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AUTOMORPHISMS OF CAYLEY GRAPHS OF METACYCLIC GROUPS OF PRIME-POWER ORDER

CAI HENG LI and HYO-SEOB SIM

To Laci Kovács on his 65th birthday

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Abstract

This paper investigates the automorphism groups of Cayley graphs of metacyclic p-groups. A characterization is given of the automorphism groups of Cayley graphs of a metacyclic p-group for odd prime p. In particular, a complete determination of the automorphism group of a connected Cayley graph with valency less than $2p$ of a nonabelian metacyclic p-group is obtained as a consequence. In subsequent work, the result of this paper has been applied to solve several problems in graph theory.

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1. Introduction

Let G be a finite group, and let S be a subset of G that does not contain the identity 1 of G. If $S = S^{-1} := \{s^{-1} \mid s \in S\}$, the graph with the vertex set G and the edge set $\{\{x, sx\} \mid x \in G,\ s \in S\}$ is called a Cayley graph of G and denoted by $\mathrm{Cay}(G, S)$.

The adjacency relation of the graph $\mathrm{Cay}(G, S)$ is uniquely determined by the group G and the subset S, and so are some simple properties of $\mathrm{Cay}(G, S)$, for example, $\mathrm{Cay}(G, S)$ is a regular graph with valency $|S|$, and $\mathrm{Cay}(G, S)$ is connected if and only if $\langle S\rangle = G$. However, to understand some further graph structure properties of the graph, for example, how symmetric the graph is, we often need to know the full

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automorphism group of $\mathrm{Cay}(G, S)$. By the definition, it is easy to see that the group G acts regularly on the vertex set G by right multiplication (that is, g acts on x as the product xg) and so G may be viewed as a regular subgroup of the automorphism group of the Cayley graph. In particular, the automorphism group of a Cayley graph acts transitively on the vertex set. But in general the problem of determining the full automorphism group of a Cayley graph is very difficult. Since a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is defined by G, a natural approach to the problem is to understand the relationship between the full automorphism group $\mathrm{Aut}\,\Gamma$ and G, for example, whether or not G, as a regular subgroup, is normal in $\mathrm{Aut}\,\Gamma$.

For convenience, a Cayley graph Γ will be called *normal* if the regular subgroup G is normal in $\mathrm{Aut}\,\Gamma$ (see [21]). The automorphism group of a normal Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is a semidirect product of the regular normal subgroup G by the subgroup $\mathrm{Aut}(G, S)$ which consists of all automorphisms of the group G that fix S setwise (see Lemma 2.1). The automorphisms of the graph Γ are therefore completely determined by automorphisms of the group G. Usually, the latter is much easier to be determined. Thus a natural problem is to determine normality of Cayley graphs for a given class of groups.

The problem determining normality of Cayley graphs of a given cyclic group of prime order was solved by Alspach [1]; some partial answers for other classes of groups to this problem can be found in several papers, for example [3, 6, 11, 20]. The main purpose of the paper is to characterize the automorphism groups of certain Cayley graphs for metacyclic groups of prime-power order, in view of normality.

For two groups G and H, let $G \rtimes H$ be a semidirect product of G by H. For a subset S of a group G, write

$$\mathrm{Aut}(G, S) := \{\theta \in \mathrm{Aut}(G) \mid \theta(S) = S\}.$$

The first result of this paper determines automorphism groups of Cayley graphs of a nonabelian metacyclic p-group in the case when p is an odd prime that does not divide the order of $\mathrm{Aut}(G, S)$.

THEOREM 1.1. *Let G be a finite nonabelian metacyclic p-group for an odd prime p, and let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of G. Assume that $\mathrm{Aut}(G, S)$ is a p'-group. Then either $\mathrm{Aut}\,\Gamma \cong G \rtimes \mathrm{Aut}(G, S)$, or $G \cong \mathbb{Z}_9 \rtimes \mathbb{Z}_{3^r}$ and $\mathrm{Aut}\,\Gamma \cong \mathrm{PSL}(2, 8) \rtimes \mathbb{Z}_{3^r}$, where $r \geq 1$.*

Note that the p'-group $\mathrm{Aut}(G, S)$ in the theorem is a cyclic group of order dividing $p-1$ (see [14, 15]). We then have a complete determination of the automorphism group of a connected Cayley graph with valency less than $2p$ for a nonabelian metacyclic p-group of odd order.

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COROLLARY 1.2. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a connected Cayley graph with valency less than* $2p$ *of a finite nonabelian metacyclic* p*-group* G *for an odd prime* p. *Then* $\mathrm{Aut}\,\Gamma \cong G \rtimes \mathrm{Aut}(G, S)$.

One of the motivations of studying normal Cayley graphs comes from some problems of graph theory. A graph is said to be *half-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive. Initiated with a question of Tutte [19, page 60], half-transitive graphs have received considerable attention for many years (see, for example, [2, 4, 16, 20]). In [13], the result of this paper has been applied to construct and characterize an interesting class of half-transitive graphs. A Cayley graph Γ of a group G is called a *graphical regular representation* of G if $\mathrm{Aut}\,\Gamma = G$. The problem of deciding whether a Cayley graph is a graphical regular representation of the corresponding group is a long-standing one, see [7]. The result given in Theorem 1.1 is used in [12] to solve the problem for metacyclic p-groups.

After we describe some background results in Section 2, we will prove Theorem 1.1 and Corollary 1.2 in Section 3.

2. Background results

We first setup some notation and terminology. Let G be a group. Denote by $\Phi(G)$ the Frattini subgroup of G. The product of all minimal normal subgroups of G is called the *socle* of G and is denoted by $\mathrm{Soc}(G)$. The automorphism group and the outer automorphism group of G are denoted by $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$, respectively. For two subgroups H and K of G, let $\mathbf{C}_H(K)$ denote the centralizer of K in H, and let $\mathbf{N}_H(K)$ denote the normalizer of K in H.

We now collect some basic results, which will be used in this paper.

LEMMA 2.1. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley graph of a finite group* G. *Then* $\mathbf{N}_{\mathrm{Aut}\,\Gamma}(G) = G \rtimes \mathrm{Aut}(G, S)$.

PROOF. Write $A := \mathrm{Aut}\,\Gamma$. The normalizer of G in the symmetric group $\mathrm{Sym}(G)$ is $G \rtimes \mathrm{Aut}(G)$ (see [5, Corollary 4.2B]). So we have

$$\mathbf{N}_A(G) = (G \rtimes \mathrm{Aut}(G)) \cap A = G \rtimes (\mathrm{Aut}(G) \cap A).$$

Obviously, $\mathrm{Aut}(G) \cap A = \mathrm{Aut}(G, S)$. □

We also note that a proof of this lemma may be found in [7, Lemma 2.1].

LEMMA 2.2. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a Cayley graph of a finite* p*-group* G. *If* $\mathrm{Aut}(G, S)$ *is a* p'*-group, then* G*, viewed as a regular subgroup, is a Sylow* p*-subgroup of* $\mathrm{Aut}\,\Gamma$.

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PROOF. Write $A = \mathrm{Aut}\,\Gamma$. Suppose that $\mathrm{Aut}(G, S)$ is a p'-group. Then $\mathbf{N}_A(G)/G$ is a p'-group. If G is not a Sylow p-subgroup of A, then G is a proper subgroup of a Sylow p-subgroup P of A. Thus $G < \mathbf{N}_P(G) \leq \mathbf{N}_A(G)$ (see [17, page 88]), which is a contradiction since $\mathbf{N}_A(G)/G$ is a p'-group. □

We also need the following facts about finite simple groups with a subgroup of prime-power index. First we prove a property about outer automorphisms and Schur multipliers of such simple groups.

LEMMA 2.3. *Let p be an odd prime. Let T be a nonabelian simple group which has a subgroup H of index $p^l > 1$, and let $M(T)$ be the Schur multiplier of T. Then*

(i) $p \nmid |M(T)|$;
(ii) *either* $p \nmid |\mathrm{Out}(T)|$ *or* $T \cong \mathrm{PSL}(2, 8)$ *and* $p^l = 3^2$.

PROOF. The finite nonabelian simple groups T with a subgroup H of prime-power index were classified by Guralnick in [9], and the Schur Multipliers of finite simple groups are completely classified, see Table 4.1 in [8, page 302]. Combining these two classifications, we only need to check the case that $T = \mathrm{PSL}(n, q)$ and $(q^n - 1)/(q - 1) = p^l$, where $q = r^f$ for some prime r and some positive integer f. It is known that

$$|M(T)| = d, \quad |\mathrm{Out}(T)| = \begin{cases} 2df & \text{if } n \geq 3, \\ df & \text{if } n = 2, \end{cases}$$

where $d = \gcd(n, q - 1)$. If $nf \leq 2$, then $T = \mathrm{PSL}(2, r)$ and $|\mathrm{Out}(T)| = 2$, so $p \nmid |\mathrm{Out}(T)|$ and $p \nmid |M(T)|$. Assume that $nf \geq 3$. If $r = 2$ and $nf = 6$, then it follows that $(q^n - 1)/(q - 1) = 3^2$ and so $T = \mathrm{PSL}(2, 8)$ and $|M(T)| = 1$ in this case. If $(r, nf) \neq (2, 6)$ then by Zigmondy Theorem (see [10, IX 8.3 and 8.4]), there is a (primitive) prime $k > nf$ such that $k \mid (r^{nf} - 1)$ but $k \nmid (r^f - 1)$. Thus $k \mid (q^n - 1)/(q - 1) = p^l$ and so $k = p$. In particular, $p > df$, and hence $p \nmid |\mathrm{Out}(T)|$ and $p \nmid |M(T)|$. □

The following lemma is an immediate consequence of Corollary 2 in Guralnick [9].

LEMMA 2.4. *Let T be a nonabelian simple group acting transitively on Ω with p^l elements for a prime p. If p does not divide the order of a point-stabilizer in T, then T acts 2-transitively on Ω.*

Finally, we observe a fact on transitive permutation groups of prime-power degree.

LEMMA 2.5. *Let p be a prime, and let A be a transitive permutation group on Ω of prime-power degree. Let B be a nontrivial subnormal subgroup of A. Then B has a proper subgroup of p-power index, and $\mathbf{O}_{p'}(B) = 1$. In particular, $\mathbf{O}_{p'}(A) = 1$.*

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PROOF. The assumption that B is subnormal in A means that there exists a series of subgroups $B \trianglelefteq B_1 \trianglelefteq \cdots \trianglelefteq B_k = A$. Let v be a point in Ω which is not fixed by B. Since A is transitive on Ω, B_{k-1} is half-transitive on Ω. Thus the B_{k-1}-orbit O_{k-1} containing v is of p-power size. Similarly, B_{k-2} is half-transitive on O_{k-1}, and thus the B_{k-2}-orbit O_{k-2} containing v is also of p-power size. Repeating this argument, we have that the B-orbit containing v is of p-power size, and so B has a proper subgroup of p-power index. So the subnormal subgroup $\mathbf{O}_{p'}(B)$ has a subgroup of p-power index, and hence $\mathbf{O}_{p'}(B) = 1$. In particular, taking $B = A$, we have that $\mathbf{O}_{p'}(A) = 1$. □

3. Proofs of the main results

In this section, we prove the main results, that is, Theorem 1.1 and Corollary 1.2. We will proceed the proofs with a series of lemmas. We recall that a metacyclic group is a group G which has a cyclic normal subgroup K such that G/K is cyclic. We notice that every subgroup and every quotient group of a metacyclic group are also metacyclic, and in particular, can be generated by at most two elements.

Let G be a finite nonabelian metacyclic p-group for an odd prime p, and let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of G. Let A denote the automorphism group of the Cayley graph Γ, and let A_1 denote the group of all automorphisms of Γ that fix the identity 1 of G. To prove Theorem 1.1, we assume that $\mathrm{Aut}(G, S)$ is a p'-group. Then by Lemma 2.2, G is a Sylow p-subgroups of $\mathrm{Aut}\,\Gamma$, or equivalently, p does not divide $|A_1|$.

LEMMA 3.1. *The graph Γ is not a complete graph.*

PROOF. Suppose that Γ is a complete graph, that is, $\Gamma = \mathrm{K}_n$, where $n = |G|$. Then $A = \mathrm{S}_n$, the symmetric group of degree n. However, a Sylow p-subgroup of S_n is not isomorphic to the nonabelian metacyclic group G, which is a contradiction. □

LEMMA 3.2. *If N be a nonabelian minimal normal subgroup of A, then $p = 3$ and $N \cong \mathrm{PSL}(2, 8)$.*

PROOF. Assume that N is non-abelian minimal normal subgroup. Then $N = T_1 \times \cdots \times T_k$, where $T_i \cong T$ for some nonabelian simple group T. By Lemma 2.5, $|N|$ is divisible by p. The normal subgroup N has a Sylow p-subgroup contained in G. Since G is metacyclic, it follows that k is at most 2. By Lemma 2.5, T_1 has a subgroup of p-power index. Thus by Lemma 2.3 (ii), either $p = 3$ and $T_1 = \mathrm{PSL}(2, 8)$ or p does not divide $|\mathrm{Out}(T_1)|$.

Assume first that p does not divide $|\mathrm{Out}(T_1)|$. Since $\mathbf{N}_A(T_1)/T_1\mathbf{C}_A(T_1)$ is isomorphic to a subgroup of $\mathrm{Out}(T_1)$, it follows that $|G|$ divides $|T_1\mathbf{C}_A(T_1)|$. Since T_1 is nonabelian simple, $T_1 \cap \mathbf{C}_A(T_1) = 1$, and hence the product $T_1\mathbf{C}_A(T_1)$ is a direct product. Suppose that $p \nmid |\mathbf{C}_A(T_1)|$. Then T_1 contains a Sylow p-subgroup of A, and hence T_1 is transitive on the vertex set G. By Lemma 2.4, noting that p does not divide $|A_1|$, T_1 is 2-transitive on the vertex set G. So Γ is a complete graph, which contradicts Lemma 3.1. Therefore, p divides $|\mathbf{C}_A(T_1)|$. Taking a Sylow p-subgroup P_1 of T_1 and a Sylow p-subgroup P_2 of $\mathbf{C}_A(T_1)$, we see that G is conjugate to $P_1 \times P_2$. Consequently, the P_i are cyclic, and so G is abelian, which is not the case.

Assume now that $p = 3$ and $T_1 = \mathrm{PSL}(2, 8)$. Consider the case where $k = 2$, namely $N = T_1 \times T_2$. As $N \cap \mathbf{C}_A(N) = 1$, $\langle N, \mathbf{C}_A(N)\rangle = N \times \mathbf{C}_A(N)$. By Lemma 2.5, if $\mathbf{C}_A(N) \neq 1$ then 3 divides $|\mathbf{C}_A(N)|$, and thus a Sylow 3-subgroup of $N \times \mathbf{C}_A(N)$ is isomorphic to $\mathbb{Z}_9 \times \mathbb{Z}_9 \times P$ for some nontrivial 3-group P, which is a contradiction since G is metacyclic. Hence $\mathbf{C}_A(N) = 1$. Write $B = \mathbf{N}_A(T_1)$. Then $B = \mathbf{N}_A(T_2)$ and B is a normal subgroup of A with index 2. Both of $\mathbf{C}_A(T_1)$ and $\mathbf{C}_A(T_2)$ are also normal in B. Since $\mathbf{C}_A(T_1) \cap \mathbf{C}_A(T_2) = \mathbf{C}_A(N) = 1$, it follows that B is isomorphic to a (subdirect) subgroup of $B/\mathbf{C}_A(T_1) \times B/\mathbf{C}_A(T_2)$, and so B is isomorphic to a subgroup of $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$. Let Q be a Sylow 3-subgroup of B. Then Q is also a Sylow 3-subgroup of A and $G \cap N$ is a normal subgroup of G isomorphic to $\mathbb{Z}_9 \times \mathbb{Z}_9$. Since G is nonabelian metacyclic, G has an element of order 27; however, there is no such an element in $\mathrm{Aut}(\mathrm{PSL}(2, 8))$, and so no such an element in $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$, a contradiction. Therefore, $k = 1$ and $N \cong \mathrm{PSL}(2, 8)$. □

We then have a consequence of Lemma 3.2.

LEMMA 3.3. *Either* $\mathrm{Soc}(A)$ *is soluble or* $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_{3^r}$ *and* $A = \mathrm{PSL}(2, 8) \rtimes \mathbb{Z}_{3^r}$, *where* $r \geq 1$.

PROOF. Suppose that $\mathrm{Soc}(A)$ is insoluble. Then by Lemma 3.2, $p = 3$ and A has a minimal normal subgroup N such that $N \cong \mathrm{PSL}(2, 8)$. Let $C = \mathbf{C}_A(N)$. Then A/C is isomorphic to a subgroup of $\mathrm{Aut}(N) \cong N \rtimes \mathbb{Z}_3$. As G is nonabelian metacyclic, it follows that $A/C \cong N \rtimes \mathbb{Z}_3$. So $A/C = L/C \rtimes B/C$, where $L/C \cong N$ and $B/C \cong \mathbb{Z}_3$. Since $L \cap N$ is normal in the simple group N, we see that $N \leq L$, and so $N \cap B = 1$. Thus $A = N \rtimes B$. Let P be a Sylow 3-subgroup of B. Then P is cyclic and $B = CP$. Let M be the normalizer of P in B. Then $M = (C \cap M)P$. Since $P/(C \cap P) \cong B/C \cong \mathbb{Z}_3$, we have $C \cap M \geq C \cap P = \Phi(P)$, the Frattini subgroup of P, and hence $M/\Phi(P) = (C \cap M)/\Phi(P) \times P/\Phi(P)$. So the subgroup $C \cap M$ acts trivially on $P/\Phi(P)$, which implies that $C \cap M$ acts trivially on P also. Thus P centralizes the normalizer M of the Sylow 3-subgroup P. It then follows from Burnside's Theorem for p-nilpotency that B is 3-nilpotent. Thus the normal Hall

$3'$-subgroup of B is a characteristic subgroup of C and so it is a normal p'-subgroup of A. By Lemma 2.5, we have $B = P$. Therefore, $A = N \rtimes P$, as desired. □

We will also prove the following lemmas.

LEMMA 3.4. *If* $\mathrm{Soc}(A)$ *is soluble, then* $\mathbf{C}_A(\mathbf{O}_p(A)) \le \mathbf{O}_p(A)$.

PROOF. Suppose that B is a normal semisimple subgroup of A. From the definition of a semisimple group, we see that $B = B'$, and $B/\mathbf{Z}(B)$ is a direct product of nonabelian simple groups. By Lemma 2.5, B has a subgroup of p-power index, and in particular, $B/\mathbf{Z}(B)$ has a subgroup of p-power index. It follows from Lemma 2.5 that $\mathbf{Z}(B)$ is a p-group. Since $B/\mathbf{Z}(B)$ is a direct product of nonabelian simple groups, we see from Lemma 2.3(i) that $p \nmid |M(B/\mathbf{Z}(B))|$. So $\mathbf{Z}(B) = 1$. Thus B is a direct product of nonabelian simple groups, and so B contains an insoluble minimal normal subgroup of A. This yields a contradiction to the assumption. Thus A has no normal semisimple subgroups. By the definition (see [18, Definition 6.10, page 452]), the generalized Fitting subgroup $\mathbf{F}^*(A)$ equals the Fitting subgroup $\mathbf{F}(A)$. By Lemma 2.5, $\mathbf{O}_{p'}(A) = 1$, and thus $\mathbf{F}^*(A) = \mathbf{F}(A) = \mathbf{O}_p(A)$. Therefore, $\mathbf{C}_A(\mathbf{O}_p(A)) \le \mathbf{O}_p(A)$. The lemma follows from Lemma 2.5. □

LEMMA 3.5. *If* $\mathrm{Soc}(A)$ *is soluble then* $G = \mathbf{O}_p(A) \trianglelefteq A$.

PROOF. Let $H = \mathbf{O}_p(A)$. It follows from Lemma 3.4 that $\mathbf{C}_A(H) \le H$. Write $V = H/\Phi(H)$ and $\overline{A} = A/\Phi(H)$. Then V may be regarded as a vector space over $\mathbb{Z}_p$. We consider the action of A on V by conjugation. Since H acts trivially on V, we have $H \le \mathbf{C}_A(V)$ and $\mathbf{C}_A(V)$ is normal in A. Suppose that H is a proper subgroup of $\mathbf{C}_A(V)$. Then $\mathbf{C}_A(V)$ has a nontrivial p'-element x. Since the p'-element x acts trivially on $H/\Phi(H)$, we see that x acts also trivially on H. So $x \in \mathbf{C}_A(H)$ but x is not contained in H. This yields a contradiction since $\mathbf{C}_A(H) \le H$. Therefore $\mathbf{C}_A(V) = H$, and so the conjugation leads a faithful representation of A/H as a subgroup of $\mathrm{GL}(V)$.

If $V \cong \mathbb{Z}_p$, then A/H is isomorphic to a subgroup of a cyclic group of order $p-1$; in this case H is the Sylow p-subgroup of A and so $G = \mathbf{O}_p(A)$. We now consider the remaining case, namely when $V \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Suppose that $H < G$. Then A/H is isomorphic to a subgroup L of $\mathrm{GL}(2,p)$. Since $H < G$, a Sylow p-subgroup of L is not normal. Then by [17, Theorem 6.17, page 404], $L \cap \mathrm{SL}(2,p)$ contains $\mathrm{SL}(2,p)$, and hence $\mathrm{SL}(2,p) \le L$. Since $1 < \mathbf{Z}(\mathrm{SL}(2,p)) \le \mathbf{O}_{p'}(L)$ for odd p, we see that $1 < \mathbf{O}_{p'}(L)$. We also have $\mathbf{O}_{p,p'}(\overline{A}) = V \rtimes Q$, where $Q \cong \mathbf{O}_{p'}(L)$. Since $\mathrm{SL}(2,p) \le L$, V is a minimal normal subgroup of $\overline{A}$. It follows from $\mathbf{Z}(\mathbf{O}_{p,p'}(\overline{A})) = \mathbf{C}_V(Q) \times \mathbf{C}_Q(V) = \mathbf{C}_V(Q)$ that $\mathbf{C}_V(Q)$ is normal in $\overline{A}$. Therefore $\mathbf{C}_V(Q) = 1$. Further, by the Frattini argument (or see [17, (8.12), page 238],

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$\overline{A} = V\mathbf{N}_{\overline{A}}(Q)$. Since $V \cap \mathbf{N}_{\overline{A}}(Q) = \mathbf{C}_V(Q) = 1$, we have $\overline{A} = V \rtimes \mathbf{N}_{\overline{A}}(Q)$, and so a Sylow p-subgroup of $\overline{A}$ is isomorphic to $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$. This is not the case since p is odd and G is a metacyclic p-group. Consequently, we have $G = \mathbf{O}_p(A) \trianglelefteq A$. $\square$

PROOF OF THEOREM 1.1. To complete the proof of Theorem 1.1, we now only need to show that $A \cong G \rtimes \mathrm{Aut}(G, S)$ when G is normal in $\mathrm{Aut}\,\Gamma$, while it follows from Lemma 2.1. So the proof of Theorem 1.1 is now complete. $\square$

We now prove Corollary 1.2.

PROOF OF COROLLARY 1.2. Since p is odd, there exists a subset T of S such that $T \cap T^{-1} = \emptyset$, $S = T \cup T^{-1}$. Since $|S| < 2p$, we have $|T| < p$. Let θ be an p-element in $\mathrm{Aut}(G, S)$. Assume that θ has an orbit of length p. Then there exists an element t in T such that both of t and t^{-1} are contained in the orbit of length p. This means that $t^{-1} = \theta^k(t)$ for some k with $1 \le k < p$. So $\langle\theta^{2k}\rangle$ stabilizes t. However, $\langle\theta^{2k}\rangle = \langle\theta\rangle$ since p does not divide $2k$. This yields a contradiction. So θ acts trivially on S. Since S generates G, we see that θ acts faithfully on S. Thus $\theta = 1$, which implies that $\mathrm{Aut}(G, S)$ is a p'-group. For $p = 3$, it easily follows from since $|S| \le 4$ that the automorphism group A is a $\{2, 3\}$-group. So A is soluble. By Theorem 1.1, we have $A = G \rtimes \mathrm{Aut}(G, S)$ for $p = 3$. If $p > 3$, the claim also follows from Theorem 1.1. $\square$

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Department of Mathematics and Statistics
The University of Western Australia
Nedlands WA 6907
Australia
e-mail: li@maths.uwa.edu.au

Division of Mathematical Sciences
Pukyong National University
Pusan 608-737
Korea
e-mail: hsim@pknu.ac.kr