform on $[r_0, R_0]$. Thus, scaling by r_{2h}, we have, again uniformly on $[r_0, R_0]$,

$$d(\partial(\psi_{2h}(\tilde{U})), \partial(\psi_{2h}(\tilde{V}_{2h}))) \to 0_+ \quad \text{as } h \to 0_+. \tag{5.9}$$

We observe that, since r_{2h} depends on R, this is the first time when the convergence being uniform on $[r_0, R_0]$ is not entirely trivial.

Further, using the Schwarz lemma for the hyperbolic metric [**CG93**, Theorems I.4.1 or I.4.2], we have that

$$\psi_{2h}(\tilde{U}) \subset \psi_{2h}(\tilde{V}_{2h}) \subset \psi_{2h}(U_R) \subset \psi_{2h}(U_{\pi/2}) \subset \psi_{2h}(\Delta_{V_{2h}}(0, (\pi/2))) = \Delta_{\mathbb{D}}(0, (\pi/2)), \tag{5.10}$$

(where we use the Schwarz Lemma for the hyperbolic metric for the last inclusion) which shows that both $\psi_{2h}(\tilde{U})$ and $\psi_{2h}(\tilde{V}_{2h})$ lie within (hyperbolic) distance $\pi/2$ of 0 within $\mathbb{D}$. Since ψ_{2h}^{-1} converges to ψ^{-1} uniformly on compact subsets of $\mathbb{D}$ by Theorem A.9, using equations (5.9) and (5.10), we have that $d(\partial\tilde{U}, \partial\tilde{V}_{2h}) \to 0_+$ uniformly on $[r_0, R_0]$ as $h \to 0_+$. Using Lemma A.4 and the fact that $\tilde{U} \subset \tilde{V}_{2h} \subset U_R \subset \Delta_U(0, (\pi/2))$, we see that we can say the same for distances with respect to the hyperbolic metric for U and that

$$\rho_U(\partial\tilde{U}, \partial\tilde{V}_{2h}) \to 0 \quad \text{as } h \to 0_+$$

uniformly on $[r_0, R_0]$.

Fix $\varepsilon_0 > 0$. Using Lemma 2.5, pick $z \in \partial\tilde{U}$ such that $\rho_U(0, z) = R''$. From above, for all h sufficiently small, we can pick $w_{2h} \in \partial\tilde{V}_{2h}$ such that

$$\rho_U(z, w_{2h}) < \frac{\varepsilon_0}{2}$$

(for any $R \in [r_0, R_0]$). Now let γ be the unique geodesic in U passing through $0, w_{2h}$. As γ must eventually leave U_R, let w be the first point on $\gamma \cap \partial U_R$ after we pass along γ from 0 to w_{2h}. Then $0, w_{2h}$, and w are on the same geodesic and w_{2h} is on the hyperbolic segment $\gamma_U[0, w]$ in U from 0 to w. We now have $\rho_U(0, w) = R$ and $\rho_U(0, w_{2h}) \geq R' = R^{\text{int}}_{(U,0)}\tilde{V}_{2h}$

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using Lemma 2.5. Then, since $w_{2h} \in \gamma_U[0, w]$, using our Claim 5.18, we have, uniformly on $[r_0, R_0]$,

$$\begin{aligned}\rho_U(w, w_{2h}) &= \rho_U(0, w) - \rho_U(0, w_{2h})\\ &\leq R - R'\\ &< \frac{\varepsilon_0}{2}\end{aligned}$$

for h sufficiently small. Further, we have

$$\begin{aligned}R - R'' &= \rho_U(0, w) - \rho_U(0, z)\\ &\leq \rho_U(z, w)\\ &\leq \rho_U(z, w_{2h}) + \rho_U(w_{2h}, w)\\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2}\\ &= \varepsilon_0\end{aligned}$$

for h sufficiently small, and thus $R - R'' \to 0$ as $h \to 0_+$ while this convergence is uniform on $[r_0, R_0]$ as desired (see Figure 6). □

By the Fitting Lemma (Lemma 5.15), there exist $\tilde{\varepsilon}_1 > 0$ and a function h defined on $(0, \tilde{\varepsilon}_1]$, both of which depend on h_0, r_0 which we fixed before the statement for which we have (by item (2) of the statement of this result) that

$$h(\varepsilon_1) \to 0_+ \quad \text{as } \varepsilon_1 \to 0_+. \tag{5.11}$$

From this, using using this function and Claim 5.19, we have that

$$R - R'' \to 0 \text{ as } \varepsilon_1 \to 0_+ \tag{5.12}$$

while this convergence is uniform on $[r_0, R_0]$.

To conclude this section of the proof, we make our final determination of the upper bound $\tilde{\varepsilon}_1$ and define the function δ on $(0, \tilde{\varepsilon}_1]$. Using the value of $\tilde{\varepsilon}_1$ above which comes from Lemma 5.15, for $\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$, set

$$\delta(\varepsilon_1) := \sup_{[r_0, R_0]} (R - R'') + 5\varepsilon_1 \tag{5.13}$$

(the justification for this definition of this function will be made clear later). Note that, in view of the above dependencies of the function h on h_0 and r_0, the function δ then depends on h_0 and the bounds r_0, R_0 for R (but not on R itself), all of which we regard as fixed in advance. It follows from equation (5.12) that $\delta(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0_+$ (we remark that this is the point where we require that the convergence above be uniform). We can then make $\tilde{\varepsilon}_1$ smaller if needed such that

$$\sup_{(0, \tilde{\varepsilon}_1]} \delta(\varepsilon_1) < \frac{r_0}{4}, \tag{5.14}$$

which ensures that $U_{R-\delta(\varepsilon_1)} \neq \emptyset$. Note from above that $\tilde{\varepsilon}_1$ will therefore also depend on h_0 and the bounds r_0, R_0 for R (in other words, we pick up an extra dependency on R_0 from the definition of the value $\delta(\varepsilon_1)$ in equation (5.13)), so that we now have the correct

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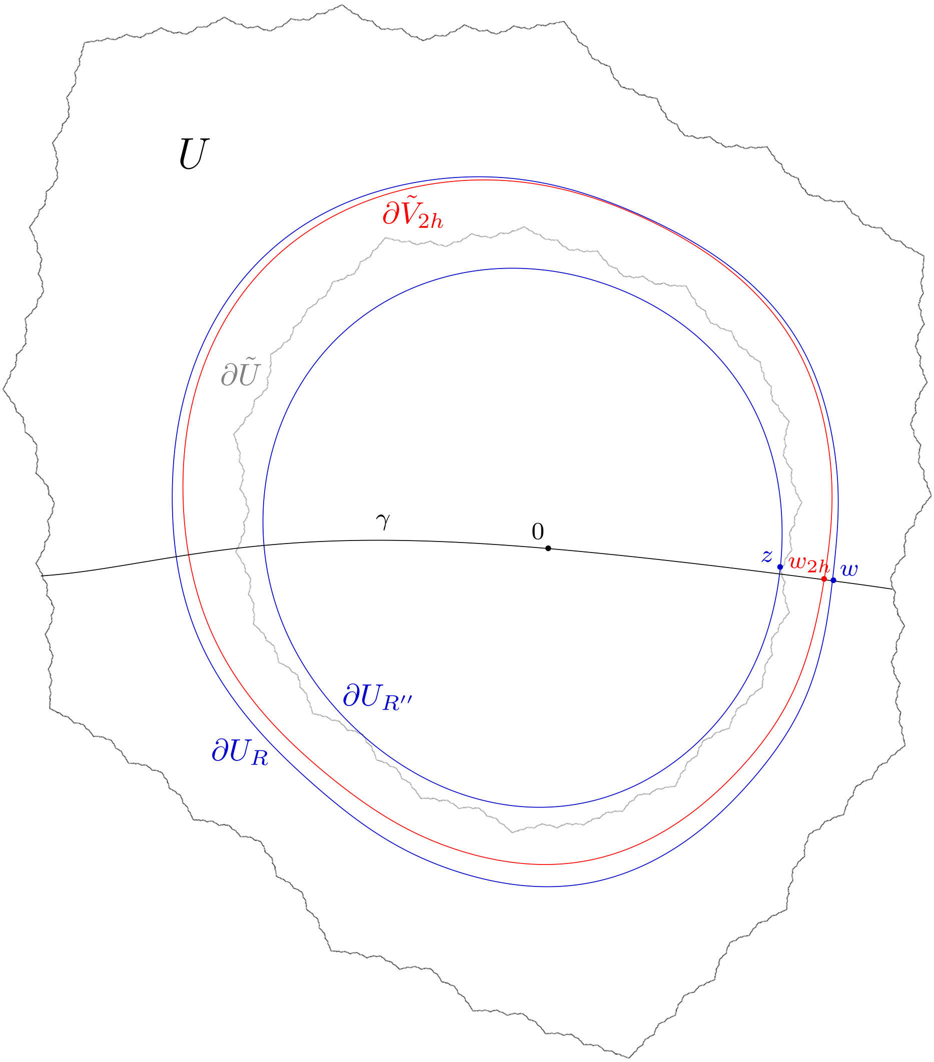

FIGURE 6. Showing $R - R'' \to 0$ as $h \to 0_+$.

dependencies for both the quantity $\tilde{\varepsilon}_1$ and the function $\delta : (0, \tilde{\varepsilon}_1] \mapsto (0, (r_0/4))$ as in the statement. Note in addition that this change in $\tilde{\varepsilon}_1$ may require us to redefine the function h above by restricting its domain of definition. Note that restricting $\tilde{\varepsilon}_1$ in this way will not violate equation (5.4) in the proof of the Fitting Lemma (Lemma 5.15) so that we can still define $h(\varepsilon_1)$ according to equation (5.5) in the proof and, in particular, equation (5.6) still holds. Note that what we are essentially doing here is defining new 'copies' of the functions $T(\varepsilon_1)$, $h(\varepsilon_1)$, $\delta(\varepsilon_1)$ with restricted domains, but the same values as the originals, and then relabeling them with the original names so that there is no danger of circular reasoning. It is also worth noting that one only needs to carry out this restriction once, after which equation (5.14) is then automatically satisfied. In addition, because the function δ depends on h_0, r_0, and R_0, the redefined function h now depends on h_0, r_0, and

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R_0 as in the statement. Lastly, note that, in particular, equation (5.14) implies that

$$U_{R-\delta} \supset U_{3r_0/4}. \tag{5.15}$$

Controlling error: 'Up': Now fix $\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$, $h = h(\varepsilon_1)$ using the function h introduced before equation (5.11) (and redefined later so as to satisfy equation (5.14)), and also fix $R \in [r_0, R_0]$ as in the statement. Recall from the discussion before Lemma 5.10 that we had $\tilde{R} = R^{\text{int}}_{(V_{2h},0)} U_R$, $\tilde{V}_{2h} = \Delta_{V_{2h}}(0, \tilde{R})$, and $\varphi_{2h} : \tilde{V}_{2h} \to V_{2h}$ which was the unique conformal map from $\tilde{V}_{2h}$ to V_{2h} normalized so that $\varphi_{2h}(0) = 0$ and $\varphi'_{2h}(0) > 0$. Recall also that ψ_{2h} is the unique normalized Riemann map which sends V_{2h} to $\mathbb{D}$. Since $\tilde{V}_{2h}$ is a hyperbolic disc about 0 in V_{2h}, it follows that, in the conformal coordinates of V_{2h}, φ_{2h} is then a dilation of $\tilde{V}_{2h}$. To estimate the error in approximating φ_{2h}, we wish to break this dilation into many smaller dilations, and apply the Polynomial Implementation Lemma (Lemma 3.9) so as to approximate each of these small dilations with a polynomial composition. The key idea here is that conformal dilations by small amounts can have larger domains of definition and, by dilating by a sufficiently small amount, we can ensure this domain of definition includes the filled Julia set and indeed all of the Green's domain V_h, which ultimately allows us to apply the Polynomial Implementation Lemma to approximate it to an arbitrarily high degree of accuracy.

As before, let $r_{2h} \in (0, 1)$ be such that $\psi_{2h}(\tilde{V}_{2h}) = \mathrm{D}(0, r_{2h})$ and recall that r_{2h} depends on ε_1, R, h_0, r_0, and R_0 (via h, R, and ψ_{2h}, noting that we can ignore the dependence of ψ_{2h} on κ in view of conformal invariance). Pick $s \in (0, 1)$ so that $\psi_{2h}(\overline{V}_h) \subset \mathrm{D}(0, s)$. Note that s depends immediately on h and ψ_{2h}, but does not depend on κ by conformal invariance, so that s also depends ultimately on ε_1, R, h_0, r_0, and R_0. Note also that we must have $s > r_{2h}$ since $V_h \supset U \supset \overline{U}_R \supset \tilde{V}_{2h}$. Now fix N such that

$$s \sqrt[N]{\frac{1}{r_{2h}}} < \sqrt{s}, \tag{5.16}$$

and note that this choice of N will depend immediately on s and r_{2h}, and thus ultimately on ε_1, R, h_0, r_0, R_0, from above.

This choice will ensure that our conformal dilations in the composition do not distort ∂V_h so much so that we no longer have a conformal annulus for interpolation when we apply the Polynomial Implementation Lemma.

Next, define on $\psi_{2h}^{-1}(\mathrm{D}(0, s))$ the map

$$g(z) = \psi_{2h}^{-1}\left(\sqrt[N]{\frac{1}{r_{2h}}}\psi_{2h}(z)\right) \tag{5.17}$$

and note in particular that g is defined and, in addition, analytic and injective on a neighborhood of $\overline{V_h}$ as $\psi_{2h}(\overline{V}_h) \subset \mathrm{D}(0, s)$ by our choice of s. Further, since ψ_{2h} fixes 0, we have $g(0) = 0$ and, given our choice of N in equation (5.16), we have

$$\overline{g(V_h)} \subset V_{2h}. \tag{5.18}$$

By conformal invariance or simply because g corresponds to a dilation by $r_{2h}^{-1/N}$ in the conformal coordinates of V_{2h}, recalling that we set $\tilde{V}_h := \varphi_{2h}^{-1}(V_h)$ (immediately before

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Lemma 5.11), we must then have that $\psi_{2h}(\tilde{U}) \subset \psi_{2h}(\tilde{V}_h) \subset \Delta_{\mathbb{D}}(0, r_{2h}s)$. Again, since g corresponds to a dilation by $r_{2h}^{-1/N}$ in the conformal coordinates of V_{2h}, the compositions $g^{\circ j}$, $0 \leq j \leq N$ (where $g^{\circ\, 0}$ is the identity) are all then defined on $\tilde{U}$ and, in particular, we have $g^{\circ N} = \varphi_{2h}$ on $\tilde{U}$. We observe that the functions $g^{\circ j}$ then form (part of) a Löwner chain on $\tilde{U}$ in a sense similar to that given in [**CDMG10**] (although these authors were working on the unit disc).

Since $R'' < R \leq R_0 \leq \pi/2$ is bounded above, the external hyperbolic radius about 0 of $U_{R''-\varepsilon_1}$ inside $U_{R''}$ (with respect to the hyperbolic metric of this slightly larger domain) can be uniformly bounded above in terms of ε_1 and the upper bound $R_0 \leq \pi/2$ for R. By the Schwarz lemma for the hyperbolic metric (e.g. [**CG93**, Theorems I.4.1 or I.4.2]), the same is true for the external hyperbolic radius about 0 of $U_{R''-\varepsilon_1}$ inside the larger (than $U_{R''}$) domain $\tilde{U}$. By conformal invariance under φ_{2h}, the same is also true for the external hyperbolic radius about 0 of $\varphi_{2h}(U_{R''-\varepsilon_1})$ inside U. We can then find an upper bound R_2 for this external hyperbolic radius which depends directly on ε_1 and the upper bound R_0 on R, and thus ultimately on ε_1, h_0, r_0, and R_0. As a result, we have

$$R^{\mathrm{ext}}_{(U,0)}\varphi_{2h}(U_{R''-3\varepsilon_1}) < R^{\mathrm{ext}}_{(U,0)}\varphi_{2h}(U_{R''-2\varepsilon_1}) < R^{\mathrm{ext}}_{(U,0)}\varphi_{2h}(U_{R''-\varepsilon_1}) \leq R_2. \tag{5.19}$$

Note that in view of equations (5.13), (5.15), $U_{R''-3\varepsilon_1} \supset U_{3r_0/4}$ and all the sets in the above are thus non-empty. Note also that this upper bound is in particular independent of N and h (recall that, in fact, we have h is a function of ε_1 in view of equation (5.5) and the discussion after equation (5.14)). We also note that, at this point, we do not actually require that R_2 be independent of h. However, we will need this later when we turn to giving upper bound $\tilde{\varepsilon}_2$ for ε_2 which has the correct dependencies as listed in the statement. Finally, we note that this upper bound is for the set $U_{R''-\varepsilon_1}$, while all we will need in this section of the proof is a bound on the slightly smaller sets $U_{R''-2\varepsilon_1}$, $U_{R''-3\varepsilon_1}$. However, we will need the bound on the larger set when it comes to the 'during' part of the proof later on.

Since, from above, the compositions $g^{\circ j}$, $0 \leq j \leq N$ are all defined on $\tilde{U} \supset U_{R''-3\varepsilon_1}$, we can now set $B := g^{\circ N}(U_{R''-3\varepsilon_1})$ and note that, since $g^{\circ N} = \varphi_{2h}$ on $U_{R''-3\varepsilon_1}$, it follows from equation (5.19) that $g^{\circ N}$ maps $U_{R''-3\varepsilon_1}$ inside U_{R_2}, which is a relatively compact subset of U. However, recalling the normalized Riemann map ψ from U to $\mathbb{D}$ (which was introduced before the start of Lemma 5.10), since $R'' \leq R_0 \leq \pi/2$, by Lemma A.6, the set $\psi_{2h}(U_{R''-3\varepsilon_1}) = \psi_{2h} \circ \psi^{-1}(\Delta_{\mathbb{D}}(0, R'' - 3\varepsilon_1))$ is star-shaped with respect to 0. Since g corresponds to a dilation by $r_{2h}^{-1/N} > 1$ in the conformal coordinates of V_{2h}, it therefore follows that the sets $g^{\circ j}(U_{R''-3\varepsilon_1})$, $0 \leq j \leq N$ (which from above are well defined) are increasing in j and therefore all contained in B. Thus, any estimate which holds on B will also automatically hold on these sets also.

Now set $A := U_{R_2+1} \supset B$ (we remark that the 'extra' 1 here is due to the fact that $\varphi_{2h} = g^{\circ N}$ is a composition of g with itself many times and each of these compositions with g will be approximated so that we need to be able to allow for the total error which arises—see the proof of Claim 5.20 below for details). Since the function g is defined on a neighborhood of $\overline{V}_h \supset U$ and this containment does not change if we increase the value of N in view of equation (5.16), it follows that the functions $\psi_{2h}^{-1}(\sqrt[N]{(1/r_{2h})}\psi_{2h}(z))$ clearly converge to the identity locally uniformly on U as $N \to \infty$ (and, in particular, for all sufficiently large N, are defined on any relatively compact subset of U and map it into another relatively compact subset).

Using a similar argument as in Lemma 3.7 and also in Step 4 of the proof of Phase I (Lemma 4.8), based on the hyperbolic density σ_U for the hyperbolic metric of U and that this density is uniformly continuous and bounded below away from 0 on any relatively compact subset of U, it follows that the functions $\psi_{2h}^{-1}(\sqrt[N]{1/r_{2h}}\psi_{2h}(z))$ converge uniformly to the identity while their hyperbolic derivatives converge uniformly to 1 on any relatively compact subset of U. Since A depends on R_2, which from above does not depend on N, it follows that, if we fix a constant $K_1 = \frac{3}{2}$, we may therefore make N larger if needed (without invalidating equation (5.16)) so that, if $\hat{A}$ is a 1-hyperbolic neighborhood of A in the hyperbolic metric of U (which implies that $\hat{A} = U_{R_2+2}$), then g still maps $\hat{A}$ into a relatively compact subset of U and we have

$$\|g^{\natural}\|_{\hat{A}} \leq K_1, \tag{5.20}$$

where, as usual, we are taking our hyperbolic derivatives with respect to the hyperbolic metric of U. Our new choice of N will depend directly on R_2 and g in addition to the old dependencies on ε_1, R, h_0, r_0, R from the discussion after equation (5.16) and so, using equation (5.17) and the dependencies of R_2 from equation (5.19), ultimately N depends on κ, ε_1, h_0, r_0, R_0, and R (this last also being via r_{2h} which we are not allowed to alter at this stage). However, since we are estimating a hyperbolic derivative here, we can eliminate the dependence on κ so that N ultimately depends on ε_1, h_0, r_0, R_0, and R (which is the same as for the original version). Note also that the function g defined in equation (5.17) is being redefined here as we are changing N (but not r_{2h} or s), but we could have introduced the requirement in equation (5.20) as part of the definition of N and thus of g by introducing the bound R_2 (which depends only on ε_1, h_0, r_0, and R_0, but not on g) earlier, so there is no danger of circular reasoning here. One also easily checks that all the properties of g listed above still remain true for the new version. This new version of g will then depend on the six quantities κ, ε_1, h_0, r_0, R_0, and R (and, in particular, we cannot eliminate the dependence on κ since the domain of g depends on κ via ψ_{2h}).

Note also that by Lemma 2.8, $\hat{A}$ is hyperbolically convex which will be useful (though not essential) when we come to apply the hyperbolic M-L estimates (Lemma 2.9) later on. Also important to note is that N is fixed from now on which means that we can choose our subsequent approximations using the Polynomial Implementation Lemma with this N in mind.

Set $\tilde{\varepsilon}_2 := 1$ and let $0 < \varepsilon_2 \leq \tilde{\varepsilon}_2$ be arbitrary and fixed (note that this upper bound $\tilde{\varepsilon}_2$ is universal, but we will be making further restrictions later in the proof to deduce the upper bound $\tilde{\varepsilon}_2$ with the same dependencies as in the statement). Define $\gamma := \partial V_h$ and $\Gamma := \partial V_{2h}$ (with positive orientations as Jordan curves with respect to the conformal annulus bounded by ∂V_h and ∂V_{2h}), and note that, since g is injective and analytic on a neighborhood of $\overline{V_h}$ while $\overline{g(V_h)} \subset V_{2h}$ from equation (5.18), we must have that $g(\gamma)$ lies inside Γ (so that (g, Id) is an admissible pair on (γ, Γ) in the sense given in Definition 3.3).

Now set ε in the statement of the Polynomial Implementation Lemma (Lemma 3.9) to be $\varepsilon_2/3(2K_1)^{N-1}K_2K_3$, where K_2 and K_3 are bounds on hyperbolic derivatives which will be chosen later. For now, we just assume that $K_i > 1$ for $i = 2, 3$ (these are just constants,

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and we can always choose a larger constant). Note that we have $\varepsilon_2/3(2K_1)^{N-1} < 1$, which implies that $\varepsilon < 1$. Further, note that $\varepsilon < \varepsilon_2$. Now, since $g(0) = 0$, $g(\gamma)$ lies inside Γ and we have the estimate in equation (5.20) on the hyperbolic derivative of g, we can apply the Polynomial Implementation Lemma (Lemma 3.9), with $\Omega = V_h$, $\Omega' = V_{2h}$, $\gamma = \Gamma_h$, $\Gamma = \Gamma_{2h}$, $f = g$, $A = U_{R_2+1}$, $\delta = 1$, $M = K_1$, and $\varepsilon = \varepsilon_2/(3(2K_1)^{N-1}K_2K_3)$ as above to g to get $n_{k_0} > 0$, and a $(17+\kappa)$-bounded finite sequence of quadratic polynomials $\{Q_m\}_{m=1}^{n_{k_0}}$ such that the composition of these polynomials, $Q_{n_{k_0}}$, is univalent on A and satisfies

$$\rho_U(Q_{n_{k_0}}(z), g(z)) < \frac{\varepsilon_2}{3(2K_1)^{N-1}K_2K_3} = \varepsilon, \quad z \in A, \tag{5.21}$$

$$\|Q^\natural_{n_{k_0}}\|_A \le K_1\bigg(1 + \frac{\varepsilon_2}{3(2K_1)^{N-1}K_2K_3}\bigg), \tag{5.22}$$

$$Q_{n_{k_0}}(0) = 0. \tag{5.23}$$

Note that by Lemma 3.9, since $M = K_1 = \frac{3}{2}$ and $\delta = 1$, n_{k_0}, and $Q_{n_{k_0}}$ depend directly on κ, K_1, ε, R_2, g, and h, one can check that n_{k_0} and $Q_{n_{k_0}}$ eventually depend on κ, ε_1, ε_2, K_2, K_3, R, h_0, r_0, and R_0 (and then ultimately on κ, ε_1, ε_2, R, h_0, r_0, R_0, and $\mathcal{E}$ via the ultimate dependencies of K_2 and K_3 in the 'during' and 'down' sections of the proof below, the dependence on $\mathcal{E}$ coming from K_2).

For $1 \le j \le N$, define $Q_{jn_{k_0}} := Q^{\circ j}_{n_{k_0}}$. We prove the following claim, which will allow us to control the error in the 'Up' portion of Phase II.

CLAIM 5.20. *For each $1 \le j \le N$, we have:*

(i) $\rho_U(Q_{jn_{k_0}}(z), g^{\circ j}(z)) < \dfrac{\varepsilon_2}{3(2K_1)^{N-j}K_2K_3} < 1, \quad z \in U_{R''-3\varepsilon_1}$;

(ii) $Q_{jn_{k_0}}(z) \in A, \quad z \in U_{R''-3\varepsilon_1}$;

(iii) $Q_{jn_{k_0}}$ *is univalent on* $U_{R''-3\varepsilon_1}$.

Proof. For the base case $j = 1$, recall that, from the discussion before the definition of R_2 given in equation (5.19), we have that the external hyperbolic radius of $U_{R''-3\varepsilon_1} \subset U_{R''-2\varepsilon_1}$ inside $\tilde{U}$ is bounded above by R_2. Since $\tilde{U} \subset U$, by the Schwarz lemma for the hyperbolic metric (e.g. [**CG93**, Theorems I.4.1 or I.4.2]), we have that $U_{R''-3\varepsilon_1} \subset U_{R_2} \subset U_{R_2+1} = A$. Part (i) then follows from equation (5.21).

For part (ii), recall that the sets $g^{\circ j}(U_{R''-3\varepsilon_1})$, $0 \le j \le N$, are increasing in j and, in view of equation (5.19), therefore, all contained in $B = g^{\circ N}(U_{R''-3\varepsilon_1}) = \varphi_{2h}(U_{R''-3\varepsilon_1}) \subset U_{R_2}$. Thus, $g(z) \in U_{R_2}$ and the result follows from equation (5.21) on recalling that $\varepsilon < 1$ and that $A = U_{R_2+1}$ contains a 1-neighborhood of B (in the hyperbolic metric of U).

Finally, part (iii) simply follows from the above fact that $Q_{n_{k_0}}$ is univalent on A, which we already saw contains $U_{R''-3\varepsilon_1}$.

Now assume the claim is true for some $1 \le j < N$. For $z \in U_{R''-3\varepsilon_1}$, we have

$$\begin{aligned}\rho_U(Q_{(j+1)n_{k_0}}(z), g^{\circ j+1}(z)) &\le \rho_U(Q_{(j+1)n_{k_0}}(z), g \circ Q_{jn_{k_0}}(z)) \\ &\quad + \rho_U(g \circ Q_{jn_{k_0}}(z), g^{\circ j+1}(z)).\end{aligned}$$

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Now $Q_{jn_{k_0}}(z) \in A$ by hypothesis, so the first term in the inequality above is less than ε by equation (5.21). In addition to $Q_{jn_{k_0}}(z) \in A$, we also have $g^{\circ j}(z) \in B \subset U_{R_2} \subset A$ (we remark that this is a place where we need to make use of the fact that the sets $g^{\circ j}(U_{R''-3\varepsilon_1}))$ are increasing in j and thus all contained in B). Using part (i) of the induction hypothesis above, equation (5.20) and the hyperbolic convexity of A which follows from Lemma 2.8, we see on applying the hyperbolic M-L estimates (Lemma 2.9) to g that the second term is less than $K_1(\varepsilon_2/3(2K_1)^{N-j}K_2K_3)$. Thus, we have

$$\begin{aligned}\rho_U(Q_{(j+1)n_{k_0}}(z), g^{\circ j+1}(z)) &< \varepsilon + K_1 \cdot \frac{\varepsilon_2}{3(2K_1)^{N-j}K_2K_3} \\ &= \frac{1}{(2K_1)^j} \cdot \frac{\varepsilon_2}{3(2K_1)^{N-(j+1)}K_2K_3} \\ &\quad + \frac{1}{2} \cdot \frac{\varepsilon_2}{3(2K_1)^{N-(j+1)}K_2K_3} \\ &< \frac{\varepsilon_2}{3(2K_1)^{N-(j+1)}K_2K_3} \\ &< 1,\end{aligned}$$

which proves part (i) in the claim using the fact that $K_1 > 1$ for the second last inequality and $\varepsilon_2 \le 1$, K_1, K_2, $K_3 > 1$ for the last inequality above.

Now $Q_{(j+1)n_{k_0}}(U_{R''-3\varepsilon_1})$ lies in a 1-neighborhood of $g^{\circ j+1}(U_{R''-3\varepsilon_1})$ by part (i) above. However, $g^{\circ j+1}(U_{R''-3\varepsilon_1}) \in B$ (where again we note that the sets $g^{\circ j}(U_{R''-3\varepsilon_1}))$ are increasing in j and thus all contained in B), while a 1-neighborhood of $B \subset U_{R_2}$ lies inside A by the definition of A and so $Q_{(j+1)n_{k_0}}(z) \in A$ if $z \in U_{R''-3\varepsilon_1}$ (note that $j+1 \le N$), which finishes the proof of part (ii). To show part (iii) and see that $Q_{(j+1)n_{k_0}}(z)$ is univalent, we obviously have $Q_{(j+1)n_{k_0}}(z) = Q_{n_{k_0}} \circ Q_{jn_{k_0}}(z)$. Since by hypothesis we have both that $Q_{jn_{k_0}}$ is univalent on $U_{R''-3\varepsilon_1}$ and $Q_{jn_{k_0}}(U_{R''-3\varepsilon_1}) \subset A$, while $Q_{n_{k_0}}$ is univalent on A by our application of the Polynomial Implementation Lemma (Lemma 3.9), we have that $Q_{(j+1)n_{k_0}}$ is univalent on $U_{R''-3\varepsilon_1}$. This completes the proof of the claim. □

For convenience, set $\mathbf{Q_1} := Q_{Nn_{k_0}}$ and recall that on $U_{R''-3\varepsilon_1} \subset \tilde{U}$, we had $g^{\circ N} = \varphi_{2h}$. From above, $\mathbf{Q_1}$ then depends on κ, ε_1, ε_2, K_2, K_3, R, h_0, r_0, and R_0 (recall that N depends on ε_1, R, h_0, r_0, and R_0, while the mapping $\varphi_{2h} = g^{\circ N}$ depends on κ, ε_1, R, h_0, r_0, and R_0). By part (iii) of Claim 5.20, $\mathbf{Q_1}$ is univalent on $U_{R''-3\varepsilon_1}$ and, on this hyperbolic disc, from part (i) of the same claim and the fact that $g^{\circ N} = \varphi_{2h}$ on $U_{R''-3\varepsilon_1}$, we have (on this set)

$$\rho_U(\mathbf{Q_1}(z), \varphi_{2h}(z)) < \frac{\varepsilon_2}{3K_2K_3}, \tag{5.24}$$

while, from part (ii) of this claim and equation (5.23),

$$\mathbf{Q_1}(z) \in A, \tag{5.25}$$

$$\mathbf{Q_1}(0) = 0. \tag{5.26}$$

The mapping φ_{2h} obviously maps $U_{R''-3\varepsilon_1}$ to $\varphi_{2h}(U_{R''-3\varepsilon_1})$ and, provided the next polynomial in our construction has the desired properties on this set, we will be able to compose in a meaningful way so that the composition also has the desired properties.

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However, in practice, we are approximating φ_{2h} with the composition $\mathbf{Q_1}$ which involves an error, and our next step is to show that we can map into the correct set $\varphi_{2h}(U_{R''-3\varepsilon_1})$ using $\mathbf{Q_1}$ provided we are wiling to 'give up' an extra ε_1. First, however, we have the following important estimates which we will need later, especially when it comes to defining the upper bound $\tilde{\varepsilon}_2$ for ε_2 to obtain the same dependencies as given in the statement.

CLAIM 5.21. *There exist $\eta_1, \eta_2 > 0$ depending on ε_1, h_0, r_0, and R_0 such that*

$$\eta_1 \le \|(\varphi_{2h}^{-1})^\natural\|_{U_{R_2+2}} \le \eta_2.$$

Proof. Recall the upper bound R_2 on $R^{\mathrm{ext}}_{(U,0)}\varphi_{2h}(R_{R''-\varepsilon_1})$ given in equation (5.19) and the normalized Riemann map ψ from U to $\mathbb{D}$ which was defined just before Lemma 5.10. Here, φ_{2h}^{-1} maps U to $\tilde{U} \subset U$ so that the conjugated mapping $\psi \circ \varphi_{2h}^{-1} \circ \psi^{-1}$ is defined on all of $\mathbb{D}$. Using equations (5.14) and (5.19), one checks easily that

$$\varphi_{2h}^{-1}(U) \supset \varphi_{2h}^{-1}(U_{R_2+2}) \supset \varphi_{2h}^{-1}(\varphi_{2h}(U_{R''-2\varepsilon_1})) = U_{R''-2\varepsilon_1} \supset U_{3r_0/4}. \tag{5.27}$$

Hence, $\psi \circ \varphi_{2h}^{-1} \circ \psi^{-1}$ maps $\mathbb{D}$ to a domain which is contained in $\mathbb{D}$ and which contains $\Delta_{\mathbb{D}}(0, 3r_0/4)$, and so by the Koebe one-quarter theorem (Theorem A.1), we then obtain (strictly positive) upper and lower bounds for the derivative $|(\psi \circ \varphi_{2h}^{-1} \circ \psi^{-1})'(0)|$ which depend only on h_0 (because we assumed $h \le h_0$) and r_0. Note that, in particular, these bounds do not depend on the values of h or R.

Since $\psi \circ \varphi_{2h}^{-1} \circ \psi^{-1}$ is defined on the whole of the unit disc, on applying the distortion theorems (Theorem A.2), we obtain strictly positive upper and lower bounds for $|(\psi \circ \varphi_{2h}^{-1} \circ \psi^{-1})'|$ on the set $\psi(U_{R_2+2}) = \Delta_{\mathbb{D}}(0, R_2 + 2)$. Since by equation (5.19), R_2 depends on ε_1 and R_0, these bounds depend on ε_1, h_0, r_0, and R_0.

Here, φ_{2h}^{-1} maps U_{R_2+2} inside $\tilde{U} \subset U_{\pi/2}$ and, as U_{R_2+2} and $U_{\pi/2}$ are both relatively compact subsets of U, $|\psi'|$ is uniformly bounded above and below away from 0 on both of these sets. We therefore obtain strictly positive upper and lower bounds for the absolute value of the Euclidean, and thus the hyperbolic derivative of φ_{2h}^{-1} on U_{R_2+2}. These bounds will depend on κ, ε_1, h_0, r_0, and R_0 (the dependence on the scaling factor κ arising via ψ). However, since we are estimating hyperbolic derivatives, we can actually eliminate the dependence on κ and the claim then follows. □

We now define our upper bound $\tilde{\varepsilon}_2$ on ε_2 by setting

$$\tilde{\varepsilon}_2 = \min\left\{1, \frac{\varepsilon_1}{\eta_2}\right\}. \tag{5.28}$$

Given the dependencies of η_2 above (in Claim 5.21) as well as the dependence of ε_1 on $\tilde{\varepsilon}_1$ which in turn depends on h_0, r_0, and R_0, this upper bound then depends on ε_1, h_0, r_0, and R_0, which is the same as given in the statement.

Now make ε_2 smaller if necessary to ensure that $\varepsilon_2 \le \tilde{\varepsilon}_2$, (note that this may require us to obtain a new composition $\mathbf{Q_1}$ as above, but since ε_2 and $\tilde{\varepsilon}_2$ in no way depend on $\mathbf{Q_1}$, there is no danger of circular reasoning).

CLAIM 5.22. *Given $\tilde{\varepsilon}_2$, $0 < \varepsilon_2 \le \tilde{\varepsilon}_2$ and $\mathbf{Q_1}$ as above, we have*

$$\mathbf{Q_1}(U_{R''-4\varepsilon_1}) \subset \varphi_{2h}(U_{R''-3\varepsilon_1}). \tag{5.29}$$

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Proof. Let $z \in U_{R''-4\varepsilon_1}$, $w \in \partial U_{R''-3\varepsilon_1}$ be arbitrary and note that $\rho_U(z, w) \geq \varepsilon_1$, while both $\varphi_{2h}(z)$ and $\varphi_{2h}(w)$ lie inside $U_{R_2} = \Delta_U(0, R_2)$ in view of equation (5.19). As φ_{2h} is a homeomorphism, we also have that $\varphi_{2h}(z) \in \operatorname{int} \varphi_{2h}(U_{R''-3\varepsilon_1})$, while $\varphi_{2h}(w) \in \partial\varphi_{2h}(U_{R''-3\varepsilon_1})$. Lemma 2.8 ensures that $\Delta_U(0, R_2 + 2)$ is hyperbolically convex and so we may apply the hyperbolic M-L estimates (Lemma 2.9) using Claim 5.21 on $\Delta_U(0, R_2 + 2)$ to φ_{2h}^{-1}. Thus, we have $\rho_U(\varphi_{2h}(z), \varphi_{2h}(w)) \geq \varepsilon_1/\eta_2$, which implies the hyperbolic distance from $\varphi_{2h}(z)$ to $\partial(\varphi_{2h}(U_{R''-3\varepsilon_1}))$ is at least ε_1/η_2.

Again let $z \in U_{R''-4\varepsilon_1}$ be arbitrary. We then have using $K_2, K_3 > 1$, and equations (5.24) and (5.28) that

$$\begin{aligned}\rho_U(\mathbf{Q_1}(z), \varphi_{2h}(z)) &< \frac{\varepsilon_2}{3K_2K_3}\\ &< \varepsilon_2\\ &\leq \frac{\varepsilon_1}{\eta_2}\end{aligned}$$

and since the hyperbolic distance from $\varphi_{2h}(z)$ to $\partial(\varphi_{2h}(U_{R''-3\varepsilon_1}))$ is at least ε_1/η_2 from above, it follows that $\mathbf{Q_1}(z)$ misses $\partial\varphi_{2h}(U_{R''-3\varepsilon_1})$. Additionally, as $\mathbf{Q_1}(U_{R''-4\varepsilon_1})$ is connected in view of part (iii) of Claim 5.20 while $\mathbf{Q_1}(0) = 0 \in \varphi_{2h}(U_{R''-3\varepsilon_1})$, it follows, since $z \in U_{R''-4\varepsilon_1}$ was arbitrary, that $\mathbf{Q_1}(U_{R''-4\varepsilon_1}) \subset \varphi_{2h}(U_{R''-3\varepsilon_1})$ and the claim follows, as desired. $\square$

Controlling error: 'During': Recall that at the start of the last section, we fixed a value of ε_1 in $(0, \tilde{\varepsilon}_1]$, which in turn fixed the value of $h = h(\varepsilon_1)$ and that we also fixed a value of $R \in [r_0, R_0]$. We now fix a function $\mathcal{E}$ as in the statement which is defined and univalent on U_R with $\mathcal{E}(0) = 0$ and $\rho_U(\mathcal{E}(z), z) < \varepsilon_1$ for $z \in U_R$. Note that, in addition to R, $\mathcal{E}$ will also depend on r_0, R_0 (via R) and also on κ, the latter arising from the fact that $\mathcal{E}$ is defined on the set U_R which depends on κ. Recall the quantity $\check{R}(h) := R^{\mathrm{ext}}_{(V_{2h},0)} V_h = R^{\mathrm{ext}}_{(\tilde{V}_{2h},0)} \tilde{V}_h$ introduced before the statement of Lemma 5.11 and the function $T : (0, \tilde{\varepsilon}_1] \mapsto (0, \infty)$ which was introduced in the statement of the Target Lemma (Lemma 5.13) and which served as a lower bound for $R^{\mathrm{int}}_{(\tilde{V}_{2h},0)}(\tilde{V}_{2h} \setminus \hat{N})$ (where $\hat{N}$ was a $2\varepsilon_1$-neighborhood of $\partial\tilde{V}_{2h}$ with respect to the hyperbolic metric of U).

Now $\tilde{U} \subset \tilde{V}_h$ (recall that $\tilde{V}_h = \varphi_{2h}^{-1}(V_h)$ was introduced immediately before Lemma 5.11), while in equations (5.5), (5.6), and (5.14), we chose $h = h(\varepsilon_1)$ (where, for convenience, we will suppress the dependence of h on ε_1) as small as possible so that $\check{R}(h) = T(\varepsilon_1)$ (cf. equation (5.6)). By item (1) of the Fitting Lemma (Lemma 5.15), we have $\overline{\tilde{V}_h} \subset \tilde{V}_{2h} \setminus \hat{N}$ (this latter set clearly being closed). Hence, the $2\varepsilon_1$-neighborhood $\hat{N}$ of $\partial\tilde{V}_{2h}$ avoids $\overline{\tilde{V}_h}$ (and hence also the smaller set $\tilde{U}$). Thus, if we let $\mathcal{O}$ be an ε_1-neighborhood (in the hyperbolic metric of U) of the closure $\overline{\tilde{V}_h}$, then, by the hypotheses on $\mathcal{E}$ in the statement, $\mathcal{E}(\mathcal{O}) \subset \tilde{V}_{2h}$, while $\mathcal{E}(\overline{\tilde{V}_h}) \subset \mathcal{E}(\tilde{V}_{2h} \setminus \hat{N})$ avoids an ε_1-neighborhood of $\partial\tilde{V}_{2h}$. In particular, again by the hypotheses on $\mathcal{E}$, $\mathcal{E}(\partial\tilde{V}_h)$ is a simple closed curve which lies inside $\partial\tilde{V}_{2h}$.

Next, on $\varphi_{2h}(\mathcal{O})$, define $\hat{\mathcal{E}} = \varphi_{2h} \circ \mathcal{E} \circ \varphi_{2h}^{-1}$. Since from above $\mathcal{E}(\mathcal{O}) \subset \tilde{V}_{2h}$, it follows that $\hat{\mathcal{E}}$ is well defined, analytic, and injective on a neighborhood of $\varphi_{2h}(\overline{\tilde{V}_h}) = \overline{V}_h$. Then

$\hat{\mathcal{E}}$ depends immediately on the six quantities κ, ε_1, h, R (these last two among other things being via the domain $\tilde{V}_{2h}$), φ_{2h}, and $\mathcal{E}$ from which one can deduce (e.g. by using the tables in the appendices) that $\hat{\mathcal{E}}$ ultimately depends on κ, ε_1, R, h_0, r_0, R_0, and $\mathcal{E}$. As before, let $\gamma = \partial V_h$, $\Gamma = \partial V_{2h}$ (again with positive orientations as Jordan curves with respect to the conformal annulus bounded by ∂V_h and ∂V_{2h}). Then, from above and again by the hypotheses on $\mathcal{E}$, since $\hat{\mathcal{E}}$ is defined on $\varphi_{2h}(\mathcal{O})$ which contains $\gamma = \partial V_h$ (since $\overline{\tilde{V}_h} \subset \mathcal{O} \subset \tilde{V}_{2h} \setminus \hat{N}$ from above), we have from above that $\hat{\mathcal{E}}(\gamma)$ lies inside Γ and so $(\hat{\mathcal{E}}, \mathrm{Id})$ is an admissible pair on (γ, Γ) in the sense given in Definition 3.3 in §3 on the Polynomial Implementation Lemma. Lastly, since φ_{2h} and $\mathcal{E}$ both fix 0, we must have that $\hat{\mathcal{E}}(0) = 0$.

By the hypotheses on $\mathcal{E}$ in the statement,

$$\mathcal{E}(U_{R''-2\varepsilon_1}) \subset U_{R''-\varepsilon_1}. \tag{5.30}$$

Since $\mathcal{O} \supset \overline{\tilde{V}_h} \supset \tilde{U} \supset U_{R''}$, it follows that $U_{R''-2\varepsilon_1} \subset \mathcal{O}$ and so $\varphi_{2h}(U_{R''-2\varepsilon_1}) \subset \varphi_{2h}(\mathcal{O})$, and from this and the definition of $\hat{\mathcal{E}} := \varphi_{2h} \circ \mathcal{E} \circ \varphi_{2h}^{-1}$,

$$\hat{\mathcal{E}}(\varphi_{2h}(U_{R''-2\varepsilon_1})) \subset \varphi_{2h}(U_{R''-\varepsilon_1}). \tag{5.31}$$

Hence, since $\hat{\mathcal{E}}$ maps the relatively compact subset $\varphi_{2h}(U_{R''-2\varepsilon_1})$ to another relatively compact subset of U, we can fix the value of $1 < K_2 < \infty$ such that

$$|\hat{\mathcal{E}}^{\natural}(z)| \leq K_2, \quad z \in \varphi_{2h}(U_{R''-2\varepsilon_1}), \tag{5.32}$$

where, as usual, we take our hyperbolic derivative with respect to the hyperbolic metric of U. Immediately, K_2 depends on ε_1, R'', φ_{2h}, and $\hat{\mathcal{E}}$. Using the chain rule in equation (2.2) for the hyperbolic derivative, it follows from equation (5.19), Claim 5.21, equation (5.30), and the hypotheses on $\mathcal{E}$ in the statement that K_2 can be bounded uniformly in terms of κ, ε_1, h_0, r_0, R_0, the function $\mathcal{E}$, as well as the particular value of R (since $\mathcal{E}$ is defined on all of U_R in the statement while the mapping φ_{2h} also depends on R). However, the dependence on κ (arising via φ_{2h}) can be eliminated since we are estimating a hyperbolic derivative. We also observe that this is the one point where we employ the full force of equation (5.19), and require an upper bound on the external hyperbolic radius of $\varphi_{2h}(U_{R''-\varepsilon_1})$ and not just $\varphi_{2h}(U_{R''-2\varepsilon_1})$ or some smaller set.

Note in particular that this bound has nothing to do with the existence of the composition $\mathbf{Q}_1$ from the last section, and so there is no danger of circular reasoning in fixing the bound K_2 at this point. Note also that this does not affect our earlier assertion that $\varepsilon < 1$ in the previous section on controlling the error for 'up'. However, the same argument as used in the proof of Claim 5.22 shows that if we set $\delta_0 = \varepsilon_1/\eta_2$, where η_2 is the upper bound on the hyperbolic derivative of φ_{2h}^{-1} from Claim 5.21, then a δ_0-hyperbolic neighborhood in U of $\varphi_{2h}(U_{R''-3\varepsilon_1})$ is contained in $\varphi_{2h}(U_{R''-2\varepsilon_1})$, while δ_0 depends on ε_1, h_0, r_0, and R_0.

Since $\hat{\mathcal{E}}(\gamma)$ lies inside Γ while $\hat{\mathcal{E}}(0) = 0$, using equation (5.32), we can then apply the Polynomial Implementation Lemma (Lemma 3.9) with $\Omega = V_h$, $\Omega' = V_{2h}$, $\gamma = \Gamma_h$, $\Gamma = \Gamma_{2h}$, $f = \hat{\mathcal{E}}$, $A = \varphi_{2h}(U_{R''-3\varepsilon_1})$, $\delta = \delta_0$, $M = K_2$, and $\varepsilon = \varepsilon_2/(3K_3)$ to construct a $(17+\kappa)$-bounded composition of quadratic polynomials, $\mathbf{Q_2}$, univalent on $\varphi_{2h}(U_{R''-3\varepsilon_1})$

such that

$$\rho_U(\mathbf{Q_2}(z), \hat{\mathcal{E}}(z)) < \frac{\varepsilon_2}{3K_3}, \quad z \in \varphi_{2h}(U_{R''-3\varepsilon_1}), \tag{5.33}$$

$$\|\mathbf{Q_2}^{\natural}\|_{\varphi_{2h}(U_{R''-3\varepsilon_1})} \le K_2\bigg(1 + \frac{\varepsilon_2}{3K_3}\bigg), \tag{5.34}$$

$$\mathbf{Q_2}(0) = 0, \tag{5.35}$$

where the bound $K_3 > 1$ is to be fixed in the next section. From the statement of Lemma 3.9, the composition $\mathbf{Q_2}$ depends directly on κ, ε_1, ε_2, K_2, K_3, η_2, φ_{2h}, R'' (via the set $\varphi_{2h}(U_{R''-3\varepsilon_1})$, and $\hat{\mathcal{E}}$. From this, one checks (e.g. using the tables) that $\mathbf{Q_2}$ depends eventually on on κ, ε_1, ε_2, K_3, R, h_0, r_0, R_0, and the function $\mathcal{E}$ (and ultimately on κ, ε_1, ε_2, R, h_0, r_0, R_0, and $\mathcal{E}$ once the dependencies of K_3 in the 'down' section of the proof below are taken into account).

Controlling error: 'Down': Recall equation (5.31), where we had that $\hat{\mathcal{E}}(\varphi_{2h}(U_{R''-2\varepsilon_1})) \subset \varphi_{2h}(U_{R''-\varepsilon_1})$. In exactly the same way, we have

$$\hat{\mathcal{E}}(\varphi_{2h}(U_{R''-3\varepsilon_1})) \subset \varphi_{2h}(U_{R''-2\varepsilon_1}). \tag{5.36}$$

Recall also that from equation (5.19), we had that $R^{\text{ext}}_{(U,0)}\varphi_{2h}(U_{R''-2\varepsilon_1}) \le R_2$. Also, by equations (5.33) and (5.36), we have that $\mathbf{Q_2}(\varphi_{2h}(U_{R''-3\varepsilon_1}))$ is contained in an $(\varepsilon_2/3)K_3$-neighborhood of $\varphi_{2h}(U_{R''-2\varepsilon_1})$ (using the hyperbolic metric of U). Thus,

$$\begin{aligned} R^{\text{ext}}_{(U,0)}\mathbf{Q_2}(\varphi_{2h}(U_{R''-3\varepsilon_1})) &\le R_2 + \frac{\varepsilon_2}{3K_3} \\ &< R_2 + \varepsilon_2, \end{aligned}$$

(recall that we assumed $K_3 > 1$) and so

$$R^{\text{ext}}_{(U,0)}\mathbf{Q_2}(\varphi_{2h}(U_{R''-3\varepsilon_1})) \le R_2 + 1 \tag{5.37}$$

as $\varepsilon_2 < 1$ using equation (5.28). Thus, $\mathbf{Q_2}(\varphi_{2h}(U_{R''-3\varepsilon_1})) \subset U_{R_2+1} \subset U_{R_2+2} \subset \overline{U} \subset V_{2h}$, while φ_{2h}^{-1} maps $U_{R_2+2} \subset U$ inside $\varphi_{2h}^{-1}(U) = \tilde{U}$, which is compactly contained in U. Using Claim 5.21, if we set $K_3 = \max\{\eta_2, \frac{3}{2}\}$ so that $K_3 > 1$, we have that

$$|(\varphi_{2h}^{-1})^{\natural}(z)| \le K_3, \quad z \in U_{R_2+2}. \tag{5.38}$$

Note that, in view of equation (5.19) and Claim 5.21, K_3 depends on ε_1, h_0, r_0, and R_0. Again, note that this bound has nothing to do with the existence of the compositions $\mathbf{Q}_1$, $\mathbf{Q}_2$ from the last sections, and so there is no danger of circular reasoning in fixing the bound K_3 at this point. Further, φ_{2h}^{-1} is analytic and injective on a neighborhood of $\overline{V}_h$ and maps ∂V_h inside $U \subset V_{2h}$ so that, if we set $\gamma = \partial V_h$, $\Gamma = \partial V_{2h}$ again with positive orientations as Jordan curves with respect to the conformal annulus bounded by ∂V_h and ∂V_{2h}, we have that $\varphi_{2h}^{-1}(\gamma)$ lies inside Γ. Thus, $(\varphi_{2h}^{-1}, \mathrm{Id})$ is easily seen to be an admissible pair on (γ, Γ), as in Definition 3.3, and we also have that $\varphi_{2h}^{-1}(0) = 0$. Using equation (5.38), we can then apply the Polynomial Implementation Lemma (Lemma 3.9) with $\Omega = V_h$, $\Omega' = V_{2h}$, $\gamma = \Gamma_h$, $\Gamma = \Gamma_{2h}$, $f = \varphi_{2h}^{-1}$, $A = U_{R_2+1}$, $\delta = 1$ (so that $\hat{A} = U_{R_2+2}$), $M = K_3$, and $\varepsilon = \varepsilon_2/3$ to construct a $(17{+}\kappa)$-bounded quadratic polynomial composition $\mathbf{Q_3}$ that is

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univalent on U_{R_2+1} for which we have

$$\rho_U(\mathbf{Q_3}(z), \varphi_{2h}^{-1}(z)) < \frac{\varepsilon_2}{3}, \quad z \in U_{R_2+1}, \tag{5.39}$$

$$\|\mathbf{Q_3}^\natural\|_{U_{R_2+1}} \le K_3\left(1 + \frac{\varepsilon_2}{3}\right), \tag{5.40}$$

$$\mathbf{Q_3}(0) = 0. \tag{5.41}$$

Note that by Lemma 3.9, since $\delta = 1$, $\mathbf{Q_3}$ depends directly on κ, K_3, ε, R_2, h (via the curves ∂V_h, ∂V_{2h}), and φ_{2h} so that one can check that $\mathbf{Q_3}$ ultimately depends on κ, ε_1, ε_2, R, h_0, r_0, and R_0.

Concluding the proof of Phase II: Now, as $\mathbf{Q_1}$, $\mathbf{Q_2}$, and $\mathbf{Q_3}$ were all constructed using the Polynomial Implementation Lemma, they are all $(17+\kappa)$-bounded compositions of quadratic polynomials. Next define the $(17 + \kappa)$-bounded composition

$$\mathbf{Q} := \mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}. \tag{5.42}$$

$\mathbf{Q}$ then has the correct coefficient bound of $17 + \kappa$ as in the statement and, checking the dependencies of each of the compositions $\mathbf{Q_i}$, $i = 1, 2, 3$, as well as those of the constants K_2, K_3, one sees that $\mathbf{Q}$ depends on κ, ε_1, ε_2, R, h_0, r_0, R_0, and $\mathcal{E}$, which is the same as given in the statement.

Using the definitions of the compositions $\mathbf{Q_1}$, $\mathbf{Q_2}$, $\mathbf{Q_3}$ (defined previously), in part (iii) of Claim 5.20, Claim 5.22, and equation (5.37) (respectively), we showed the following:

(1) $\mathbf{Q_1}$ is univalent on $U_{R''-3\varepsilon_1} \supset U_{R''-4\varepsilon_1}$ and $\mathbf{Q_1}(U_{R''-4\varepsilon_1}) \subset \varphi_{2h}(U_{R''-3\varepsilon_1})$;

(2) $\mathbf{Q_2}$ is univalent on $\varphi_{2h}(U_{R''-3\varepsilon_1})$ and $\mathbf{Q_2}(\varphi_{2h}(U_{R''-3\varepsilon_1})) \subset U_{R_2+1}$;

(3) $\mathbf{Q_3}$ is univalent on U_{R_2+1}.

Combining these three observations, and recalling the definition of $\delta(\varepsilon_1) = \sup_{[r_0,R_0]}(R - R'') + 5\varepsilon_1$ which we set in equation (5.13) at the end of the section on ideal loss of domain, we see that the composition $\mathbf{Q}$ is univalent on $U_{R''-4\varepsilon_1}$ and therefore univalent on a neighborhood of $\overline{U}_{R''-5\varepsilon_1} \supset \overline{U}_{R-\delta(\varepsilon_1)}$ (this is the reason why the function $\delta : (0, \tilde{\varepsilon}_1] \mapsto (0, (r_0/4))$ was defined the way it was and, in particular, why we needed to include an 'extra' ε_1 in our definition of δ), which gives part (i) in the statement. As all compositions were created with the Polynomial Implementation Lemma, we have using equations (5.26), (5.35), and (5.41) that $\mathbf{Q}(0) = 0$, which gives part (iii) in the statement.

The last thing we need to do is then establish part (ii) in the statement. Recall that in equation (5.14), we chose $\tilde{\varepsilon}_1$ sufficiently small such that, in particular, $\delta(\varepsilon_1) < r_0/4$, which ensured that $U_{R-\delta(\varepsilon_1)} \neq \emptyset$.

Then for $z \in \overline{U}_{R-\delta(\varepsilon_1)} \subset \overline{U}_{R''-5\varepsilon_1} \subset U_{R''-4\varepsilon_1}$, we have

$$\begin{aligned}\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) \le{}& \rho_U(\mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z)) \\ &+ \rho_U(\varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}(z)) \\ &+ \rho_U(\varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}, \mathcal{E}(z)).\end{aligned} \tag{5.43}$$

We now estimate the three terms on the right-hand side of the inequality above. We have that $z \in \overline{U}_{R''-5\varepsilon_1} \subset U_{R''-4\varepsilon_1}$, so $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-3\varepsilon_1})$ by Claim 5.22. Then $\mathbf{Q_2} \circ \mathbf{Q_1}(z) \in U_{R_2+1}$ by equation (5.37). Thus,

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$$\rho_U(\mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z)) < \frac{\varepsilon_2}{3} \tag{5.44}$$

by equation (5.39). For the second term, we still have $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-3\varepsilon_1})$ and $\mathbf{Q_2} \circ \mathbf{Q_1}(z) \in U_{R_2+1} \subset U_{R_2+2}$ as above. Also, we have $\hat{\mathcal{E}} \circ \mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-2\varepsilon_1}) \subset U_{R_2} \subset U_{R_2+2}$ by equations (5.19) and (5.36). Thus, using the hyperbolic convexity lemma (Lemma 2.8) and the hyperbolic M-L estimates (Lemma 2.9) applied to φ_{2h}^{-1} on U_{R_2+2}, by equations (5.33) and (5.38), we have

$$\begin{aligned}\rho_U(\varphi_{2h}^{-1} \circ \mathbf{Q_2} \circ \mathbf{Q_1}(z), \varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}(z)) &< K_3 \cdot \frac{\varepsilon_2}{3K_3} \\ &< \frac{\varepsilon_2}{3}.\end{aligned} \tag{5.45}$$

For the third term, we note that $\mathcal{E}(z) = \varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \varphi_{2h}$ on the set $\mathcal{O} \supset \tilde{V}_h \supset \overline{\tilde{U}} \supset \overline{U}_{R''-5\varepsilon_1} \supset \overline{U}_{R-\delta(\varepsilon_1)}$ (where we remind the reader that $\mathcal{O}$ is an ε_1-neighborhood of $\overline{\tilde{V}}_h$ in the hyperbolic metric of U) so that $\mathcal{E}$ and $\varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}$ differ in the first mapping of the composition. We still have $\mathbf{Q_1}(z) \in \varphi_{2h}(U_{R''-3\varepsilon_1})$ by Claim 5.22, and clearly $\varphi_{2h}(z) \in \varphi_{2h}(\overline{U}_{R''-5\varepsilon_1}) \subset \varphi_{2h}(U_{R''-3\varepsilon_1})$. We need to take care to ensure that we have at least a local version of hyperbolic convexity when it comes to applying the hyperbolic M-L estimates for $\hat{\mathcal{E}}$ and φ_{2h}^{-1}. By equation (5.24) (and the fact that $K_2, K_3 > 1$), we have that $\mathbf{Q_1}(z) \in \Delta_U(\varphi_{2h}(z), \varepsilon_2)$. Since by equation (5.28), $\varepsilon_2 \le \tilde{\varepsilon}_2 \le 1$, it follows from equation (5.19) that this hyperbolic disc is in turn contained in U_{R_2+1}.

Recall that by Claim 5.21, we had η_2 depending only on ε_1, h_0, r_0, and R_0 for which we had, in particular, $\|(\varphi_{2h}^{-1})^\natural\|_{\Delta_U(0,R_2+2)} \le \eta_2$. If we now apply the hyperbolic convexity lemma (Lemma 2.8) and the hyperbolic M-L estimates (Lemma 2.9) for the function φ_{2h}^{-1} on the ball $\Delta_U(\varphi_{2h}(z), \varepsilon_2)$, we have that $\varphi_{2h}^{-1}(\Delta_U(\varphi_{2h}(z), \varepsilon_2)) \subset \Delta_U(z, \eta_2\varepsilon_2) \subset \Delta_U(z, \varepsilon_1)$, the last inclusion following from equation (5.28), which implies that $\varepsilon_2 \le \tilde{\varepsilon}_2 \le \varepsilon_1/\eta_2$. Thus, $\varphi_{2h}^{-1}(\Delta_U(\varphi_{2h}(z), \varepsilon_2)) \subset \Delta_U(z, \varepsilon_1) \subset U_{R''-4\varepsilon_1} \subset U_{R''-3\varepsilon_1}$ so that $\Delta_U(\varphi_{2h}(z), \varepsilon_2) \subset \varphi_{2h}(U_{R''-3\varepsilon_1})$. We also know using equation (5.32) that $|\hat{\mathcal{E}}^\natural|$ is bounded above on $\varphi_{2h}(U_{R''-2\varepsilon_1}) \supset \varphi_{2h}(U_{R''-3\varepsilon_1}) \supset \Delta_U(\varphi_{2h}(z), \varepsilon_2)$.

Thus, by equations (5.19) and (5.36), we have $\hat{\mathcal{E}}(\Delta_U(\varphi_{2h}(z), \varepsilon_2)) \subset \hat{\mathcal{E}}(\varphi_{2h}(U_{R''-3\varepsilon_1})) \subset \varphi_{2h}(U_{R''-2\varepsilon_1}) \subset U_{R_2} \subset U_{R_2+2}$ so that, in particular, $\hat{\mathcal{E}}(\varphi_{2h}(z))$ and $\hat{\mathcal{E}}(\mathbf{Q_1}(z))$ both lie in U_{R_2+2}, while we know $|(\varphi_{2h}^{-1})^\natural|$ is bounded above on U_{R_2+2} using equation (5.38). Then, using equations (5.24), (5.32), and (5.38), and combining the hyperbolic convexity lemma (Lemma 2.8) and the hyperbolic M-L estimates (Lemma 2.9), applied first to $\hat{\mathcal{E}}$ on $\Delta_U(\varphi_{2h}(z), \varepsilon_2) \subset \varphi_{2h}(U_{R''-2\varepsilon_1})$ and then to φ_{2h}^{-1} on U_{R_2+2}, we have

$$\begin{aligned}\rho_U(\varphi_{2h}^{-1} \circ \hat{\mathcal{E}} \circ \mathbf{Q_1}, \mathcal{E}(z)) &< K_3 \cdot K_2 \cdot \frac{\varepsilon_2}{3K_2K_3} \\ &< \frac{\varepsilon_2}{3}.\end{aligned} \tag{5.46}$$

Finally, using equations (5.43), (5.44), (5.45), and (5.46), we have

$$\rho_U(\mathbf{Q}(z), \mathcal{E}(z)) < \varepsilon_2,$$

which establishes part (ii) in the statement and completes the proof of Phase II. □

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Before going on to §6, we close with a couple of observations. It is possible if one wishes to find a bound on the absolute value of the hyperbolic derivative of the composition $\mathbf{Q}$ above on $\overline{U}_{R''-\delta(\varepsilon_1)}$ which is uniform in terms of the constants κ, ε_1, ε_2, h_0, r_0, and R_0 (the hardest part of this is controlling the hyperbolic derivative of $\mathbf{Q_1}$ which can best be done using equation (5.24) and Claim 5.21 combined with Lemma A.4 and the version of Cauchy's integral formula for derivatives—e.g. [**Con78**, Corollary IV.5.9]).

However, we do not actually require estimates on the size of $\mathbf{Q}^\natural$. The reason for this is that the purpose of Phase II is to correct the error from a previous Phase I (Lemma 4.8) approximation which essentially resets the error of which we need to keep a track. However, as we saw, this Phase II correction itself generates an error which is then passed through the next Phase I approximation. To control this, then, we do need an estimate on the hyperbolic derivative of the Phase I composition (which is item (4) in the statement of Phase I).

6. *Proof of the main theorem*

In this section, we prove Theorem 1.3. The proof of the theorem will follow from a large inductive argument. First, however, we need one more technical lemma. Recall the Siegel disc U for P and that, for $R > 0$, $U_R = \Delta_U(0, R)$ is used to denote the hyperbolic disc of radius R about 0 with respect to the hyperbolic metric of U.

LEMMA 6.1. (The Jordan curve argument) *Let U and U_R be as above. Given $0 < \varepsilon < R$, suppose g is a univalent function defined on a neighborhood of $\overline{U}_R$ such that $g(0) = 0$ and $\rho_U(g(z), z) \le \varepsilon$ on ∂U_R. Then, $g(U_R) \supset U_{R-\varepsilon}$.*

Proof. The function g is a homeomorphism and is bounded on $\overline{U}_R$, so that it maps ∂U_R to $g(\partial U_R) = \partial(g(U_R))$ which is a Jordan curve in $\mathbb{C}$, while U_R gets mapped to the bounded complementary component of this Jordan curve in view of the Jordan curve theorem (e.g. [**Mun00**, Theorem 63.4] or [**New51**, Theorem V.10.2]). Then $0 = g(0)$ lies in $g(U_R)$ and thus inside $\partial(g(U_R))$, and since this curve avoids $U_{R-\varepsilon}$, all of the connected set $U_{R-\varepsilon}$ lies inside $\partial(g(U_R))$. Hence, $U_{R-\varepsilon} \subset g(U_R)$. □

LEMMA 6.2. *There exist:*

(a) *a sequence of positive real numbers $\{\varepsilon_k\}_{k=1}^{\infty}$ which converges to 0;*
(b) *a sequence $\{J_i\}_{i=1}^{\infty}$ of natural numbers, a positive constant $\kappa_0 \ge 576$, and a sequence of compositions of quadratic polynomials $\{\mathbf{Q^i}\}_{i=1}^{\infty}$;*
(c) *a sequence of strictly decreasing hyperbolic radii $\{R_i\}_{i=0}^{\infty}$; and*
(d) *a sequence of strictly increasing hyperbolic radii $\{S_i\}_{i=0}^{\infty}$,*

such that:

(1) *for each $i \ge 0$, $S_i < 1/10 < \frac{1}{5} < R_i$;*
(2) *for each $i \ge 1$, $\mathbf{Q^i}$ is a composition of J_i ($17+\kappa_0$)-bounded quadratic polynomials with $\mathbf{Q^i}(0) = 0$;*
(3) *for each $i \ge 1$, $\mathbf{Q^i} \circ \cdots\cdots \circ \mathbf{Q^1}(U_{1/20}) \subset U_{S_i} \subset U_{1/10}$; and*
(4) *for each $i \ge 1$ and $1 \le m \le J_i$, if $\mathbf{Q^i_m}$ denotes the partial composition of the first m quadratics of $\mathbf{Q^i}$, then, for all $f \in \mathcal{S}$ and for all $i = 2k + 1$ odd, there exists $1 \le$*

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$m_k \leq J_i$ such that, for all $z \in U_{1/20}$, we have

$$\rho_U(\mathbf{Q^i_{m_k}} \circ \mathbf{Q^{i-1}} \circ \cdots \cdots \circ \mathbf{Q^1}, f(z)) < \varepsilon_{k+1}.$$

Let J_i be the integers and $\mathbf{Q}^i$ the polynomial compositions from part (b) of the statement above. For $i = 0$, set $T_0 = 0$ and, for each $i \geq 1$, set $T_i = \sum_{j=1}^{i} J_j$. Given this, we define a sequence $\{P_m\}_{m=1}^{\infty}$ in the following natural way: for $m \geq 1$, let $i \geq 1$ be the largest index such that $T_{i-1} < m$ so that $T_{i-1} < m \leq T_i = T_{i-1} + J_i$. Then simply let P_m be the $(m - T_{i-1})$th quadratic in the composition $\mathbf{Q}^i$ (which is a composition of J_i quadratic polynomials).

The next lemma then follows as an immediate corollary (using items (2), (3), and (4) above).

LEMMA 6.3. *There exists a sequence of quadratic polynomials $\{P_m\}_{m=1}^{\infty}$ such that the following hold:*

(1) *$\{P_m\}_{m=1}^{\infty}$ is $(17+\kappa_0)$-bounded;*
(2) *$Q_m(U_{1/20}) \subset U_{1/10}$ for infinitely many m;*
(3) *for all $f \in \mathcal{S}$, there exists a subsequence $\{Q_{m_k}\}_{k=1}^{\infty}$ which converges uniformly to f on $U_{1/20}$ as $k \to \infty$.*

Proof of Lemma 6.2. We begin by fixing the values of the constants in the statements of Phases I and II (Lemmas 4.8 and 5.17). Starting with Phase II, let $h_0 = 1$ be the maximum value for the Green's function G and let $r_0 = 1/20$, $R_0 = \frac{1}{4} < \pi/2$ be the upper and lower bounds for the hyperbolic radii we consider in applying Phase II. We will also use $R_0 = \frac{1}{4}$ when we apply Phase I and we set $\kappa = \kappa_0 = \kappa_0(\frac{1}{4}) \geq 576$ for both Phases I and II.

Let $C := 7$ be the bound on the hyperbolic derivative from item (4) of the statement of Phase I and let $\tilde{\varepsilon}_1 > 0$ and $\delta(x)$ be the function defined on $(0, \tilde{\varepsilon}_1]$ measuring loss of hyperbolic radius from the statement of Phase II, both of which are determined by the values of h_0, r_0, and R_0 which we have just fixed. The reader might find it helpful to consult the block diagram for the scheme of the proof in Figure 7 for orientation in what follows. The proof of Lemma 6.2 will follow quickly from the following claim, which we prove by induction.

CLAIM 6.4. *There exist inductively defined infinite sequences of positive real numbers $\{\varepsilon_k\}_{k=1}^{\infty}$, $\{\eta_k\}_{k=1}^{\infty}$, and $\{\sigma_k\}_{k=1}^{\infty}$, sequences of hyperbolic radii $\{R_i\}_{i=0}^{\infty}$ and $\{S_i\}_{i=0}^{\infty}$, integers $\{J_i\}_{i=1}^{\infty}$, and polynomial compositions $\{\mathbf{Q^i}\}_{i=1}^{\infty}$ such that, for each $n \in \mathbb{N}$, the following hold.*

(i) *The sequences $\{\varepsilon_k\}_{k=1}^{n}$, $\{\eta_k\}_{k=1}^{n}$, and $\{\sigma_k\}_{k=1}^{n}$ satisfy*

$$\eta_k = \begin{cases} \dfrac{4\varepsilon_1}{3} + \delta(\varepsilon_1), & k = 1, \\ \left(\dfrac{4}{3} + \dfrac{1}{3C}\right)\varepsilon_k + \delta(\varepsilon_k), & 2 \leq k \leq n, \end{cases}$$

$$\sigma_k = \begin{cases} \dfrac{4\varepsilon_1}{3}, & k = 1, \\ \left(\dfrac{4}{3} + \dfrac{1}{3C}\right)\varepsilon_k, & 2 \leq k \leq n, \end{cases}$$

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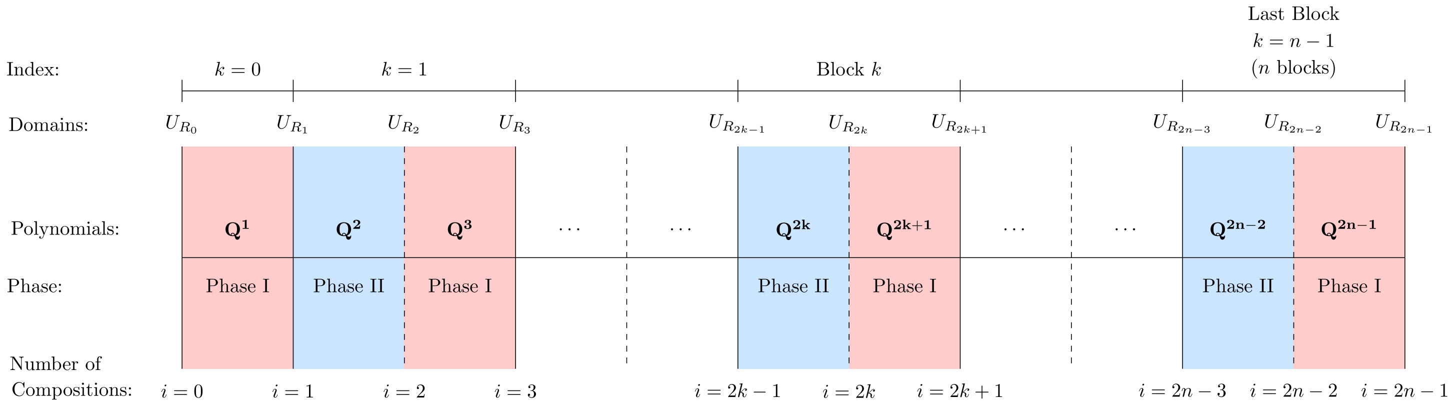

FIGURE 7. A block diagram illustrating the induction scheme.

where in addition, we require that $0 < \varepsilon_k < \sigma_k < \eta_k < 1/40 \cdot 2^k$ *and that* $\varepsilon_k \leq \tilde{\varepsilon}_1$ *for each* $1 \leq k \leq n$.

(ii) *The sequence* $\{R_i\}_{i=0}^{2n-1}$ *is strictly decreasing and is given by* $R_0 = \frac{1}{4}$, $R_1 = \frac{1}{4} - (\varepsilon_1/3)$, *and*

$$R_i = \begin{cases} \dfrac{1}{4} - \left(\displaystyle\sum_{j=1}^{k} \eta_j\right) - \dfrac{\varepsilon_{k+1}}{3C}, & i = 2k \text{ for some } 1 \leq k \leq n-1, \\[2ex] \dfrac{1}{4} - \left(\displaystyle\sum_{j=1}^{k} \eta_j\right) - \left(\dfrac{1}{3} + \dfrac{1}{3C}\right)\varepsilon_{k+1}, & i = 2k+1 \text{ for some} \\ & 1 \leq k \leq n-1. \end{cases}$$

The sequence $\{S_i\}_{i=0}^{2n-1}$ *is strictly increasing and is given by* $S_0 = 1/20$, $S_1 = (1/20) + (\varepsilon_1/3)$, *and*

$$S_i = \begin{cases} \dfrac{1}{20} + \left(\displaystyle\sum_{j=1}^{k} \sigma_j\right) + \dfrac{\varepsilon_{k+1}}{3C}, & i = 2k \text{ for some } 1 \leq k \leq n-1, \\[2ex] \dfrac{1}{20} + \left(\displaystyle\sum_{j=1}^{k} \sigma_j\right) + \left(\dfrac{1}{3} + \dfrac{1}{3C}\right)\varepsilon_{k+1}, & i = 2k+1 \text{ for some} \\ & 1 \leq k \leq n-1. \end{cases}$$

(iii) $1/20 \leq S_i < 1/10 < \frac{1}{5} < R_i \leq \frac{1}{4}$ *for each* $0 \leq i \leq 2n-1$.

(iv) *For each* $1 \leq i \leq 2n-1$, $\mathbf{Q^i}$ *is a (17+κ_0)-bounded composition of* J_i *quadratic polynomials with* $\mathbf{Q^i}(0) = 0$.

(v) *For each* $1 \leq i \leq 2n-1$, *the branch of* $(\mathbf{Q^i})^{-1}$ *which fixes* 0 *is well defined and univalent on* U_{R_i}, *and maps* U_{R_i} *inside* $U_{R_{i-1}}$. *The branch of* $(\mathbf{Q^i} \circ \cdots\cdots \circ \mathbf{Q^2} \circ \mathbf{Q^1})^{-1}$ *which fixes* 0 *is then also well defined and univalent on* U_{R_i}.

(vi) *For each* $1 \leq i \leq 2n-1$, $\mathbf{Q^i}$ *is univalent on* $U_{S_{i-1}}$ *and*

$$\mathbf{Q^i}(U_{S_{i-1}}) \subset U_{S_i}.$$

Thus, $\mathbf{Q^i} \circ \cdots\cdots \circ \mathbf{Q^1}$ *is univalent on* $U_{1/20}$ *and*

$$\mathbf{Q^i} \circ \cdots\cdots \circ \mathbf{Q^1}(U_{1/20}) \subset U_{S_i} \subset U_{1/10}.$$

(vii) *If* $i = 2k$ *with* $1 \leq k \leq n-1$ *is even, and* $z \in U_{R_{i-1}-\delta(\varepsilon_k)}$,

$$\rho_U(\mathbf{Q^i}(z), (\mathbf{Q^{i-1}} \circ \cdots\cdots \circ \mathbf{Q^1})^{-1}(z)) < \frac{\varepsilon_{k+1}}{3C},$$

where we use the same branch of $(\mathbf{Q^{i-1}} \circ \cdots\cdots \circ \mathbf{Q^1})^{-1}$ *which fixes* 0 *from part (v) above.*

For the final two hypotheses, let $i = 2k+1$ *with* $0 \leq k \leq n-1$ *be odd.*

(viii) *If* $z \in U_{R_i}$, *using the same inverse branch mentioned in statement (v), we have*

$$\rho_U((\mathbf{Q^i} \circ \cdots\cdots \circ \mathbf{Q^1})^{-1}(z), z) < \varepsilon_{k+1}.$$

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(ix) *If, for each* $1 \le m \le J_i$, $\mathbf{Q^i_m}$ *denotes the partial composition of the first m quadratics of* $\mathbf{Q^i}$, *then for all* $f \in \mathcal{S}$, *there exists* $1 \le m \le J_i$, *such that, for all* $z \in U_{1/20}$, *we have*

$$\rho_U(\mathbf{Q^i_m} \circ \mathbf{Q^{i-1}} \circ \cdots \cdots \circ \mathbf{Q^1}(z), f(z)) < \varepsilon_{k+1}.$$

Remarks.

(1) Statements (i)–(iii) are designed for keeping track of the domains on which estimates are holding and, in particular, to ensure that these domains do not get too small and that the constants ε_i which keep track of the accuracy of the approximations do indeed tend to 0. The outer radii R_i are chosen primarily so that the image of U_{R_i} under the inverse branch of $\mathbf{Q^i}$ which fixes 0 is contained in $U_{R_{i-1}}$ (this is statement (v) above). This allows us to compose the inverses of these compositions and then approximate this composition of inverses by means of Phase II. The inner radii S_i are chosen primarily so that the image of $U_{S_{i-1}}$ under the polynomial composition $\mathbf{Q^i}$ lies inside U_{S_i} (this is statement (vi) above). This allows us to compose these polynomial compositions and gives us our iterates which remain bounded and approximate the elements of $\mathcal{S}$.

(2) Statement (vii) is a 'Phase II' statement regarding error correction using Phase II of the inverse of an earlier polynomial composition. Effectively, the Phase II correction compensates for the error in the previous Phase I composition, whose deviation from the identity is measured in statement (viii) above.

(3) Statements (viii) and (ix) are 'Phase I' statements. Statement (viii) is a bound on the error to be corrected by the next Phase II approximation. Statement (ix) is the key element for proving Theorem 1.3.

(4) It follows readily from statement (i) that the sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ converges to 0 exponentially fast, which gives item (a) in the statement of the lemma. Item (b) follows from statement (iv) and our choice of κ_0, while items (c) and (d) follow from statement (ii).

(5) Part (1) of the second part of the statement of the lemma follows from statement (iii) above while part (2) of the statement follows from statement (iv). Lastly, part (3) follows from statement (vi), while part (4) follows from statement (ix).

Proof. Base case: $n = 1$. Recall the bound $\tilde{\varepsilon}_1 > 0$ and function $\delta(x)$ defined on $(0, \tilde{\varepsilon}_1]$ whose existence is given by Phase II (recall that we have fixed the values of h_0, r_0, R_0 at the start of the proof) and that $\delta(x) \to 0$ as $x \to 0_+$. We can then pick $0 < \varepsilon_1 \le \tilde{\varepsilon}_1$ such that if we set

$$\eta_1 = \frac{4}{3}\varepsilon_1 + \delta(\varepsilon_1),$$
$$\sigma_1 = \frac{4}{3}\varepsilon_1,$$

then we can ensure that $0 < \varepsilon_1 < \sigma_1 < \eta_1 < 1/(40 \cdot 2) = 1/80$. This verifies statement (i). Now recall that we already set $R_0 = \frac{1}{4}$, let $S_0 = 1/20$, and set

$$R_1 = \frac{1}{4} - \frac{\varepsilon_1}{3},$$
$$S_1 = \frac{1}{20} + \frac{\varepsilon_1}{3},$$

which verifies statement (ii) and then statement (iii) follows easily.

Applying Lemma 2.1, we choose an $\varepsilon_1/3$-net $\{f_0, f_1, \ldots, f_{N_1+1}\}$ for $\mathcal{S}$ (consisting of elements of $\mathcal{S}$) on $U_{1/2}$, where $N_1 = N_1(\varepsilon_1) \in \mathbb{N}$, and with $f_0 = f_{N_1+1} = \mathrm{Id}$. Apply Phase I (Lemma 4.8) for this collection of functions with $R_0 = \frac{1}{4}$, $\varepsilon = \varepsilon_1/3$, to obtain $M_1 \in \mathbb{N}$ and a $(17 + \kappa_0)$-bounded finite sequence $\{P_m\}_{m=1}^{(N_1+1)M_1}$ of quadratic polynomials both of which depend directly on R_0, κ_0, N_1, the functions $\{f_i\}_{i=0}^{N_1+1}$, and ε, and thus ultimately on ε_1 and $\{f_i\}_{i=0}^{N_1+1}$ (recall that we set $h_0 = 1$, $r_0 = 1/20$, $R_0 = \frac{1}{4}$, as well as $\kappa_0 = \kappa_0(1/4)$ at the start of the proof in equation (6)) such that, for $1 \le i \le N_1 + 1$, if we let $\mathbf{Q}^1_m$, $1 \le m \le J_1$ denote the composition of the first m polynomials of this sequence, we have:

(1) $\mathbf{Q^1_{iM_1}}(0) = 0$;
(2) $\mathbf{Q^1_{iM_1}}$ is univalent on $U_{1/2}$;
(3) $\rho_U(f_i(z), \mathbf{Q^1_{iM_1}}(z)) < \varepsilon_1/3$ on $U_{1/2}$;
(4) $\|(\mathbf{Q^1_{iM_1}})^\natural\|_{U_{1/4}} \le C$.

Now set $\mathbf{Q^1} = Q_{(N_1+1)M_1}$. By item (1), $\mathbf{Q^1}(0) = 0$ and, as Phase I guarantees $\mathbf{Q^1}$ is $(17 + \kappa_0)$-bounded, on setting $J_1 = (N_1 + 1)M_1$, statement (iv) is verified.

Now we have that each $\mathbf{Q^1_{iM_1}}$ is univalent on $U_{1/2} \supset \overline{U}_{1/4} = \overline{U}_{R_0}$ by item (2) above. Further, by item (3), if $\rho_U(0, z) = \frac{1}{4}$, then $\rho_U(\mathbf{Q^1}(z), z) < \varepsilon_1/3$, so by item (1) and the Jordan curve argument (Lemma 6.1), $\mathbf{Q^1}(U_{R_0}) \supset U_{R_1}$. The branch of $(\mathbf{Q^1})^{-1}$ which fixes 0 is then well defined and univalent on U_{R_1} and maps this set inside U_{R_0}. With this, we have verified statement (v).

Likewise, if $\rho_U(0, z) < 1/20$, then $\rho_U(\mathbf{Q^1}(z), 0) < (1/20) + (\varepsilon_1/3)$. This implies $\mathbf{Q^1}(U_{S_0}) \subset U_{S_1}$ and, since by statement (iii), $S_1 < 1/10$ while by item (2) above, $\mathbf{Q^1}$ is univalent on $U_{1/2} \supset U_{1/20}$, which verifies statement (vi). We observe that hypothesis (vii) is vacuously true as it is concerned only with Phase II.

Now let $z \in U_{R_1}$. Using the same branch of $(\mathbf{Q^1})^{-1}$ as in statement (v), it follows from statement (v) that we can write $z = \mathbf{Q^1}(w)$ for some $w \in U_{R_0}$ and that by item (2) above, this w is unique. Since $f_{N_1+1} = \mathrm{Id}$, it follows from item (3) above that

$$\begin{aligned}\rho_U((\mathbf{Q^1})^{-1}(z), z) &= \rho_U(w, \mathbf{Q^1}(w)) \\ &< \frac{\varepsilon_1}{3} \\ &< \varepsilon_1\end{aligned}$$

which verifies statement (viii).

Finally, let $z \in U_{1/20}$. For $f \in \mathcal{S}$, let f_i be a member of the net for which $\rho_U(f(w), f_i(w)) < \varepsilon_1/3$ on $U_{1/2} \supset U_{1/20}$, and, using item (3), let $\mathbf{Q^1_{iM_1}}$ be a partial

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composition which satisfies $\rho_U(\mathbf{Q}^{\mathbf{1}}_{\mathbf{m}}(w), f_i(w)) < \varepsilon_1/3$ on $U_{1/2} \supset U_{1/20}$ Then, on setting $m = iM_1$,

$$\begin{aligned}\rho_U(\mathbf{Q}^{\mathbf{1}}_{\mathbf{m}}(z), f(z)) &\le \rho_U(\mathbf{Q}^{\mathbf{1}}_{\mathbf{m}}(z), f_i(z)) + \rho_U(f_i(z), f(z))\\ &\le \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3}\\ &< \varepsilon_1,\end{aligned}$$

which verifies statement (ix) and completes the base case.

Induction hypothesis: Assume statements (i)–(ix) hold for some arbitrary $n \ge 1$.

Induction step: We now show this is true for $n+1$.

Since the above hypotheses hold for n, we have already defined $R_{2n-1} = R_{2n-2} - (\varepsilon_n/3)$. Using statement (viii) for n with $i = 2n-1$, we have

$$\rho_U((\mathbf{Q}^{\mathbf{2n-1}} \circ \cdots\cdots \circ \mathbf{Q}^{\mathbf{1}})^{-1}(z), z) < \varepsilon_n, \quad z \in U_{R_{2n-1}}, \tag{6.1}$$

where of course we are using the branch of $(\mathbf{Q}^{\mathbf{2n-1}} \circ \cdots\cdots \circ \mathbf{Q}^{\mathbf{1}})^{-1}$ from statement (v) which fixes 0.

Recalling that the function $\delta : (0, \tilde{\varepsilon}_1] \mapsto (0, (r_0/4))$ in Phase II (Lemma 5.17) has a limit of 0 from the right, we can pick $\varepsilon_{n+1} > 0$ sufficiently small such that $\varepsilon_{n+1} \le \tilde{\varepsilon}_1$, and if we set

$$\eta_{n+1} = \left(\frac{4}{3} + \frac{1}{3C}\right)\varepsilon_{n+1} + \delta(\varepsilon_{n+1}), \tag{6.2}$$

$$\sigma_{n+1} = \left(\frac{4}{3} + \frac{1}{3C}\right)\varepsilon_{n+1}, \tag{6.3}$$

then we can ensure

$$0 < \varepsilon_{n+1} < \sigma_{n+1} < \eta_{n+1} < \frac{1}{40 \cdot 2^{n+1}}, \tag{6.4}$$

which verifies statement (i) for $n+1$. If we now apply Phase II, with κ_0, h_0, r_0, R_0 as above, $R = R_{2n-1}$, $\varepsilon_1 = \varepsilon_n$ (recall that $\varepsilon_n \le \tilde{\varepsilon}_1$ in view of hypothesis (i) for n), $\varepsilon_2 = \varepsilon_{n+1}/3C$, and $\mathcal{E} = (\mathbf{Q}^{\mathbf{2n-1}} \circ \cdots \circ \mathbf{Q}^{\mathbf{1}})^{-1}$, and make use of equation (6.1), we can find a $(17+\kappa_0)$-bounded composition of quadratic polynomials $\mathbf{Q}^{\mathbf{2n}}$ which depends immediately on κ_0, ε_n, ε_{n+1}, R, and $\mathcal{E}$, and thus ultimately on $\{\varepsilon_k\}_{k=1}^{n+1}$ and $\{\mathbf{Q}^{\mathbf{i}}\}_{i=1}^{2n-1}$, such that $\mathbf{Q}^{\mathbf{2n}}$ is univalent on a neighborhood of $\overline{U}_{R_{2n-1}-\delta(\varepsilon_n)}$, satisfies $\mathbf{Q}^{\mathbf{2n}}(0) = 0$, and

$$\rho_U(\mathbf{Q}^{\mathbf{2n}}(z), (\mathbf{Q}^{\mathbf{2n-1}} \circ \cdots \circ \mathbf{Q}^{\mathbf{1}})^{-1}(z)) < \frac{\varepsilon_{n+1}}{3C}, \quad z \in \overline{U}_{R_{2n-1}-\delta(\varepsilon_n)}, \tag{6.5}$$

which verifies statement (vii) for $n+1$. Note that, because of the upper bound $\tilde{\varepsilon}_2$ in the statement of Phase II, we may need to make ε_{n+1} smaller, if necessary. However, this does not affect the estimates on η_{n+1} or σ_{n+1} or any of the other dependencies for $\mathbf{Q}^{\mathbf{2n}}$ above. Finally, $\mathbf{Q}^{\mathbf{2n}}(0) = 0$ from above, so that, if we let J_{2n} be the number of quadratics in $\mathbf{Q}^{\mathbf{2n}}$, we see that the first half of statement (iv) for $n+1$ is also verified. Now set

$$R_{2n} = R_{2n-1} - \varepsilon_n - \delta(\varepsilon_n) - \frac{\varepsilon_{n+1}}{3C}, \tag{6.6}$$

$$S_{2n} = S_{2n-1} + \varepsilon_n + \frac{\varepsilon_{n+1}}{3C}. \tag{6.7}$$

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We observe that the ε_n change in radius above is required in view of statement (viii) which measures how much the function $(\mathbf{Q^{2n-1}} \circ \cdots\cdots \circ \mathbf{Q^1})^{-1}$ which we are approximating moves points on $U_{R_{2n-1}-\delta(\varepsilon_n)}$, the $\delta(\varepsilon_n)$ change is the loss of domain incurred by Phase II, while the additional $\varepsilon_{n+1}/3C$ is to account for the error in the Phase II approximation (the factor of C arising from the fact that this error needs to be passed through a subsequent Phase I to verify statement (ix) for $n+1$).

A final observation worth making is that here we are dealing with a loss of radius in passing from R_{2n-1} to R_{2n} arising from two distinct sources—the initial loss of domain by an amount $\delta(\varepsilon_n)$ arising from the need to make a Phase II approximation, and the subsequent losses of ε_n and $\varepsilon_{n+1}/3C$ which arise via the Jordan curve argument (Lemma 6.1) due the amount that $\mathbf{Q^{2n}}$ moves points on $U_{R_{2n-1}-\delta(\varepsilon_n)}$ (for details, see equation (6.10) below as well as the discussions immediately preceding and succeeding this inequality).

One easily checks that using hypotheses (i) and (ii) for n that

$$
\begin{aligned}
R_{2n} &= \left(\frac{1}{4} - \left(\sum_{j=1}^{n-1} \eta_j\right) - \left(\frac{1}{3} + \frac{1}{3C}\right)\varepsilon_n\right) - \varepsilon_n - \delta(\varepsilon_n) - \frac{\varepsilon_{n+1}}{3C} \\
&= \frac{1}{4} - \left(\sum_{j=1}^{n} \eta_j\right) - \frac{\varepsilon_{n+1}}{3C},
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
S_{2n} &= \left(\frac{1}{20} + \left(\sum_{j=1}^{n-1} \sigma_j\right) + \left(\frac{1}{3} + \frac{1}{3C}\right)\varepsilon_n\right) + \varepsilon_n + \frac{\varepsilon_{n+1}}{3C} \\
&= \frac{1}{20} + \left(\sum_{j=1}^{n} \sigma_j\right) + \frac{\varepsilon_{n+1}}{3C},
\end{aligned}
\tag{6.9}
$$

which verifies the first half of statement (ii) for $n+1$. We also observe at this stage that one can verify that the total loss of radius on passing from R_{2n-2} to R_{2n} is $(\varepsilon_n/3) + (\varepsilon_n + \delta(\varepsilon_n) + \varepsilon_{n+1}/3C) = (4\varepsilon_n/3 + \delta(\varepsilon_n)) + \varepsilon_{n+1}/3C$, which explains the form of the constants η_i in statement (i) of the induction hypothesis. A similar argument also accounts for the other constants σ_i in statement (i). Further, clearly $R_{2n} \leq \frac{1}{4}$ and, using statement (iii) for n and equation (6.4),

$$
\begin{aligned}
R_{2n} &= \frac{1}{4} - \left(\sum_{j=1}^{n} \eta_j\right) - \frac{\varepsilon_{n+1}}{3C} \\
&> \frac{1}{4} - \left(\sum_{j=1}^{n} \frac{1}{40 \cdot 2^j}\right) - \frac{1}{40 \cdot 2^{n+1}} \\
&= \frac{1}{4} - \frac{1}{40}\left(1 - \frac{1}{2^n} + \frac{1}{2^{n+1}}\right) \\
&> \frac{1}{5}.
\end{aligned}
$$

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The calculation for S_{2n} is similar, and thus we have verified the first half of statement (iii) for $n+1$. Combining equations (6.1) and (6.5), we have, on $\overline{U}_{R_{2n-1}-\delta(\varepsilon_n)}$,

$$\begin{aligned}\rho_U(\mathbf{Q^{2n}}(z), z) &\leq \rho_U(\mathbf{Q^{2n}}(z), (\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z)) \\ &\quad + \rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) \\ &< \frac{\varepsilon_{n+1}}{3C} + \varepsilon_n. \end{aligned} \tag{6.10}$$

This, combined with the Jordan curve argument (Lemma 6.1), the fact that $\mathbf{Q_{2n}}(0) = 0$, and equation (6.6) implies that

$$\mathbf{Q^{2n}}(U_{R_{2n-1}-\delta(\varepsilon_n)}) \supset U_{R_{2n-1}-\delta(\varepsilon_n)-\varepsilon_n-(\varepsilon_{n+1}/3C)} = U_{R_{2n}}, \tag{6.11}$$

and, since from above $\mathbf{Q^{2n}}$ is univalent on a neighborhood of $\overline{U}_{R_{2n-1}-\delta(\varepsilon_n)}$, the branch of $(\mathbf{Q^{2n}})^{-1}$ which fixes 0 is well defined on $U_{R_{2n}}$ and maps this set inside $U_{R_{2n-1}-\delta(\varepsilon_n)} \subset U_{R_{2n-1}}$, which verifies the first half of statement (v) for $n+1$.

By equation (6.6), the first half of statement (iii) for $n+1$, and statement (iii) for n, $R_{2n-1} - \delta(\varepsilon_n) > R_{2n} > \frac{1}{5} > S_{2n-1}$ so that $\mathbf{Q^{2n}}$ is univalent on $U_{S_{2n-1}}$. It then follows using equations (6.7) and (6.10) that

$$\mathbf{Q^{2n}}(U_{S_{2n-1}}) \subset U_{S_{2n-1}+\varepsilon_n+\varepsilon_{n+1}/3C} = U_{S_{2n}}.$$

Since, by the first half of statement (iii) for $n+1$, $S_{2n} < 1/10$ and, together with statement (vi) for n, this verifies half of statement (vi) for $n+1$ and finishes the Phase II portion of the induction step.

Now again apply Lemma 2.1 to construct an $\varepsilon_{n+1}/3$-net $\{f_0, f_1, \ldots, f_{N_{n+1}+1}\}$ for $\mathcal{S}$ (which again consists of elements of $\mathcal{S}$) on $U_{1/2}$, where we obtain $N_{n+1} = N_{n+1}(\varepsilon_{n+1}) \in \mathbb{N}$ and require $f_0 = f_{N_{n+1}+1} = \mathrm{Id}$. We apply Phase I (Lemma 4.8) with $R_0 = \frac{1}{4}$ and $\varepsilon = \varepsilon_{n+1}/3$ for this collection of functions to obtain $M_{n+1} \in \mathbb{N}$, and a $(17+\kappa_0)$-bounded sequence of quadratic polynomials $\{P_m\}_{m=1}^{(N_{n+1}+1)M_{n+1}}$ both of which depend directly on R_0, κ_0, N_{n+1}, the functions $\{f_i\}_{i=0}^{N_{n+1}+1}$, and ε, and thus ultimately on ε_{n+1} and $\{f_i\}_{i=0}^{N_{n+1}+1}$.

Now let $J_{2n+1} = M_{n+1}(N_{n+1}+1)$ be the number of quadratics and denote similarly to before the composition of the first m of these quadratics by $\mathbf{Q_m^{2n+1}}$. By Phase I, these compositions satisfy, for each $1 \leq i \leq N_{n+1}+1$:

(1) $\mathbf{Q^{2n+1}_{iM_{n+1}}}(0) = 0$;

(2) $\mathbf{Q^{2n+1}_{iM_{n+1}}}$ is univalent on $U_{1/2}$;

(3) $\rho_U(f_i(z), \mathbf{Q^{2n+1}_{iM_{n+1}}}(z)) < \varepsilon_{n+1}/3$, $z \in U_{1/2}$;

(4) $\|(\mathbf{Q^{2n+1}_{iM_{n+1}}})^\natural\|_{U_{1/4}} \leq C$.

Now set $\mathbf{Q^{2n+1}} := \mathbf{Q^{2n+1}_{(N_{n+1}+1)M_{n+1}}}$. The polynomial composition $\mathbf{Q^{2n+1}}$ is then a $(17+\kappa_0)$-bounded composition of J_{2n+1} quadratic polynomials which by item (1) satisfies $\mathbf{Q^{2n+1}}(0) = 0$. This then verifies statement (iv) for $n+1$.

Next, we define

$$R_{2n+1} = R_{2n} - \frac{\varepsilon_{n+1}}{3}, \tag{6.12}$$

$$S_{2n+1} = S_{2n} + \frac{\varepsilon_{n+1}}{3}. \tag{6.13}$$

We observe that the $\varepsilon_{n+1}/3$ change in radius above is required in view of item (3) above. One easily checks, using the above and equations (6.8) and (6.9),

$$\begin{aligned}
R_{2n+1} &= \left(\frac{1}{4} - \sum_{j=1}^{n} \eta_j - \frac{\varepsilon_{n+1}}{3C}\right) - \frac{\varepsilon_{n+1}}{3} \\
&= \frac{1}{4} - \sum_{j=1}^{n} \eta_j - \left(\frac{1}{3} + \frac{1}{3C}\right)\varepsilon_{n+1}, \\
S_{2n+1} &= \left(\frac{1}{20} + \sum_{j=1}^{n} \sigma_j + \frac{\varepsilon_{n+1}}{3C}\right) + \frac{\varepsilon_{n+1}}{3} \\
&= \frac{1}{20} + \sum_{j=1}^{n} \sigma_j + \left(\frac{1}{3} + \frac{1}{3C}\right)\varepsilon_{n+1}.
\end{aligned}$$

Thus, we have verified statement (ii) for $n+1$ and a similar calculation (again using the first half of statement (iii) for $n+1$ and equation (6.4)) to that for verifying the first half of statement (iii) for $n+1$ allows us to complete the verification of statement (iii) for $n+1$.

By items (1) and (3) above applied to the function $f_{N_{n+1}+1} = \mathrm{Id}$, together with statement (iii) for $n+1$, equation (6.12), and Lemma 6.1, we have

$$\mathbf{Q^{2n+1}}(U_{R_{2n}}) \supset U_{R_{2n}-\varepsilon_{n+1}/3} = U_{R_{2n+1}}, \tag{6.14}$$

while $\mathbf{Q^{2n+1}}$ is univalent on a neighborhood of this set by item (2). Hence, the branch of $(\mathbf{Q^{2n+1}})^{-1}$ which fixes 0 is well defined and univalent on $U_{R_{2n+1}}$, and maps $U_{R_{2n+1}}$ inside $U_{R_{2n}}$ which then verifies statement (v) for $n+1$.

By item (2) above and the first half of statement (iii) for $n+1$, $\mathbf{Q^{2n+1}}$ is univalent on $U_{1/2} \supset U_{1/10} \supset U_{S_{2n}}$. Again by item (3) applied to the function $f_{N_{n+1}+1} = \mathrm{Id}$, statement (iii) for $n+1$, and equation (6.13), we see

$$\mathbf{Q^{2n+1}}(U_{S_{2n}}) \subset U_{S_{2n}+\varepsilon_{n+1}/3} = U_{S_{2n+1}}.$$

By statement (iii) for $n+1$, we have $U_{S_{2n+1}} \subset U_{1/10}$ and, together with statement (vi) for n, this verifies statement (vi) for $n+1$.

Now let $w \in U_{R_{2n}}$. Using the same branch of $(\mathbf{Q^{2n}})^{-1}$ which fixes 0 as in the first part of statement (v) for $n+1$, by equation (6.11), $(\mathbf{Q^{2n}})^{-1}(w) = \zeta$ for some (unique) $\zeta \in U_{R_{2n-1}-\delta(\varepsilon_n)}$ and thus $w = \mathbf{Q^{2n}}(\zeta)$. Then, by equation (6.5),

$$\begin{aligned}
&\rho_U((\mathbf{Q^{2n}} \circ \mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(w), w) \\
&\quad = \rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n}})^{-1}(w), w) \\
&\quad = \rho_U((\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(\zeta), \mathbf{Q^{2n}}(\zeta)) \\
&\quad < \frac{\varepsilon_{n+1}}{3C}.
\end{aligned} \tag{6.15}$$

By equation (6.14), if now $z \in U_{R_{2n+1}} \subset \mathbf{Q^{2n+1}}(U_{R_{2n}})$, then, using the same inverse branch as in statement (v) which fixes 0, $(\mathbf{Q^{2n+1}})^{-1}(z) = w$ for some (unique) $w \in U_{R_{2n}}$ and thus

$z = \mathbf{Q^{2n+1}}(w)$. By item (3) above and statement (iii) for $n+1$, since $f_{N_{n+1}+1} = \mathrm{Id}$,

$$\begin{aligned}\rho_U((\mathbf{Q^{2n+1}})^{-1}(z), z) &= \rho_U(w, \mathbf{Q^{2n+1}}(w)) \\ &< \frac{\varepsilon_{n+1}}{3}. \end{aligned} \tag{6.16}$$

Then, if we now take $z \in U_{R_{2n+1}}$ and we let $w = (\mathbf{Q^{2n+1}})^{-1}(z) \in U_{R_{2n}}$ again as above using the branch of $(\mathbf{Q^{2n+1}})^{-1}$ which fixes 0, using equations (6.15) and (6.16), we have

$$\begin{aligned}&\rho_U((\mathbf{Q^{2n+1}} \circ \mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1}(z), z) \\ &\quad = \rho_U((\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n+1}})^{-1}(z), z) \\ &\quad \le \rho_U((\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n+1}})^{-1}(z), (\mathbf{Q^{2n+1}})^{-1}(z)) \\ &\qquad + \rho_U((\mathbf{Q^{2n+1}})^{-1}(z), z) \\ &\quad = \rho_U((\mathbf{Q^{2n}} \circ \mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(w), w) + \rho_U((\mathbf{Q^{2n+1}})^{-1}(z), z) \\ &\quad < \frac{\varepsilon_{n+1}}{3C} + \frac{\varepsilon_{n+1}}{3} < \varepsilon_{n+1}. \end{aligned} \tag{6.17}$$

This verifies statement (viii).

Now by statements (iii), (vi) for n together with equation (6.6), we have

$$(\mathbf{Q^{2n-1}} \circ \cdots\cdots \circ \mathbf{Q^1})(U_{1/20}) \subset U_{S_{2n-1}} \subset U_{1/10} \subset U_{R_{2n}} \subset U_{R_{2n-1}-\delta(\varepsilon_n)}, \tag{6.18}$$

while, again by statement (vi) for n, the forward composition $\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1}$ is univalent on $U_{1/20}$. Lastly, applying statement (v) for n, we see that the branch of $(\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}$ which fixes 0 is well defined and univalent on $U_{R_{2n-1}} \supset \overline{U}_{R_{2n}-\delta(\varepsilon_n)}$. Combining these three observations, we have the cancellation property

$$(\mathbf{Q^{2n-1}} \circ \cdots\cdots \circ \mathbf{Q^1})^{-1} \circ (\mathbf{Q^{2n-1}} \circ \cdots\cdots \circ \mathbf{Q^1}) = \mathrm{Id} \quad \text{on } U_{1/20}. \tag{6.19}$$

Let $z \in U_{1/20}$ and set $\zeta = (\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})(z)$. Then from equation (6.19), we have $(\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(\zeta) = z$ while by equation (6.18), $\zeta \in U_{R_{2n-1}-\delta(\varepsilon_n)}$. We then calculate, using equation (6.5),

$$\begin{aligned}\rho_U(\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1}(z), z) &= \rho_U(\mathbf{Q^{2n}}(\zeta), (\mathbf{Q^{2n-1}} \circ \cdots \circ \mathbf{Q^1})^{-1}(\zeta)) \\ &< \frac{\varepsilon_{n+1}}{3C}. \end{aligned} \tag{6.20}$$

Now let $f \in \mathcal{S}$ be arbitrary. Let $f_i \in \mathcal{S}$ be an element of the $\varepsilon_{n+1}/3$-net which approximates f to within $\varepsilon_{n+1}/3$ on $U_{1/2} \supset U_{1/20}$. Let $\mathbf{Q^{2n+1}_{iM_{n+1}}}$ be a partial composition of $\mathbf{Q^{2n+1}}$ which approximates f_i to within $\varepsilon_{n+1}/3$ also on $U_{1/2} \supset U_{1/20}$ using item (3) above and let $m = iM_{n+1}$ so that $\mathbf{Q^{2n+1}_m} = \mathbf{Q^{2n+1}_{iM_{n+1}}}$.

Applying statement (vi) for $n+1$ gives us that $\mathbf{Q^{2n}} \circ \cdots \circ \mathbf{Q^1}(z) \in U_{1/10} \subset U_{1/4}$. Then, using the hyperbolic convexity of $U_{1/4}$ (which follows from Lemma 2.8), the hyperbolic M-L estimates (Lemma 2.9), equation (6.20), items (3), (4), and the fact that f_i approximates f, we have

$$\begin{aligned}&\rho_U(\mathbf{Q^{2n+1}_m} \circ \mathbf{Q^{2n}} \circ \cdots\cdots \circ \mathbf{Q^1}(z), f(z)) \\ &\quad \le \rho_U(\mathbf{Q^{2n+1}_m} \circ \mathbf{Q^{2n}} \circ \cdots\cdots \circ \mathbf{Q^1}(z), \mathbf{Q^{2n+1}_m}(z))\end{aligned}$$

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$$+ \rho_U(\mathbf{Q_m^{2n+1}}(z), f_i(z)) + \rho_U(f_i(z), f(z))$$
$$\leq C \cdot \frac{\varepsilon_{n+1}}{3C} + \frac{\varepsilon_{n+1}}{3} + \frac{\varepsilon_{n+1}}{3}$$
$$= \varepsilon_{n+1},$$

which verifies statement (ix). Note that the first term uses Lemmas 2.8, 2.9, equation (6.20), and item (4), the second uses item (3), and the third uses the net approximation. This completes the proof of the claim. □

Lemma 6.2 now follows. □

We are now finally in a position to prove the main result of this paper.

Proof of Theorem 1.3. Let $f \in \mathcal{S}$ be arbitrary. Let $\{P_m\}_{m=1}^{\infty}$ be the sequence of quadratic polynomials which exists in view of Lemma 6.3 and which is bounded by part (1) of the statement. By Proposition 1.2 and part (2) of the statement, $U_{1/20}$ is contained in a bounded Fatou component V for this sequence. By part (3) of the statement, there exists a subsequence $\{Q_{m_k}\}_{k=1}^{\infty}$ of $\{Q_m\}_{m=1}^{\infty}$ such that the sequence of compositions $\{Q_{m_k}\}_{k=1}^{\infty}$ converges locally uniformly to f on $U_{1/20}$. Since $\{Q_{m_k}\}_{k=1}^{\infty}$ is normal on V, we may pass to a further subsequence, if necessary, to ensure this subsequence of iterates will converge locally uniformly on all of V. By the identity principle, the limit must then be f. In fact, since every such convergent subsequence must have limit f, it follows readily that $\{Q_{m_k}\}_{k=1}^{\infty}$ converges locally uniformly to f on all of V. □

Finally, we arrive at the last result of this paper, Theorem 1.4. We note that the proof of this result is not simply a 'change of coordinates' applied to Theorem 1.3. While it is straightforward to make a change of coordinates to transform one function from the family $\mathcal{N}$ to a member of $\mathcal{S}$ which we can then approximate, there are, in general, many functions in $\mathcal{N}$, and each of these requires, in general, a different change of coordinates. We will use a density argument to approximate all the necessary changes of coordinates using a countable set. The proof therefore requires that one successfully integrates *two* approximation schemes, one for the changes of coordinates and the other for the approximations of suitable functions from $\mathcal{N}$ using Theorem 1.3, with the first approximation scheme operating on a longer time scale than the second. Essentially, the proof says that one has to first wait until one has approximately the right change of coordinates after which one chooses the right time when one also has approximately the right function from $\mathcal{S}$.

Proof of Theorem 1.4. Let $r > 0$ be such that $\mathrm{D}(z_0, r) \subset \Omega$. Then the function

$$g(w) = \frac{f(rw + z_0) - f(z_0)}{rf'(z_0)}, \quad w \in \mathbb{D} \tag{6.21}$$

belongs to $\mathcal{S}$, while f can clearly be recovered from g using the formula

$$f(z) = rf'(z_0)g\left(\frac{z - z_0}{r}\right) + f(z_0), \quad z \in \mathrm{D}(z_0, r). \tag{6.22}$$

Since $\mathcal{N}$ is locally bounded and all limit functions are non-constant, using Hurwitz's theorem e.g. [**Con78**, Theorem VII.2.5 and also Corollary IV.5.9], we can find $K \geq 1$

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such that, for all $f \in \mathcal{N}$, we have

$$\frac{1}{K} \leq |f'(z_0)| \leq K, \quad |f(z_0)| \leq K. \tag{6.23}$$

Then, if we let X be the subset of $\mathbb{C}^2$ given by $X = \{(f'(z_0), f(z_0)), f \in \mathcal{N}\}$, we can clearly pick a sequence $\{(\alpha_n, \beta_n)\}_{n=1}^{\infty}$ which densely approximates all of X and such that, for all n,

$$\frac{1}{2K} \leq |\alpha_n| \leq 2K, \quad |\beta_n| \leq 2K.$$

We next wish to apply a suitable affine conjugacy to the polynomial sequence $\{P_m\}_{m=1}^{\infty}$ of Theorem 1.3 to construct the sequence $\{\tilde{P}_m\}_{m=1}^{\infty}$ needed to prove the current result. To this end, define $\varphi_0(w) = rw + z_0$, and $\varphi_n(w) = r\alpha_n w + \beta_n$ for $n \geq 1$. Recall the compositions $\{\mathbf{Q^i}\}_{i=1}^{\infty}$ from Lemma 6.2 and that each $\mathbf{Q^i}$ was a $(17 + \kappa_0)$-bounded composition of J_i quadratic polynomials.

As we did before the statement of Lemma 6.3, for $i = 0$, set $T_0 = 0$ and, for each $i \geq 1$, set $T_i = \sum_{j=1}^{i} J_j$. Recall that these compositions $\{\mathbf{Q^i}\}_{i=1}^{\infty}$ then gave rise the the polynomial sequence $\{P_m\}_{m=1}^{\infty}$ of Lemma 6.3 and ultimately Theorem 1.3.

For $m = 1$, we define $\tilde{P}_1 = \varphi_1 \circ P_1 \circ \varphi_0^{-1}$. For $m > 1$, let $i \geq 1$ be the largest index such that $T_{i-1} < m$. For $i = 2k$ even, we define $\tilde{P}_m$ by

$$\tilde{P}_m = \begin{cases} \varphi_{k+1} \circ P_m \circ \varphi_k^{-1}, & m = T_{i-1} + 1, \\ \varphi_{k+1} \circ P_m \circ \varphi_{k+1}{}^{-1}, & T_{i-1} + 1 < m \leq T_i, \end{cases} \tag{6.24}$$

while for $i = 2k + 1$ odd, we set

$$\tilde{P}_m = \varphi_{k+1} \circ P_m \circ \varphi_{k+1}{}^{-1}, \quad T_{i-1} + 1 \leq m \leq T_i. \tag{6.25}$$

Then (whether i is even or odd), if as usual we let $Q_m = P_m \circ \cdots \cdots \circ P_2 \circ P_1$ and $\tilde{Q}_m = \tilde{P}_m \circ \cdots \cdots \circ \tilde{P}_2 \circ \tilde{P}_1$, then

$$\tilde{Q}_m = \varphi_{k+1} \circ Q_m \circ \varphi_0^{-1}. \tag{6.26}$$

Recall the Fatou component $V \supset U_{1/20} \ni 0$ from the proof of Theorem 1.3. Since the family $\{\varphi_n\}_{n=0}^{\infty}$ is bi-equicontinuous in the sense that the family $\{\varphi_n\}_{n=0}^{\infty}$ as well as the family of inverses $\{\varphi_n^{-1}\}_{n=0}^{\infty}$ are both equicontinuous and locally bounded on $\mathbb{C}$, it follows from [**Com03**, Proposition 2.1] that $W = \varphi_0(V)$ is a bounded Fatou component for the sequence $\{\tilde{P}_m\}_{m=1}^{\infty}$ which contains $\varphi_0(U_{1/20})$.

Let $\varepsilon > 0$. It follows from applying the local equivalence of the Euclidean and hyperbolic metrics from Lemma A.4 to items (a), (b), and part (4) of Lemma 6.2, there exists j_0 such that, for each $j \geq j_0$, there exists $\tilde{m}_j$, $1 \leq \tilde{m}_j \leq J_{2j+1}$ such that for $w \in U_{1/20}$, we have

$$|\mathbf{Q}^{\mathbf{2j+1}}_{\tilde{\mathbf{m}}_{\mathbf{j}}} \circ \mathbf{Q^{2j}} \circ \cdots \circ \mathbf{Q^1}(w) - g(w)| < \frac{\varepsilon}{2Kr}, \tag{6.27}$$

where, as before, $\mathbf{Q}^{\mathbf{2j+1}}_{\tilde{\mathbf{m}}_{\mathbf{j}}}$ denotes the partial composition of the first $\tilde{m}_j$ quadratics of $\mathbf{Q^{2j+1}}$.

Next, using the approximation property of the sequence $\{(\alpha_n \beta_n)\}_{n=1}^{\infty}$ to all of the set X above, we can find a subsequence $\{(\alpha_{n_k}, \beta_{n_k})\}_{j=1}^{\infty}$ which converges to $(f'(z_0), f(z_0))$.

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Hence, we can find k_0 such that for all $k \geq k_0$, if $|w| \leq 1/288 \leq 2/\kappa_0$, we have

$$|\varphi_{n_k}(w) - (rf'(z_0)w + f(z_0))| < \frac{\varepsilon}{2}. \tag{6.28}$$

Now let $z \in \varphi_0(U_{1/20})$ be arbitrary, and let k_0 be sufficiently large so that $n_{k_0} \geq j_0$. Then, for each $k \geq k_0$, if we let $i = 2n_k + 1$ so that $i = 2j + 1$, where $j = n_k$, so that by equation (6.26) and the construction of the sequence $\{P_m\}_{m=1}^{\infty}$ from just before Lemma 6.3,

$$\tilde{Q}_{T_{2n_k}+\tilde{m}_{n_k}} = \varphi_{n_k+1} \circ Q_{T_{2n_k}+\tilde{m}_{n_k}} \circ \varphi_0^{-1} = \varphi_{n_k+1} \circ \mathbf{Q}^{\mathbf{2n_k+1}}_{\tilde{\mathbf{m}}_{\mathbf{n_k}}} \circ \mathbf{Q}^{\mathbf{2n_k}} \circ \cdots \circ \mathbf{Q}^{\mathbf{1}} \circ \varphi_0^{-1}$$

and, using equations (6.22) and (6.26),

$$\begin{aligned}
&|\tilde{Q}_{T_{2n_k}+\tilde{m}_{n_k}}(z) - f(z)| \\
&\quad = |\varphi_{n_k+1} \circ Q_{T_{2n_k}+\tilde{m}_{n_k}} \circ \varphi_0^{-1}(z) - (rf'(z_0) \cdot g \circ \varphi_0^{-1}(z) + f(z_0))| \\
&\quad \leq |\varphi_{n_k+1} \circ Q_{T_{2n_k}+\tilde{m}_{n_k}} \circ \varphi_0^{-1}(z) - (rf'(z_0) \cdot Q_{T_{2n_k}+\tilde{m}_{n_k}} \circ \varphi_0^{-1}(z) + f(z_0))| \\
&\qquad + |(rf'(z_0) \cdot Q_{T_{2n_k}+\tilde{m}_{n_k}} \circ \varphi_0^{-1}(z) + f(z_0)) - (rf'(z_0) \cdot g \circ \varphi_0^{-1}(z) + f(z_0))|.
\end{aligned}$$

Recall that we chose $\kappa_0 \geq 576$ in Lemma 6.2. From this, it follows that $Q_{T_{2n_k}+\tilde{m}_{n_k}} \circ \varphi_0^{-1}(z) \in Q_{T_{2n_k}+\tilde{m}_{n_k}}(U_{1/20}) \subset Q_{T_{2n_k}+\tilde{m}_{n_k}}(V) \subset \mathrm{D}(0, (1/288))$ so that the first term on the right-hand side of the above is less than $\varepsilon/2$ in view of equation (6.28). In addition, it follows from equation (6.27) that the second term is bounded above by $r|f'(z_0)|(\varepsilon/2Kr) \leq \varepsilon/2$ in view of equation (6.23). Thus, if for $k \geq 1$, we set $m_k := T_{2n_k} + \tilde{m}_{n_k}$, then for $k \geq k_0$, we have

$$|\tilde{Q}_{m_k}(z) - f(z)| < \varepsilon$$

and, since $\varepsilon > 0$ was arbitrary, $\{\tilde{Q}_{m_k}\}_{k=1}^{\infty}$ converges uniformly to f on $\varphi_0(U_{1/20})$. The same argument using the identity principle as at the end of the proof of Theorem 1.3 shows that $\{\tilde{Q}_{m_k}\}_{k=1}^{\infty}$ converges locally uniformly on W to f and, as $f \in \mathcal{N}$ was arbitrary, this completes the argument. □

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A.1. *Appendix. Known results*

A.1.1. *Classical results on $\mathcal{S}$.* We now state some common results regarding the class $\mathcal{S}$. These can be found in many texts, in particular, [**CG93**]. Before we state the first result, let us establish some notation. Throughout, let $\mathbb{D}$ be the unit disk and let $\mathrm{D}(z, R)$ be the (open) Euclidean disk centered at z of radius R. The following is [**CG93**, Theorem I.1.3].

THEOREM A.1. (The Koebe one-quarter theorem) *If $f \in \mathcal{S}$, then $f(\mathbb{D}) \supset \mathrm{D}(0, \frac{1}{4})$.*

Also of great importance are the well-known distortion theorems [CG93, Theorem I.1.6].

THEOREM A.2. (The distortion theorems) *If* $f \in \mathcal{S}$, *then*

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3},$$
$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

The above implies immediately that $\mathcal{S}$ is a normal family in view of Montel's theorem. More precisely, we have the following [CG93, Theorem I.1.10].

COROLLARY A.3. *The family* $\mathcal{S}$ *is normal, and the limit of any sequence in* $\mathcal{S}$ *belongs to* $\mathcal{S}$.

A.1.2. *The hyperbolic metric.* One of the key tools we will be using is the following relationship between the hyperbolic and Euclidean metrics (see [CG93, Theorem I.4.3]).

LEMMA A.4. *Let* $D \subsetneq \mathbb{C}$ *be a simply connected domain and let* $z \in D$. *Then,*

$$\frac{1}{2}\frac{|dz|}{\delta_D(z)} \le d\rho_D(z) \le 2\frac{|dz|}{\delta_D(z)}.$$

We remark that there is also a more general version of this theorem for hyperbolic domains in $\mathbb{C}$ which are not necessarily simply connected (again see [CG93, Theorem I.4.3]). However, for the purposes of this paper, we will consider only simply connected domains which are proper subsets of $\mathbb{C}$. The advantage of this is that there is always a unique geodesic segment joining any two distinct points, and we can use the length of this segment to measure hyperbolic distance.

A.1.3. *Star-shaped domains.* Recall that a domain $D \subset \mathbb{C}$ is said to be *star-shaped* with respect to some point $z_0 \in D$ if, for every point $z \in D$, $[z_0, z] \subset D$, where $[z_0, z]$ denotes the Euclidean line segment from z_0 to z. We have the following classical result which will be of use to us later in the 'up' section of the proof of Phase II (Lemma 5.17).

LEMMA A.5. [Dur83, Corollary to Theorem 3.6] *For every radius* $r \le \rho := \tanh(\pi/4) = 0.655\ldots$, *each function* $f \in \mathcal{S}$ *maps the Euclidean disc* $\mathrm{D}(0, r)$ *to a domain which is starlike with respect to the origin. This is false for every* $r > \rho$.

Since this value of r corresponds via the formula $\rho_{\mathbb{D}}(0, z) = \log(1 + |z|)/(1 - |z|)$ to a hyperbolic radius about 0 of exactly $\pi/2$, we have the following easy consequence.

LEMMA A.6. *If* f *is univalent on* $\mathbb{D}$, $z_0 \in \mathbb{D}$, $r \le \pi/2$, *and* $\Delta_{\mathbb{D}}(z_0, r)$ *denotes the hyperbolic disc in* $\mathbb{D}$ *of radius* r *about* z_0, *then the image* $f(\Delta_{\mathbb{D}}(z_0, r))$ *is star-shaped with respect to* $f(z_0)$.

The important property of star-shaped domains for us is that, if we dilate such a domain about its center point by an amount greater than 1, then the enlarged domain will cover the

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original. More precisely, if X is star-shaped with respect to z_0, $r > 1$, and we let rX be the domain $rX := \{z : (z - z_0)/r + z_0 \in X\}$, then $X \subset rX$. Again, this is something we will make use of in the 'up' portion of the proof of Phase II (Lemma 5.17).

A.1.4. *The Carathéodory topology.* The Carathéodory topology is a topology on pointed domains, which consist of a domain and a marked point of the domain which is referred to as the base point. In [**Car52**], Constantin Carathéodory defined a suitable topology for simply connected pointed domains for which convergence in this topology is equivalent to the convergence of suitably normalized inverse Riemann maps. The work was then extended in an appropriate sense to hyperbolic domains by Adam Epstein in his Ph.D thesis [**Eps93**]. This work was expanded upon further still by the first author [**Com13a**, **Com14**]. This is a supremely useful tool in non-autonomous iteration where the domains on which certain functions are defined may vary. We follow [**Com13a**] for the following discussion. Recall that a *pointed domain* is an ordered pair (U, u) consisting of an open connected subset U of $\hat{\mathbb{C}}$, (possibly equal to $\hat{\mathbb{C}}$ itself) and a point u in U.

Definition A.7. We say that $(U_m, u_m) \to (U, u)$ in the *Carathéodory topology* if:

(1) $u_m \to u$ in the spherical topology;

(2) for all compact sets $K \subset U$, $K \subset U_m$ for all but finitely many m;

(3) for any *connected* (spherically) open set N containing u, if $N \subset U_m$ for infinitely many m, then $N \subset U$.

We also wish to consider the degenerate case where $U = \{u\}$. In this case, condition (2) is omitted (U has no interior of which we can take compact subsets) while condition (3) becomes

(3) for any *connected* (spherically) open set N containing u, N is contained in at most finitely many of the sets U_m.

Convergence in the Carathéodory topology can also be described using the *Carathéodory kernel*. Originally defined by Carathéodory himself in [**Car52**], one first requires that $u_m \to u$ in the spherical topology. If there is no open set containing u which is contained in the intersection of all but finitely many of the sets U_m, then one defines the kernel of the sequence $\{(U_m, u_m)\}_{m=1}^{\infty}$ to be $\{u\}$. Otherwise, one defines the Carathéodory kernel as the largest domain U containing u with the property (2) above. It is easy to check that a largest domain does indeed exist. Carathéodory convergence can also be described in terms of the Hausdorff topology. We have the following theorem in [**Com13a**].

THEOREM A.8. *Let $\{(U_m, u_m)\}_{m=1}^{\infty}$ be a sequence of pointed domains and (U, u) be another pointed domain where we allow the possibility that $(U, u) = (\{u\}, u)$. Then the following are equivalent:*

(1) *$(U_m, u_m) \to (U, u)$;*

(2) *$u_m \to u$ in the spherical topology and $\{(U_m, u_m)\}_{m=1}^{\infty}$ has Carathéodory kernel U as does every subsequence;*

(3) *$u_m \to u$ in the spherical topology and, for any subsequence where the complements of the sets U_m converge in the Hausdorff topology (with respect to the spherical*

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metric), U corresponds with the connected component of the complement of the Hausdorff limit which contains u (this component being empty in the degenerate case $U = \{u\}$).

Of particular use to us will be the following theorem in [**Com13a**] regarding the equivalence of Carathéodory convergence and the local uniform convergence of suitably normalized covering maps, most of which was proved by Adam Epstein in his PhD thesis [**Eps93**].

THEOREM A.9. *Let $\{(U_m, u_m)\}_{m\geq 1}$ be a sequence of pointed hyperbolic domains and for each m, let π_m be the unique normalized covering map from $\mathbb{D}$ to U_m satisfying $\pi_m(0) = 0$, $\pi_m'(0) > 0$.*

Then (U_m, u_m) converges in the Carathéodory topology to another pointed hyperbolic domain (U, u) if and only if the mappings π_m converge with respect to the spherical metric uniformly on compact subsets of $\mathbb{D}$ to the covering map π from $\mathbb{D}$ to U satisfying $\pi(0) = u$, $\pi'(0) > 0$.

In addition, in the case of convergence, if D is a simply connected subset of U and $v \in D$, then locally defined branches ω_m of π_m^{-1} on D for which $\omega_m(v)$ converges to a point in $\mathbb{D}$ will converge locally uniformly with respect to the spherical metric on D to a uniquely defined branch ω of π^{-1}.

Finally, if π_m converges with respect to the spherical topology locally uniformly on $\mathbb{D}$ to the constant function u, then (U_m, u_m) converges to $(\{u\}, u)$.

A.2. *Appendix. Glossary of symbols*

We will be using many different symbols repeatedly throughout this exposition. For clarity of exposition, we have gathered them into the following table.

Symbol	Description	Defined in §
λ	$\lambda = e^{2\pi i((\sqrt{5}-1)/2)}$: the irrational multiplier for the fixed point 0 in the Siegel disc polynomial	1.2.1, 3.1, 4.1, 5, 5.1
P_λ	$P_\lambda(z) = \lambda z(1 - z)$: the unscaled Siegel disc polynomial	1.2.1, 3.1, 4.1, 4.1, 5, 5.1
$\mathcal{K}_\lambda$	The filled Julia set for P_λ	3.1
U_λ	The Siegel disc for P_λ	1.2.1, 3.1, 4.1
$\tilde{U}_R$	The disc of hyperbolic radius $R > 0$ about 0 in U	4.1, 5.1
$\tilde{r}_0$	$\tilde{r}_0 := \mathrm{dist}(\partial \tilde{U}_R, \partial U_\lambda)$: the Euclidean distance between ∂U_λ and $\partial \tilde{U}$	4.1
ψ_λ	The unique Riemann map from U_λ to $\mathbb{D}$ satisfying $\psi_\lambda(0) = 0$ and $\psi_\lambda'(0) > 0$	4.1
κ	Scaling factor	3.1, 4.1, 4.1, 4.2, 5.1
P	$P(z) = (1/\kappa)P_\lambda(\kappa z) = \lambda z(1 - \kappa z)$: the scaled version of P_λ	1.2.1, 3.1, 4.1

(*Continued*)

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Symbol	Description	Defined in §
$\mathcal{K}$	The filled Julia set for P	1.2.1, 3.1, 4.1
U	The Siegel disc for P	1.2.1, 3.1, 4.1
U_R	The disc of hyperbolic radius $R > 0$ about 0 in U	4.1
r_0	$r_0 := \mathrm{dist}(\partial U_R, \partial U)$: the Euclidean distance between ∂U_R and ∂U	4.1
ψ	The unique Riemann map from U to $\mathbb{D}$ satisfying $\psi(0) = 0$ and $\psi'(0) > 0$	4.1, 5.1
G	$G(z)$: the Green's function for P	3.1, 5, 5.1
V_h	The Green's domain $\{z \in \mathbb{C} \ : \ G(z) < h\}$ (for $h > 0$)	5.1
V_{2h}	The Green's domain $\{z \in \mathbb{C} \ : \ G(z) < 2h\}$ (for $h > 0$)	5.1, 5.1
$\tilde{R}$	$\tilde{R} := R^{\mathrm{int}}_{(V_{2h},0)} U_R$: the internal hyperbolic radius of U_R in V_{2h} about 0	5.1
$\tilde{V}_{2h}$	$\tilde{R} := \tilde{V}_{2h} := \Delta_{V_{2h}}(0, \tilde{R})$: the hyperbolic disc of radius of $\tilde{R}$ in V_{2h} about 0	5.1
φ_{2h}	The unique Riemann map from $\tilde{V}_{2h}$ to V_{2h} satisfying $\varphi_{2h}(0) = 0$, $\varphi'_{2h}(0) > 0$.	5.1, 5.2
ψ_{2h}	The unique Riemann map from V_{2h} to $\mathbb{D}$ satisfying $\psi_{2h}(0) = 0$, $\psi'_{2h}(0) > 0$.	5.1, 5.2
R'	$R' := R^{\mathrm{int}}_{(U,0)} \tilde{V}_{2h}$: the internal hyperbolic radius of $\tilde{V}_{2h}$ in U about 0	5.2, 5.2, 5.2
$\tilde{U}$	$\tilde{U} := \varphi_{2h}^{-1}(U)$: the inverse image of U under φ_{2h}	5.2
R''	$R'' := R^{\mathrm{int}}_{(U,0)} \tilde{U}$: the internal hyperbolic radius of $\tilde{U}$ in U about 0	5.2, 5.2, 5.2

A.3. *Appendix. Dependency tables*

The proofs of the three key steps in this paper, namely the Polynomial Implementation Lemma (Lemma 3.9), Phase I (Lemma 4.8), and Phase II (Lemma 5.17) involve many quantities and functions which are defined in terms of other quantities introduced earlier (and occasionally later) in the proofs of these results. To fully understand these quantities and avoid any danger of circular reasoning, we feel it is therefore important, if not indispensable, that we provide full tables for all three of these results detailing the dependencies of the most important objects in their statements and proofs (see Tables 1–3).

The objects in each table are for the most part listed in the order in which they appear in the proof of the corresponding result as well as the statements and proofs of the supporting lemmas which lead up to it. The tables for Polynomial Implementation Lemma (Lemma 3.9) and Phase I (Lemma 4.8) each have five columns. To determine the dependencies for a given object (given as ultimate dependency in the third column), one looks at the immediate dependencies (second column) for that row. One then reads off the dependencies for each object in this column from the column entries for ultimate dependencies (third column) for the earlier lines in the table for these objects, and the combined list of these dependencies for every object then forms the new entry in the third column.

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TABLE 1. Dependencies table for the Polynomial Implementation Lemma (Lemma 3.9).

Quantity/ Function	Immediate Dependency	Ultimate Dependency	Role	Defined in §
γ, Γ [$*$]			Boundary curves of Jordan domains Ω, Ω'	3.1, 3.1
f [$*$]	γ, Γ	γ, Γ	Function defined and univalent on a neighborhood of $\overline{\Omega}$	3.1, 3.1
F	γ, Γ, f	γ, Γ, f	Quasiconformal interpolation between f and Id	3.1, 3.1, 3.1
κ [$*$]			Scaling factor for P_λ	3.1
γ, Γ (scale fixed) [$*$]			Boundary curves of Jordan domains Ω, Ω'	3.1, 3.1
f (scale fixed) [$*$]	κ, γ, Γ	κ, γ, Γ	Function defined and univalent on a neighborhood of $\overline{\Omega}$	3.1, 3.1
F (scale fixed)	κ, γ, Γ, f	κ, γ, Γ, f	Quasiconformal interpolation between f and Id	3.1, 3.1
N [$*$]			Iterative time at which interpolation is constructed	3.1
$\{\psi_m^N\}_{m=0}^{\mathbb{N}}$	κ, γ, Γ, f, N	κ, γ, Γ, f, N	Quasiconformal changes of coordinates which fix 0	3.1
$\{P_m^N\}_{m=0}^{\mathbb{N}}$	κ, γ, Γ, f, N	κ, γ, Γ, f, N	Conjugated version of P which fixes 0	3.1
A [$*$]	κ, f	κ, f	Relatively compact set on which polynomial approximation is constructed.	3.2
δ [$*$]	f	f	Enlargement of hyperbolic neighborhood	3.2
M [$*$]	f	f	Bound on hyperbolic derivative	3.2
ε [$*$]			Error bound of polynomial approximation	3.2
k_0	κ, γ, Γ, f, A, δ, M, ε, $\{(\psi_0^{n_k})^{-1}\}_{k=1}^{\infty}$	κ, γ, Γ, f, A, δ, M, ε	Minimum number of compositions of P required for desired approximations	3.2
k_1 [$*$]	k_0	κ, γ, Γ, f, A, δ, M, ε	Length of finite polynomial sequence	3.2
$\{P_m^{n_{k_1}}\}_{m=1}^{n_{k_1}}$	κ, γ, Γ, f, A, δ, M, ε, k_0, k_1, $\{\psi_m^{n_{k_1}}\}_{m=0}^{n_{k_1}}$	κ, γ, Γ, f, A, δ, M, ε, k_1	Finite polynomial sequence possessing desired approximation properties	3.2

TABLE 2. Dependencies table for Phase I (Lemma 4.8).

Quantity/ Function	Immediate Dependency	Ultimate Dependency	Role	Defined in §
κ (first version) [$*$]			Scaling factor for P_λ	4.1
R_0 [$*$]			Radius of hyperbolic disc about 0 in U	4.1, 4.2
$\tilde{r}_0$ (function of R) [$*$]			$d(\partial \tilde{U}_R, \partial \tilde{U})$	4.1
r_0 (function of R) [$*$]	κ	κ	$d(\partial U_R, \partial U)$	4.1
κ_0	R_0	R_0	Minimum necessary scaling to control distortion	4.1, 4.1, 4.2
κ (redefined) [$*$]	κ_0	R_0	$\kappa \geq \kappa_0$	4.1, 4.1, 4.2
ε (initial version) [$*$]			Upper bound on error of approximation	4.2
$N+1$ [$*$]			Number of functions being approximated	4.2
$\{f_i\}_{i=1}^{N+1}$ [$*$]			The finite sequence of functions being approximated	4.2
ε (first redefinition) [$*$]	R_0	R_0	$\varepsilon < R_0$	4.2
M_N	κ, R_0, N, $\{f_i\}_{i=1}^{N+1}$, ε	κ, R_0, N, $\{f_i\}_{i=1}^{N+1}$, ε	Number of quadratics needed to approximate each $f_i \circ f_{i-1}^{-1}$	4.2
$\{P_m\}_{m=1}^{(N+1)M_N}$	κ, R_0, N, $\{f_i\}_{i=1}^{N+1}$, ε	κ, R_0, N, $\{f_i\}_{i=1}^{N+1}$, ε	Finite polynomial sequence with desired approximation properties	4.2
σ	R_0	R_0	Inf. of hyperbolic density σ_U on U_{4R_0}	4.2
δ_0	R_0	R_0	$d(\partial U_{R_0}, \partial U_{(3/2)R_0})$	4.2
ε (second redefinition) [$*$]	κ, R_0	κ, R_0	Continuity estimate for σ_U on U_{4R_0}	4.2

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TABLE 3. Dependencies table for Phase II (Lemma 5.17).

Quantity/ Function	Immediate Dependency	Intermediate Dependency	Ultimate Dependency	Role	Defined in §
κ [$*$]				Scaling factor for P_λ	5.1
r_0 [$*$]				Lower bound for hyperbolic radius	5.1, 5.1
d_0	κ, r_0		κ, r_0	$d(0, \partial U_R) \geq d_0$	5.1
h_0 [$*$]				Upper bound for value of Green's function for P	5.1, 5.1, 5.2
D_0	κ, h_0		κ, h_0	$\delta_{V_{2h}}(z) \leq D_0,\ z \in U, h \leq h_0$	5.1
ρ_0	d_0, D_0		h_0, r_0	$R_{(V_{2h},0)}(\tilde{V}_{2h}) \geq \rho_0$	5.1
V_{2h}	h		$\kappa, \varepsilon_1, h_0, r_0, R_0$	Domain bounded by Green's line for P with value $2h$	5.1
V_h	h		$\kappa, \varepsilon_1, h_0, r_0, R_0$	Domain bounded by Green's line for P with value h	5.1
$\check{R}$ (function of h)	h_0		h_0	$\check{R}(h) = R^{\text{ext}}_{(V_{2h},0)} V_h$	5.1
R_0 [$*$]				Upper bound for hyperbolic radius	5.1
R [$*$]	r_0, R_0		r_0, R_0	Hyperbolic radius of disc on which $\mathcal{E}$ is defined (value is fixed at the start of 'up')	5.1, 5.2, 5.2
$\mathcal{E}$ [$*$]	κ, R, r_0, R_0		κ, R, r_0, R_0	Error we wish to correct (fixed at the start of 'during')	5.2, 5.2
$\tilde{\varepsilon}_1$ (first version)	ρ_0		h_0, r_0	$\tilde{V}_{2h} \setminus \hat{N} \neq \emptyset$	5.1
ε_1 [$*$]	$\tilde{\varepsilon}_1$		h_0, r_0	$\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$	5.1, 5.1

(*Continued*)

TABLE 3. Dependencies table for Phase II (Lemma 5.17) – Continued.

Quantity/ Function	Immediate Dependency	Intermediate Dependency	Ultimate Dependency	Role	Defined in §
$\tilde{\varepsilon}_1$ (first restriction)	$\tilde{\varepsilon}_1$ (previous), κ, d_0		h_0, r_0	$\tilde{\varepsilon}_1 < \kappa d_0/8D_1$	5.1, 5.1
ε_1 (restricted) [$*$]	$\tilde{\varepsilon}_1$		h_0, r_0	$\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$	5.1, 5.1
T (function of ε_1)	κ, $\tilde{\varepsilon}_1$, d_0		h_0, r_0	$R^{\text{int}}_{(\tilde{V}_{2h},0)}(\tilde{V}_{2h} \setminus \hat{N}) \geq T(\varepsilon_1)$,	5.1, 5.1
$\tilde{\varepsilon}_1$ (second restriction)	$\tilde{\varepsilon}_1$ (previous), T, $\check{R}$, h_0		h_0, r_0	$\min_{0<\varepsilon_1\leq\tilde{\varepsilon}_1} T(\varepsilon_1) \geq$ $\min_{0<h\leq h_0} \check{R}(h)$	5.1, 5.1
ε_1 (restricted) [$*$]	$\tilde{\varepsilon}_1$		h_0, r_0	$\varepsilon_1 \in (0, \tilde{\varepsilon}_1]$	5.1
h (function of ε_1)	$\tilde{\varepsilon}_1$, T, $\check{R}$, h_0		h_0, r_0	$\tilde{V}_{h(\varepsilon_1)} \subset \tilde{V}_{2h(\varepsilon_1)} \setminus \hat{N}, \quad 0 < \varepsilon_1 \leq \tilde{\varepsilon}_1$	5.1, 5.1
δ (function of ε_1)	$\tilde{\varepsilon}_1$, h, r_0, R_0		h_0, r_0, R_0	Measures total loss of domain	5.2
$\tilde{\varepsilon}_1$ (third restriction)	$\tilde{\varepsilon}_1$ (previous), r_0, δ	$\tilde{\varepsilon}_1$ (previous), r_0, $\sup_{(0,\tilde{\varepsilon}_1]} \delta(\varepsilon_1)$	h_0, r_0, R_0	$U_{R-\delta(\varepsilon_1)} \neq \emptyset$	5.2
ε_1 (restricted) [$*$]	$\tilde{\varepsilon}_1$		h_0, r_0, R_0	Maximum size of error to be corrected in Phase II (fixed at the start of 'up')	5.2, 5.2
h (domain restricted)	$\tilde{\varepsilon}_1$, T		h_0, r_0, R_0	$\tilde{V}_{h(\varepsilon_1)} \subset \tilde{V}_{2h(\varepsilon_1)} \setminus \hat{N}, 0 < \varepsilon_1 \leq \tilde{\varepsilon}_1$	5.1, 5.1, 5.2
ψ	κ		κ	Normalized Riemann map from U to $\mathbb{D}$	5.1
ψ_{2h}	κ, h		κ, ε_1, h_0, r_0, R_0	Normalized Riemann map from V_{2h} to $\mathbb{D}$	5.1, 5.2

(*Continued*)

TABLE 3. Dependencies table for Phase II (Lemma 5.17) – Continued.

Quantity	Immediate Dependency	Intermediate Dependency	Ultimate Dependency	Role	Defined in §
φ_{2h}	κ, h, R		κ, ε_1, R, h_0, r_0, R_0	Normalized Riemann map from $\tilde{V}_{2h}$ to V_{2h}	5.1, 5.2
$\tilde{V}_{2h}$	h, R		κ, ε_1, R, h_0, r_0, R_0	Hyperbolic 'incircle' of U_R in V_{2h}	5.1
$\tilde{V}_h$	ψ_{2h}, h		κ, ε_1, R, h_0, r_0, R_0	$\psi_{2h}(V_h)$	5.1
R'	$\tilde{V}_{2h}$, R		ε_1, R, h_0, r_0, R_0	$R' = R^{\text{int}}_{(U,0)}\tilde{V}_{2h}$	5.2, 5.2, 5.2
R''	h, R, φ_{2h}	κ, ε_1, R, h_0, r_0, R_0	ε_1, R, h_0, r_0, R_0	$R'' = R^{\text{int}}_{(U,0)}\tilde{U}$	5.2, 5.2, 5.2
r_{2h}	$\tilde{V}_{2h}$, ψ_{2h}		ε_1, R, h_0, r_0, R_0	$\psi_{2h}(\tilde{V}_{2h}) = \mathrm{D}(0, r_{2h})$	5.2, 5.2
s	V_h, ψ_{2h}		ε_1, R, h_0, r_0, R_0	$\psi_{2h}(\overline{V}_h) \subset \mathrm{D}(0, s)$	5.2
N	r_{2h}, s		ε_1, R, h_0, r_0, R_0	$s\sqrt[N]{1/r_{2h}} < \sqrt{s}$	5.2
g	ψ_{2h}, r_{2h}, s, N		κ, ε_1, R, h_0, r_0, R_0	Conjugated expansion map	5.2
R_2	ε_1, R_0	ε_1, $\tilde{\varepsilon}_1$, R_0	ε_1, h_0, r_0, R_0	$R^{\text{ext}}_{(U,0)}\varphi_{2h}(U_{R''-\varepsilon_1}) \le R_2$	5.2
K_1				$K_1 = \frac{3}{2}$	5.2
N (redefined)	s, ψ_{2h}, r_{2h}, R_2	κ, ε_1, R, h_0, r_0, R_0	ε_1, R, h_0, r_0, R_0	$\Vert g^\natural\Vert_{\hat{A}} \le K_1$	5.2
g (redefined)	s, ψ_{2h}, r_{2h}, N, R_2,		κ, ε_1, R, h_0, r_0, R_0	$\Vert g^\natural\Vert_{\hat{A}} \le K_1$	5.2
$\tilde{\varepsilon}_2$ (first version)				$\tilde{\varepsilon}_2 = 1$	5.2
ε_2 [*]	$\tilde{\varepsilon}_2$		$\tilde{\varepsilon}_2$	$\varepsilon_2 \in (0, \tilde{\varepsilon}_2]$	5.2
$\mathbf{Q_1}$	κ, ε_2, K_1, K_2, K_3, R_2, h, g	κ, ε_1, ε_2, K_1, K_2, K_3, R, h_0, r_0, R_0	κ, ε_1, ε_2, R, h_0, r_0, R_0, $\mathcal{E}$	Approximates φ_{2h} on $U_{R''-3\varepsilon_1}$	5.2

(*Continued*)

TABLE 3. Dependencies table for Phase II (Lemma 5.17) – Continued.

Quantity	Immediate Dependency	Intermediate Dependency	Ultimate Dependency	Role	Defined in §
η_1, η_2	φ_{2h}, R_2	$\kappa, \varepsilon_1, R, h_0, r_0, R_0$	$\varepsilon_1, h_0, r_0, R_0$	$\eta_1 \leq \|(\varphi_{2h}^{-1})^\natural\|_{U_{R_2+2}} \leq \eta_2$	5.2
$\tilde{\varepsilon}_2$ (restricted)	$\kappa, \varepsilon_1, \eta_2$		$\varepsilon_1, h_0, r_0, R_0$	Maximum size of error of approximation using $\mathbf{Q}$	5.2
ε_2 [$*$]	$\tilde{\varepsilon}_2$		$\varepsilon_1, h_0, r_0, R_0$	Upper bound for error of approximation using $\mathbf{Q}$	5.2
$\hat{\mathcal{E}}$	$\kappa, \varepsilon_1, h, R, \varphi_{2h}, \mathcal{E}$		$\kappa, \varepsilon_1, R, h_0, r_0, R_0, \mathcal{E}$	Conjugated error $\hat{\mathcal{E}} = \varphi_{2h} \circ \mathcal{E} \circ \varphi_{2h}^{-1}$	5.2
K_2	$\varepsilon_1, R'', \varphi_{2h}, \hat{\mathcal{E}}$		$\varepsilon_1, R, h_0, r_0, R_0, \mathcal{E}$	$\|\hat{\mathcal{E}}^\natural(z)\| \leq K_2$ for $z \in \varphi_{2h}(U_{R''-2\varepsilon_1})$	5.2
$\mathbf{Q_2}$	$\kappa, \varepsilon_1, \varepsilon_2, K_2, K_3, \eta_2, \varphi_{2h}, R'', \hat{\mathcal{E}}$	$\kappa, \varepsilon_1, \varepsilon_2, K_2, K_3, R, h_0, r_0, R_0, \mathcal{E}$	$\kappa, \varepsilon_1, \varepsilon_2, R, h_0, r_0, R_0, \mathcal{E}$	Approximates the conjugated error $\hat{\mathcal{E}}$ on $\varphi_{2h}(U_{R''-3\varepsilon_1})$	5.2
K_3	η_2, R_2		$\varepsilon_1, h_0, r_0, R_0$	$\|(\varphi_{2h}^{-1})^\natural(z)\| \leq K_3$ for $z \in U_{R_2+2}$	5.2
$\mathbf{Q_3}$	$\kappa, K_3, \varepsilon_2, R_2, h, \varphi_{2h}$	$\kappa, \varepsilon_1, \varepsilon_2, K_3, R, h_0, r_0, R_0$	$\kappa, \varepsilon_1, \varepsilon_2, R, h_0, r_0, R_0$	Approximates φ_{2h}^{-1} on U_{R_2+1}	5.2
$\mathbf{Q}$	$\mathbf{Q_1}, \mathbf{Q_2}, \mathbf{Q_3}$		$\kappa, \varepsilon_1, \varepsilon_2, R, h_0, r_0, R_0, \mathcal{E}$	$\mathbf{Q} := \mathbf{Q_3} \circ \mathbf{Q_2} \circ \mathbf{Q_1}$	5.2

Due to the more complicated nature of the proof, the table for Phase II has an extra column for intermediate dependencies. However, the determination of the ultimate dependencies is done similarly to before where, instead, one looks at the ultimate dependencies (fourth column) for each quantity in the second and third columns for the row containing a given object and the combined list gives the entry for the ultimate dependencies for that object (which is in the fourth column). One exception to this is where, for objects depending on the constants K_2, K_3, one needs to look at later entries in the table for these constants (as explained in the proof of Lemma 5.17, there is no danger of circular reasoning here). For intermediate dependencies, when listed, these are the same as ultimate dependencies but involve extra quantities which are then eliminated by monotonicity or some uniformization procedure such as taking a maximum (e.g. η_1, η_2) or by being determined later as is the case with K_2, K_3. Another exception is when some of the ultimate dependencies appear to be 'missing'. Most of these are instances of where the scaling factor κ is omitted because an estimate involving the hyperbolic metric of U which will not depend on κ.

One final remark concerns those objects which appear in the columns for ultimate dependencies. As a matter of logical necessity, these fall into two categories—objects which are defined or whose value is determined before the proof (e.g. κ, R_0) and universally quantified objects appearing in the statement (e.g. ε_1, ε_2). Both of these are indicated in the tables by their appearance in the first column being marked by an asterisk. However, these objects also generally come with bounds which have dependencies of their own which then forces these objects to inherit these dependencies. A good way to think of this is to view these results of determining these ultimate dependencies as methods or routines within a computer program where the objects in the ultimate dependencies are then free objects which are either defined outside the method and then passed to it as parameters or alternatively set within the method itself.

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