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$B^0_{(s)}$-$\bar{B}^0_{(s)}$ mixing, kaon CP violation

In this chapter, we provide the basics of the phenomenological description of the B^0-$\bar{B}^0$ and K^0-$\bar{K}^0$ systems, and summarize the different results obtained from QCD spectral sum rules (QSSR), on the bag constant parameters entering in the analysis of the $B^0_{(s)}$-$\bar{B}^0_{(s)}$ mass differences, and on different operators entering in the analysis of kaon CP violation. There is practically no theory behind this description. It is only based on first principles: the superposition principle, Lorentz invariance, and general invariance properties under the P, C and T symmetries. The basic idea is to reduce the description of this system to a minimum of phenomenological parameters which, eventually, an underlying theory, like the Standard Model (SM) should be able to predict.

56.1 Standard formalism

This section has been inspired from the lectures given in [500].

56.1.1 Phenomenology of B^0-$\bar{B}^0$ and K^0-$\bar{K}^0$ mixings

In the absence of the weak interactions, the K^0 and $\bar{K}^0$[1] particles produced by the strong interactions are stable eigenstates of strangeness with eigenvalues ± 1. In the presence of the weak interaction they become unstable. The states with an exponential time dependence law (τ is the proper time):

$$|K_L\rangle \rightarrow e^{-iM_L\tau}\ |K_L\rangle \quad \text{and} \quad |K_S\rangle \rightarrow e^{-iM_S\tau}\ |K_S\rangle\,, \tag{56.1}$$

are linear superpositions of the eigenstates of strangeness:

$$|K_L\rangle = \frac{1}{\sqrt{|\,p\,|^2+|\,q\,|^2}}(p\ |\ K^0\rangle + q\ |\bar{K}^0\rangle) \tag{56.2}$$

$$|K_S\rangle = \frac{1}{\sqrt{|\,p\,|^2+|\,q\,|^2}}(p\ |\ K^0\rangle - q\ |\bar{K}^0\rangle)\,, \tag{56.3}$$

where p and q are complex numbers and CPT invariance, which is a property of the SM in

[1] Discussions for the B^0 and $\bar{B}^0$ particles are very similar.

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any case, has been assumed. The parameters $M_{L,S}$ in Eq. (56.1) are also complex:

$$M_{L,S} = m_{L,S} - \frac{i}{2}\,\Gamma_{L,S}\,, \tag{56.4}$$

with $m_{L,S}$ the masses and $\Gamma_{L,S}$ the decay widths of the long-lived and short-lived neutral kaon states.

As we shall see, experimentally, the $|K_S\rangle$ and $|K_L\rangle$ states are very close to the CP eigenstates:

$$\left|K_1^0\right\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \quad \text{and} \quad \left|K_2^0\right\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \tag{56.5}$$

with:

$$\mathrm{CP}\left|K_1^0\right\rangle = +\left|K_1^0\right\rangle \quad \text{and} \quad \mathrm{CP}\left|K_2^0\right\rangle = -\left|K_2^0\right\rangle. \tag{56.6}$$

This is characterized by the small complex parameter $\tilde{\epsilon}$:

$$\tilde{\epsilon} = \frac{p-q}{p+q}\;; \tag{56.7}$$

in terms of which:

$$|K_{L,S}\rangle = \frac{1}{\sqrt{1+|\tilde{\epsilon}\,|^2}}\left(\left|K_{2,1}^0\right\rangle + \tilde{\epsilon}\left|K_{1,2}^0\right\rangle\right). \tag{56.8}$$

According to Eqs. (56.1) and (56.2), a state initially pure $|K^0\rangle$ evolves, in a period of time τ to a state which is a superposition of $|K^0\rangle$ and $|\bar{K}^0\rangle$:

$$|K^0\rangle \to \frac{1}{2}[e^{-iM_L\tau} + e^{-iM_S\tau}]\,|K^0\rangle + \frac{1}{2}\frac{p}{q}[e^{-iM_L\tau} - e^{-iM_S\tau}]\,|\bar{K}^0\rangle\,; \tag{56.9}$$

and, likewise:

$$|\,\bar{K}^0\rangle \to \frac{1}{2}[e^{-iM_L\tau} + e^{-iM_S\tau}]\,|\,\bar{K}^0\rangle + \frac{1}{2}\frac{q}{p}[e^{-iM_L\tau} - e^{-iM_S\tau}]\,|\,K^0\rangle\,. \tag{56.10}$$

For a small period of time $\delta\tau$ we then have:

$$|\,K^0\rangle \to |\,K^0\rangle - i\delta\tau(\mathcal{M}_{11}\,|\,K^0\rangle + \mathcal{M}_{12}\,|\,\bar{K}^0\rangle)\,; \tag{56.11}$$
$$|\,\bar{K}^0\rangle \to |\,\bar{K}^0\rangle - i\delta\tau(\mathcal{M}_{21}\,|\,K^0\rangle + \mathcal{M}_{22}\,|\,\bar{K}^0\rangle), \tag{56.12}$$

where:

$$\mathcal{M}_{ij} = \frac{1}{2}\begin{pmatrix} M_L + M_S & \frac{p}{q}(M_L - M_S) \\ \frac{q}{p}(M_L - M_S) & M_L + M_S \end{pmatrix}. \tag{56.13}$$

This is the complex mass matrix of the $K^0 - \bar{K}^0$ system.

In full generality, the mass matrix $\mathcal{M}_{ij}$ admits a decomposition, similar to the one of the complex parameters $M_{L,S}$ in Eq. (56.4), in terms of an absorptive part Γ_{ij} and a dispersive

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part M_{ij}:

$$\mathcal{M}_{ij} = M_{ij} - \frac{i}{2}\Gamma_{ij} \,. \tag{56.14}$$

In a given quantum field theory, like, for example, the Standard Electroweak Model, the complex $K^0 - \bar{K}^0$ mass matrix is defined via the transition matrix T which characterizes S-matrix elements. More precisely, the off-diagonal absorptive matrix element Γ_{12} for example, is given by the sum of products of on-shell matrix elements:

$$\Gamma_{12} = \sum_{\Gamma} \int d\Gamma (\langle \Gamma \mid T \mid \bar{K}^0 \rangle)^* \langle \Gamma \mid T \mid K^0 \rangle \,, \tag{56.15}$$

where the sum is extended to all possible states $\mid \Gamma\rangle$ to which the states $\mid K^0\rangle$ and $\mid \bar{K}^0\rangle$ can decay. The symbol $d\Gamma$ denotes the phase space measure appropriate to the particle content of the state Γ. The corresponding matrix element M_{12} is defined by the dispersive principal part integral:

$$M_{12} = \frac{1}{\pi}\wp \int ds \frac{1}{m_K^2 - s}\Gamma_{12}(s) + \text{'local} - \text{terms'} \,. \tag{56.16}$$

The fact that $\mathcal{M}_{11} = \mathcal{M}_{22}$ in Eq. (56.13) is a consequence of CPT invariance. In general, if we have a transition between an initial state $\mid IN\rangle$ and a final state $\mid FN\rangle$, CPT invariance relates the matrix elements of this transition to the one between the corresponding CPT-transformed states $\mid \overline{FN}^{'}\rangle$ and $\mid \overline{IN}^{'}\rangle$, where $\mid \overline{IN}^{'}\rangle$ denotes the state obtained from $\mid IN\rangle$ by interchanging all particles into antiparticles (this is the meaning of the bar symbol in $\overline{IN}$), and taking the mirror image of the kinematic variables: $[(E, \vec{p}) \to (E, -\vec{p})\,; (\sigma^0, \vec{\sigma}) \to (-\sigma^0, \vec{\sigma})]$, as well as their motion reversal image: $[(E, \vec{p}) \to (E, -\vec{p})\,; (\sigma^0, \vec{\sigma}) \to (\sigma^0, -\vec{\sigma})]$. (These kinematic changes are the meaning of the prime symbol in $IN^{'}\rangle$.)

Altogether, CPT invariance implies then:

$$\langle FN \mid T \mid IN\rangle = \langle \overline{IN}^{'} \mid T \mid \overline{FN}^{'}\rangle \,. \tag{56.17}$$

Since, for the K^0-states: $\mid (\bar{K}^0)^{'}\rangle = \mid K^0\rangle$, the CPT invariance relation implies:

$$\mathcal{M}_{11} = \mathcal{M}_{22} \,. \tag{56.18}$$

The off–diagonal matrix elements in Eq. (56.13) are also related by CPT invariance, plus the hermiticity property of the T-matrix in the absence of strong final-state interactions; certainly the case when the $\mid IN\rangle$ and $\mid FN\rangle$ states are $\mid K^0\rangle$ and $\mid \bar{K}^0\rangle$. In general, in the absence of strong final-state interactions, we have:

$$\langle \overline{IN}^{'} \mid T \mid \overline{FN}^{'}\rangle = (\langle \overline{FN}^{'} \mid T \mid \overline{IN}^{'}\rangle)^* \,. \tag{56.19}$$

This relation, together with the CPT invariance relation in Eq. (56.17) implies then:

$$\mathcal{M}_{12} = (\mathcal{M}_{21})^* \,. \tag{56.20}$$

There are a number of interesting constraints between the various phenomenological parameters we have introduced. With M_{12} and Γ_{12} defined in Eqs. (56.16) and (56.15) and using Eqs. (56.13), (56.4) and (56.7), we have:

$$\frac{q}{p} = \frac{1-\tilde{\epsilon}}{1+\tilde{\epsilon}} = \frac{1}{2}\frac{\Delta m + i\frac{1}{2}\Delta\Gamma}{M_{12} - \frac{i}{2}\Gamma_{12}} = \frac{M_{21} - \frac{i}{2}\Gamma_{21}}{\frac{1}{2}\left(\Delta m + \frac{i}{2}\Delta\Gamma\right)}\,, \tag{56.21}$$

where

$$\Delta m \equiv m_L - m_S \quad \text{and} \quad \Delta\Gamma \equiv \Gamma_S - \Gamma_L. \tag{56.22}$$

As already discussed, CPT invariance implies:

$$M_{21} = (M_{12})^* \quad \text{and} \quad \Gamma_{21} = (\Gamma_{12})^*\,. \tag{56.23}$$

Experimentally, the masses $m_{L,S}$ and widths $\Gamma_{L,S}$ are well measured, and in what follows they will be used as known parameters. (There is no way for theory at present to do better than experiments in the determination of these parameters . . .) The precise values for the masses and widths can be found in PDG [16]. Nevertheless, it is important to keep in mind some orders of magnitude:

$$\Gamma_S^{-1} \simeq 0.9 \times 10^{-10}\mathrm{s}\,; \tag{56.24}$$

$$\Gamma_L \simeq 1.7 \times 10^{-3}\Gamma_S\,; \tag{56.25}$$

$$\Delta m \simeq 0.5\Gamma_S. \tag{56.26}$$

56.1.2 The Bell–Steinberger unitarity constraint

Let us consider a state $|\,\Psi\rangle$ to be an arbitrary superposition of the short-lived and long-lived kaon states:

$$|\,\Psi\rangle = \alpha\,|\,K_S\rangle + \beta\,|\,K_L\rangle\,. \tag{56.27}$$

The total decay rate of this state must be compensated by a decrease of its norm:

$$\sum_\Gamma |\,\langle\Gamma\,|\,T\,|\,\Psi\rangle\,|^2 = -\frac{d}{d\tau}\,|\,\Psi\,|^2\,. \tag{56.28}$$

The change in rate is governed by the mass matrix defined by Eq. (56.11). Equating terms proportional to $|\,\alpha\,|^2$ and $|\,\beta\,|^2$ in both sides of Eq. (56.28) results in the trivial relations:

$$\Gamma_L = \sum_\Gamma \int d\Gamma\,|\,\langle\Gamma\,|\,T\,|\,K_L\rangle\,|^2\,, \tag{56.29}$$

$$\Gamma_S = \sum_\Gamma \int d\Gamma\,|\,\langle\Gamma\,|\,T\,|\,K_S\rangle\,|^2\,. \tag{56.30}$$

The mixed terms, proportional to $\alpha\beta^*$ and $\alpha^*\beta$, lead however to a highly non-trivial relation, first derived by Bell and Steinberger [807]:

$$-i(M_L^* - M_S)\langle K_L\,|\,K_S\rangle = \sum_\Gamma \int d\Gamma\,(\langle\Gamma\,|\,T\,|\,K_L\rangle)^*\,\langle\Gamma\,|\,T\,|\,K_S\rangle\,. \tag{56.31}$$

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Notice that

$$\langle K_L \mid K_S \rangle = \frac{\mid p \mid^2 - \mid q \mid^2}{\mid p \mid^2 + \mid q \mid^2} = \frac{2\mathrm{Re}\tilde{\epsilon}}{1 + \mid \tilde{\epsilon} \mid^2} . \tag{56.32}$$

The LHS of Eq. (56.31) can be expressed in terms of measurable physical parameters with the result:

$$\left(\frac{\Gamma_S + \Gamma_L}{2} - i\Delta m \right) \frac{2\mathrm{Re}\tilde{\epsilon}}{1 + \mid \tilde{\epsilon} \mid^2} = \sum_\Gamma \int d\Gamma \left(\langle \Gamma \mid T \mid K_L \rangle \right)^* \langle \Gamma \mid T \mid K_S \rangle . \tag{56.33}$$

The RHS of this equation can be bounded, using the Schwartz inequality, with the result:

$$\left| \frac{\Gamma_S + \Gamma_L}{2} - i\Delta m \right| \frac{2\mathrm{Re}\tilde{\epsilon}}{1 + \mid \tilde{\epsilon} \mid^2} \leq \sqrt{\Gamma_L \Gamma_S} . \tag{56.34}$$

Inserting the experimental values for $\Gamma_{S,L}$ and Δm, results in an interesting bound for the non-orthogonality of the K_L and K_S states [see Eq. (56.32)]:

$$\frac{2\mathrm{Re}\tilde{\epsilon}}{1 + \mid \tilde{\epsilon} \mid^2} \leq 2.9 \times 10^{-2} , \tag{56.35}$$

indicating also that the admixture of $K_1^0(K_2^0)$ in $K_L(K_S)$ has to be rather small.

It is possible to obtain further information from the unitarity constraint in Eq. (56.33), if one uses the experimental fact that the 2π states are by far the dominant terms in the sum over hadronic states Γ.

One can then write the RHS of Eq. (56.33) in the form:

$$\sum_{\pi\pi} \int d(\pi\pi) \left(\langle \pi\pi \mid T \mid K_L \rangle \right)^* \langle \pi\pi \mid T \mid K_S \rangle + \gamma \Gamma_S . \tag{56.36}$$

It is possible to obtain a bound for γ, by considering other states than 2π in the sum of the RHS in Eq. (56.33) and applying the Schwartz inequality to individual sets of states separated by selection rules.

The contribution from the various semi-leptonic modes, for example, is known to be smaller than:

$$\left| \sum_{\text{lep.modes}} \int \cdots \right| \ll 10^{-3} \Gamma_S ; \tag{56.37}$$

and the contribution from the 3π-states:

$$\left| \sum_{3\pi} \int \cdots \right| \ll 10^{-3} \Gamma_S . \tag{56.38}$$

We conclude that, to a good approximation, we can restrict the Bell–Steinberger relation to 2π-states. We shall later come back to this inequality, but first we have to discuss the phenomenology of the dominant $K \to \pi\pi$ transitions.

56.1.3 $K \to 2\pi$ amplitudes

In the limit where CP is conserved the states $K_S(K_L)$ become eigenstates of CP; namely, the states $K_1^0(K_2^0)$ introduced in Eq. (56.5) with eigenvalues CP $= +1$(CP $= -1$). On the other hand a state of two-pions with total angular momentum $J = 0$ has CP $= +1$. Therefore, the observation of a transition from the long-lived component of the neutral kaon system to a two-pion final state is evidence for CP violation. The first observation of such a transition to the $\pi^+\pi^-$ mode was made by Christenson *et al.* [808] in 1964, with the result:

$$\frac{\Gamma_L(+,-)}{\Gamma_L(\text{all})} = (2 \pm 0.4) \times 10^{-3} \,. \tag{56.39}$$

Since then the transition to the $\pi^0\pi^0$ mode has also been observed, as well as the phases of the amplitude ratios:

$$\eta_{+-} = \frac{\langle \pi^+\pi^- \mid T \mid K_L\rangle}{\langle \pi^+\pi^- \mid T \mid K_S\rangle} \quad \text{and} \quad \eta_{00} = \frac{\langle \pi^0\pi^0 \mid T \mid K_L\rangle}{\langle \pi^0\pi^0 \mid T \mid K_S\rangle} \,, \tag{56.40}$$

with the results [16]:

$$\eta_{+-} = (2.269 \pm 0.023) \times 10^{-3} e^{i(44.3\pm0.8)^\circ} \,; \tag{56.41}$$

$$\eta_{00} = (2.259 \pm 0.023) \times 10^{-3} e^{i(43.3\pm1.3)^\circ} \,. \tag{56.42}$$

In order to make a phenomenological analysis of $K \to \pi\pi$ transitions, it is convenient to express the states $\mid \pi^+\pi^-\rangle$ and $\mid \pi^0\pi^0\rangle$ in terms of well defined isospin $I = 0$, and $I = 2$ states. (The $I = 1$ state in this case is forbidden by Bose statistics.):

$$\mid +-\rangle = \sqrt{\frac{2}{3}} \mid 0\rangle + \sqrt{\frac{1}{3}} \mid 2\rangle \,; \tag{56.43}$$

$$\mid 00\rangle = \sqrt{\frac{2}{3}} \mid 2\rangle - \sqrt{\frac{1}{3}} \mid 0\rangle \,. \tag{56.44}$$

The reason for introducing pure isospin states, is that the matrix elements of transitions from K^0 and the $\bar{K}^0$ states to the same $(\pi\pi)_I$-state can be related by *CPT* invariance plus Watson's theorem on final-state interactions. The relation in question is the following:

$$e^{-2i\delta_I}\langle I \mid T \mid K^0\rangle = (\langle I \mid T \mid \bar{K}^0\rangle)^* \,, \tag{56.45}$$

where δ_I denotes the appropriate $J = 0$, isospin I $\pi\pi$ phase-shift at the energy of the neutral kaon mass.

The proof of this relation is rather simple. With $S = 1 + iT$, the unitarity of the S matrix, $SS^\dagger = 1$, implies:

$$T^\dagger T = i(T^\dagger - T) \,. \tag{56.46}$$

If one takes matrix elements of this operator relation between an initial state K^0, and a final 2π-state with isospin I, we then have:

$$\sum_F \langle I \mid T^\dagger \mid F\rangle\langle F \mid T \mid K^0\rangle = i\langle I \mid T^\dagger \mid K^0\rangle - i\langle I \mid T \mid K^0\rangle \,, \tag{56.47}$$

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where we have inserted a complete set of states $\sum \mid F\rangle\langle F \mid= 1$ between T and $T^\dagger$. The crucial observation is that, in the strong interaction sector of the S matrix, only the state $F = I$ can contribute to the $T^\dagger$-matrix element. All the other states are suppressed by selection rules; for example, the 3π-states have opposite G-parity than the 2π-states; the $\pi l \nu$-states are not related to 2π-states by the strong interactions alone; etc. Then, introducing the $\pi\pi$ phase-shift definition:

$$\langle I \mid S \mid I\rangle = e^{2i\delta_I} , \tag{56.48}$$

results in the relation:

$$\begin{aligned} i(e^{-2i\delta_I} - 1)\langle I \mid T \mid K^0\rangle &= i\langle I \mid T^\dagger \mid K^0\rangle - i\langle I \mid T \mid K^0\rangle , \\ &= i(\langle K^0 \mid T \mid I\rangle)^* - i\langle I \mid T \mid K^0\rangle . \end{aligned} \tag{56.49}$$

We can next use CPT invariance [recall Eq. (56.17), which in our case implies the relation: $\langle K^0 \mid T \mid I\rangle^* = (\langle I \mid T \mid \bar{K}^0\rangle)^*$.] The result in Eq. (56.45) then follows.

As a consequence of the relation we have proved, we can use in full generality the following parametrization for $K^0(\bar{K}^0) \to (\pi\pi)_I$ amplitudes:

$$\langle I \mid T \mid K^0\rangle = iA_I e^{i\delta_I} ; \tag{56.50}$$

$$\langle I \mid T \mid \bar{K}^0\rangle = -iA_I^* e^{i\delta_I} . \tag{56.51}$$

One possible quantity we can introduce to characterize the amount of CP violation in $K \to 2\pi$ transitions is the parameter:

$$\epsilon = \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]} . \tag{56.52}$$

This parameter is related to the $\tilde{\epsilon}$–parameter introduced in Eq. (56.7); as well as to the complex A_0-amplitude defined in Eqs. (56.50) and (56.51), in the following way:

$$\epsilon = \frac{(1+\tilde{\epsilon})A_0 - (1-\tilde{\epsilon})A_0^*}{(1+\tilde{\epsilon})A_0 + (1-\tilde{\epsilon})A_0^*} . \tag{56.53}$$

namely:

$$\epsilon = \frac{\tilde{\epsilon} + i\frac{\text{Im}A_0}{\text{Re}A_0}}{1 + i\tilde{\epsilon}\frac{\text{Im}A_0}{\text{Re}A_0}} . \tag{56.54}$$

This is a good place to comment on the history of phase conventions in neutral K-decays. In their pioneering paper on the phenomenology of the $K - \bar{K}$ system, Wu and Yang [809] chose to freeze the arbitrary relative phase between the K^0 and $\bar{K}^0$ states, with the choice $\text{Im}A_0 = 0$. With this convention, $\epsilon = \tilde{\epsilon}$. In fact, the parameter ϵ is phase-convention independent; while neither $\tilde{\epsilon}$, nor A_I are. Indeed, under a small arbitrary phase change of the K^0-state:

$$\mid K^0\rangle \to e^{-i\varphi} \mid K^0\rangle , \tag{56.55}$$

the parameters A_I, M_{12}, and $\tilde{\epsilon}$ change as follows:

$$\mathrm{Im}A_I \to \mathrm{Im}A_I - \varphi \mathrm{Re}A_I \,; \tag{56.56}$$

$$\mathrm{Im}M_{12} \to \mathrm{Im}M_{12} + \varphi \Delta m \,; \tag{56.57}$$

$$\tilde{\epsilon} \to \tilde{\epsilon} + i\varphi \,; \tag{56.58}$$

while ϵ remains invariant. The Wu–Yang phase convention was made prior to the development of the electroweak theory. In the standard model, the conventional way by which the freedom in the choice of relative phases of the quark fields has been frozen, is not compatible with the Wu–Yang convention. Since ϵ is convention independent, we shall keep it as one of the fundamental parameters. Then, however, we need a second parameter which characterizes the amount of *intrinsic* CP violation specific to the $K \to 2\pi$ decay, by contrast to the CP violation in the $K^0 - \bar{K}^0$ mass matrix. The parameter we are looking for has to be sensitive then to the lack of relative reality of the the two isospin amplitudes A_0 and A_2. This is the origin of the famous ϵ'–parameter, which we shall next discuss.

In general, we can define three independent ratios of the $K_{L,S} \to (2\pi)_{I=0,2}$ transition amplitudes. One is the ϵ parameter in Eq. (56.52). Two other natural ratios are

$$\frac{A[K_L \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} \quad \text{and} \quad \omega \equiv \frac{A[K_S \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} \,. \tag{56.59}$$

Both ratios can be expressed in terms of the $\tilde{\epsilon}$ parameter introduced in Eq. (56.7), and the complex A_I amplitudes defined in Eqs. (56.50) and (56.51):

$$\begin{aligned}\frac{A[K_L \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} &= \frac{(1+\tilde{\epsilon})A_2 - (1-\tilde{\epsilon})A_2^*}{(1+\tilde{\epsilon})A_0 + (1-\tilde{\epsilon})A_0^*} e^{i(\delta_2-\delta_0)} \\ &= \frac{i\frac{\mathrm{Im}A_2}{\mathrm{Re}A_0} + \tilde{\epsilon}\frac{\mathrm{Re}A_2}{\mathrm{Re}A_0}}{1 + i\tilde{\epsilon}\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}} e^{i(\delta_2-\delta_0)} \,;\end{aligned} \tag{56.60}$$

and:

$$\begin{aligned}\omega \equiv \frac{A[K_S \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} &= \frac{(1+\tilde{\epsilon})A_2 + (1-\tilde{\epsilon})A_2^*}{(1+\tilde{\epsilon})A_0 + (1-\tilde{\epsilon})A_0^*} e^{i(\delta_2-\delta_0)} \\ &= \frac{\frac{\mathrm{Re}A_2}{\mathrm{Re}A_0} + \tilde{\epsilon}\frac{\mathrm{Im}A_2}{\mathrm{Re}A_0}}{1 + i\tilde{\epsilon}\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}} e^{i(\delta_2-\delta_0)} \,.\end{aligned} \tag{56.61}$$

The ϵ' parameter is then defined as the following combination of these ratios:

$$\epsilon' = \frac{1}{\sqrt{2}} \left(\frac{A[K_L \to (\pi\pi)_{I=2}]}{A[K_S \to (\pi\pi)_{I=0}]} - \epsilon \times \omega \right) . \tag{56.62}$$

From these results, and using the expression for ϵ we obtained in Eq. (56.54), we finally get:

$$\epsilon' = \frac{i}{\sqrt{2}} \frac{(1-\tilde{\epsilon}^2)e^{i(\delta_2-\delta_0)}}{(\mathrm{Re}A_0 + i\tilde{\epsilon}\mathrm{Im}A_0)^2} (\mathrm{Im}A_2 \mathrm{Re}A_0 - \mathrm{Im}A_0 \mathrm{Re}A_2) \,, \tag{56.63}$$

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an expression which clearly shows the proportionality to the lack of relative reality between the A_0 and A_2 amplitudes.

We shall next establish contact with the parameters η_{+-} and η_{00}, which were introduced in Eq. (56.40), and which are directly accessible to experiment. Using Eqs. (56.43), (56.44), as well as the definitions of ϵ, ϵ', and ω above, one finds:

$$\eta_{+-} = \epsilon + \epsilon' \frac{1}{1 + \frac{1}{\sqrt{2}}\omega} ; \tag{56.64}$$

$$\eta_{00} = \epsilon - 2\epsilon' \frac{1}{1 - \sqrt{2}\omega} . \tag{56.65}$$

So far, we have made no approximations in our phenomenological analysis of the $K^0 - \bar{K}^0$ mass matrix and $K \to 2\pi$ decays. It is however useful to try to thin down in some way the exact expressions we have derived, by taking into account the relative size of the various phenomenological parameters which appear in the expressions above. The strategy will be to neglect first, terms which are products of CP violation parameters. For example, in Eq. (56.61), we have introduced the parameter ω, which a priori we can reasonably expect to be dominated by the term:

$$\omega \simeq \frac{\mathrm{Re}A_2}{\mathrm{Re}A_0} e^{i(\delta_2 - \delta_0)}, \tag{56.66}$$

where experimentally [16]:

$$\delta_2 - \delta_0 = -(42 \pm 4)^0 . \tag{56.67}$$

We can justify this approximation by the fact that non-leptonic $\Delta I = \frac{3}{2}$ transitions, although suppressed with respect to the $\Delta I = \frac{1}{2}$ transitions, are nevertheless larger than the observed CP violation effects. Notice that the amplitude A_2 is responsible for the deviation from an exact $\Delta I = \frac{1}{2}$ rule. The ratio $\frac{\mathrm{Re}A_2}{\mathrm{Re}A_0}$ can be obtained from the experimentally known branching ratios $\Gamma(K_S \to \pi^+\pi^-)$ and $\Gamma(K_S \to \pi^0\pi^0)$.

More precisely, correcting for the phase–space effects, one must compare the normalized decay rates:

$$\gamma(1,2) \equiv \frac{\Gamma(K \to \pi_1\pi_2)}{\frac{1}{16\pi M}\sqrt{1 - \frac{(m_1+m_2)^2}{M^2}}\sqrt{1 - \frac{(m_1-m_2)^2}{M^2}}} , \tag{56.68}$$

where the denominator here is the two-body phase space factor for the mode $K \to \pi_1\pi_2$, (M is the mass of the K-particle and $m_{1,2}$ the pion masses.) Then, we have:

$$\frac{\gamma_S(+-)}{2\gamma_S(00)} = 1 + 3\sqrt{2}\frac{\mathrm{Re}A_2}{\mathrm{Re}A_0}\cos(\delta_2 - \delta_1) + \mathcal{O}\left(\frac{\alpha}{\pi}\right) . \tag{56.69}$$

Experimentally, from the PDG [16], one finds:

$$\frac{\gamma_S(+-)}{2\gamma_S(00)} = 1.109 \pm 0.012 , \tag{56.70}$$

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and using the present experimental information on $(\delta_2 - \delta_1)$, we find, with neglect of radiative corrections:

$$\frac{\text{Re}A_2}{\text{Re}A_0} = (+22.2)^{-1} . \tag{56.71}$$

We shall discuss later some of the qualitative dynamical explanations, within the standard model, of how this small number appears. It is fair to say however, that a reliable calculation of this ratio is still lacking at present. Using the approximations:

$$\tilde{\epsilon}\,\text{Im}A_0 \ll \text{Re}A_0 \quad \text{and} \quad \tilde{\epsilon}^2 \ll 1 , \tag{56.72}$$

we can rewrite ϵ' in a simpler form:

$$\epsilon' \simeq \frac{1}{\sqrt{2}} e^{i(\delta_2 - \delta_0 + \frac{\pi}{2})} \frac{\text{Re}A_2}{\text{Re}A_0} \left(\frac{\text{Im}A_2}{\text{Re}A_2} - \frac{\text{Im}A_0}{\text{Re}A_0} \right) , \tag{56.73}$$

clearly showing the fact that ϵ' *is proportional to direct CP–violation in* $K \to 2\pi$ *transitions and is also suppressed by the* $\Delta I = \frac{1}{2}$ *selection rule.*

The same approximations in Eq. (56.72), when applied to ϵ, lead to:

$$\epsilon \simeq \tilde{\epsilon} + i \frac{\text{Im}A_0}{\text{Re}A_0} . \tag{56.74}$$

Let us next go back to the mass matrix equations in Eq. (56.21) which, expanding in powers of $\tilde{\epsilon}$, we can rewrite as follows:

$$1 - 2\tilde{\epsilon} \simeq \frac{\text{Re}M_{12} - \frac{i}{2}\text{Re}\Gamma_{12}}{\frac{1}{2}\left(\Delta m + \frac{i}{2}\Delta\Gamma\right)} - i \frac{\text{Im}M_{12} - \frac{i}{2}\text{Im}\Gamma_{12}}{\frac{1}{2}\left(\Delta m + \frac{i}{2}\Delta\Gamma\right)} . \tag{56.75}$$

To a first approximation, neglecting CP violation effects altogether, we find that:

$$\text{Re}M_{12} \simeq \frac{\Delta m}{2} \quad \text{and} \quad \text{Re}\Gamma_{12} \simeq -\frac{\Delta\Gamma}{2} . \tag{56.76}$$

If furthermore, we restrict the sum over intermediate states in Γ_{12} [see Eq. (56.15)] to 2π states, an approximation which we have already seen to be rather good [see Eqs. (56.37) and (56.38)] we can write

$$\Gamma_{12} \simeq (-i A_0^* e^{i\delta_0})^* i A_0 e^{i\delta_0} = -(\text{Re}A_0 + i\text{Im}A_0)^2 , \tag{56.77}$$

from where it follows that:

$$\frac{\text{Im}\Gamma_{12}}{\text{Re}\Gamma_{12}} \simeq \frac{2\text{Re}A_0\text{Im}A_0}{\text{Re}A_0^2 + \text{Im}A_0^2} \simeq 2\frac{\text{Im}A_0}{\text{Re}A_0} . \tag{56.78}$$

Then, using the empirical fact that $\Delta m \simeq \frac{\Gamma_S}{2}$, and $\Gamma_L \ll \Gamma_S$, we finally arrive at the simplified expression:

$$\tilde{\epsilon} \simeq \frac{1}{1+i} \left(i \frac{\text{Im}M_{12}}{\Delta m} + \frac{\text{Im}A_0}{\text{Re}A_0} \right) , \tag{56.79}$$

and, using Eq. (56.74):

$$\epsilon \simeq \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(\frac{\text{Im}M_{12}}{\Delta m} + \frac{\text{Im}A_0}{\text{Re}A_0} \right) . \tag{56.80}$$

This is as much as one can do, within a strict phenomenological analysis of the CP violation in K decays. We have reduced the problem to the knowledge of two parameters: ϵ in Eq. (56.80), and ϵ' in Eq. (56.73). Their present experimental values are [16,599]:

$$\epsilon \simeq (2.280 \pm 0.013) \times 10^{-3} e^{i(43.5\pm0.1)^0} , \qquad \text{Re}(\epsilon'/\epsilon) \simeq (17.2 \pm 1.8) \times 10^{-4} . \tag{56.81}$$

We shall come back to these parameters in the next section. There, we shall discuss what predictions for these fundamental parameters can be made at present within the framework of the Standard Model. As we shall see, the main difficulty comes from the lack of quantitative understanding of the low-energy sector of the strong interactions. In terms of QCD, the sector in question is the one of the interactions between the states with lowest masses: the octet of the pseudoscalar particles (π, K, η) and presumably the singlet (σ, η').

56.2 $B^0_{(s)}$-$\bar{B}^0_{(s)}$ mixing

56.2.1 Introduction

$B^0_{(s)}$ and $\bar{B}^0_{(s)}$ are not eigenstates of the weak Hamiltonian, such that their oscillation frequency is governed by their mass difference ΔM_q. The measurement by the UA1 collaboration [810] of a large value of ΔM_d was the *first* indication of an heavy top quark mass. In the SM, the mass difference is approximately given by [665,475]:

$$\Delta M_q \simeq \frac{G_F^2}{4\pi^2} M_W^2 |V_{tq} V_{tb}^*|^2 S_0 \left(\frac{m_t^2}{M_W^2} \right) \eta_B C_B(\nu) \frac{1}{2M_{B_q}} \langle \bar{B}^0_q | \mathcal{O}_q(\nu) | B^0_q \rangle , \tag{56.82}$$

where the $\Delta B = 2$ local operator $\mathcal{O}_q$ is defined as:

$$\mathcal{O}_q(x) \equiv (\bar{b}\gamma_\mu L q)(\bar{b}\gamma_\mu L q) , \tag{56.83}$$

with: $L \equiv (1 - \gamma_5)/2$ and $q \equiv d, s,$; S_0, η_B and $C_B(\nu)$ are short-distance quantities and Wilson coefficients which are calculable perturbatively [811,475,665,812], while the matrix element $\langle \bar{B}^0_q | \mathcal{O}_q | B^0_q \rangle$ requires non-perturbative QCD calculations, and is usually parametrized for $SU(N)_c$ colours as:

$$\langle \bar{B}^0_q | \mathcal{O}_q | B^0_q \rangle = N_c \left(1 + \frac{1}{N_c} \right) f^2_{B_q} M^2_{B_q} B_{B_q} . \tag{56.84}$$

f_{B_q} is the B_q decay constant normalized as $f_\pi = 92.4$ MeV, and B_{B_q} is the so-called bag parameter which is $B_{B_q} \simeq 1$ if one uses a vacuum saturation of the matrix element. From Eq. (56.82), it is clear that the measurement of ΔM_d provides a measurement of the CKM

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mixing angle $|V_{td}|$ if one uses $|V_{tb}| \simeq 1$. One can also extract this quantity from the ratio:

$$\frac{\Delta M_s}{\Delta M_d} = \left|\frac{V_{ts}}{V_{td}}\right|^2 \frac{M_{B_d}}{M_{B_s}} \frac{\langle \bar{B}^0_s | \mathcal{O}_s | B^0_s \rangle}{\langle \bar{B}^0_d | \mathcal{O}_d | B^0_d \rangle} \equiv \left|\frac{V_{ts}}{V_{td}}\right|^2 \frac{M_{B_d}}{M_{B_s}} \xi^2 \,, \tag{56.85}$$

since in the SM with three generations and unitarity constraints, $|V_{ts}| \simeq |V_{cb}|$. Here:

$$\xi \equiv \sqrt{\frac{g_s}{g_d}} \equiv \frac{f_{B_s}\sqrt{B_{B_s}}}{f_B\sqrt{B_B}} \,. \tag{56.86}$$

The great advantage of Eq. (56.85) compared with the former relation in Eq. (56.82) is that in the ratio, different systematics in the evaluation of the matrix element tends to cancel out, thus providing a more accurate prediction. However, unlike $\Delta M_d = 0.473(17)$ ps^{-1}, which is measured with a good precision [16], the determination of ΔM_s is an experimental challenge due to the rapid oscillation of the B^0_s-$\bar{B}^0_s$ system. At present, only a lower bound of 13.1 ps^{-1} is available at the 95% confidence level from experiments [16], but this bound already provides a strong constraint on $|V_{td}|$.

56.2.2 Two-point function sum rule

Pich [813] has extended the analysis of the K^0-$\bar{K}^0$ systems of [814], using a two-point correlator of the four-quark operators in the analysis of the quantity $f_B\sqrt{B_B}$ which governs the B^0-$\bar{B}^0$ mass difference. The two-point correlator defined as:

$$\psi_H(q^2) \equiv i \int d^4x \, e^{iqx} \, \langle 0|\mathcal{T}\mathcal{O}_q(x)(\mathcal{O}_q(0))^\dagger|0\rangle \,, \tag{56.87}$$

is built from the $\Delta B = 2$ weak operator defined previously. Its QCD expression is given in the chapter on the two-point function. The hadronic part of the spectral function can be conveniently parametrized using the effective realization [813]:

$$\mathcal{O}_q^{\rm eff} = \frac{2}{3}\left(g_B \equiv f^2_{B-q} B_{B_q}\right) \partial_\mu B^0_q \partial^\mu B^0_q + \cdots \,, \tag{56.88}$$

where $\cdots$ corresponds to higher mass hadronic states. It leads to the general form [814]:

$$\begin{aligned}
\frac{1}{\pi}{\rm Im}\hat{\Psi}^{\rm had}(t) \;=\; & \theta\big(t - 4M_B^2\big)\frac{2}{9}\left(\frac{g_B}{4\pi}\right)^2 t^2 \cdot \int_{t_{10}}^{(\sqrt{t}-\sqrt{t_{20}})^2} dt_1 \int_{t_{20}}^{(\sqrt{t}-\sqrt{t_1})^2} dt_2 \, \lambda^{1/2}\left(1, \frac{t_1}{t}, \frac{t_2}{t}\right) \\
& \cdot \left\{ \left(\frac{t_1}{t} + \frac{t_2}{t} - 1\right)^2 \frac{1}{\pi}{\rm Im}\Pi^{(0)}(t_1)\frac{1}{\pi}{\rm Im}\Pi^{(0)}(t_2) \right. \\
& + 2\lambda\left(1, \frac{t_1}{t}, \frac{t_2}{t}\right) \frac{1}{\pi}{\rm Im}\Pi^{(1)}(t_1)\frac{1}{\pi}{\rm Im}\Pi^{(0)}(t_2) \\
& \left. + \left[\left(\frac{t_1}{t} + \frac{t_2}{t} - 1\right)^2 + 8\frac{t_1 t_2}{t^2}\right]\frac{1}{\pi}{\rm Im}\Pi^{(1)}(t_1)\frac{1}{\pi}{\rm Im}\Pi^{(1)}(t_2) \right\} \\
& + \Theta(t - t_c)\frac{1}{\pi}{\rm Im}\Psi^{QCD}(t),
\end{aligned} \tag{56.89}$$

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where the index $i = 0, 1$ refers to the hadronic states with spin 0, 1, and:

$$\frac{1}{\pi}\text{Im}\Pi^{(i)}(t) \equiv \frac{1}{\pi}\text{Im}\Pi_V^{(i)}(t) + \frac{1}{\pi}\text{Im}\Pi_A^{(i)}(t) , \tag{56.90}$$

are the correlators associated to the vector (index V) and axial-vector (index A) currents; $\lambda^{1/2}$ is the usual phase space factor. In the following, we shall retain the contributions from the B-$\bar{B}$ and B^*-$\bar{B}^*$ states, and we (reasonnably) assume that:

$$g_B \simeq g_{B^*} , \tag{56.91}$$

which is supported by the HQET and QSSR results ($f_B \approx f_{B^*}$) and the vacuum saturation assumption ($B_B \approx B_{B^*} \approx 1$) a posteriori recovered from our analysis. The corresponding Laplace (resp. moment) sum rules are:

$$\mathcal{L}(\tau) = \int_{4M_B^2}^{\infty} dt\ \text{e}^{-t\tau}\text{Im}\psi_H(t) , \qquad \mathcal{M}_n = \int_{4M_B^2}^{\infty} dt\ t^n\ \text{Im}\psi_H(t) , \tag{56.92}$$

The two-point function approach is very convenient due to its simple analytic properties which is not the case for the approach based on three-point functions.[2] However, it involves non-trivial QCD calculations which become technically complicated when one includes the contributions of radiative corrections due to non-factorizable diagrams. These perturbative radiative corrections due to *factorizable and non-factorizable* diagrams have been already computed in [816] (referred as NP), where it has been found that the factorizable corrections are large while the non-factorizable ones are negligibly small. NP analysis has confirmed the estimate in [323] from lowest order calculations, where under some assumptions on the contributions of higher mass resonances to the spectral function, the value of the bag constant B_B has been found to be:

$$B_{B_d}\left(4m_b^2\right) \simeq (1 \pm 0.15) . \tag{56.93}$$

This value is comparable with the value $B_{B_d} = 1$ from the vacuum saturation estimate, which is expected to be quite a good approximation due to the relative high scale of the B-meson mass. Equivalently, the corresponding RGI quantity is:

$$\hat{B}_{B_d} \simeq (1.5 \pm 0.2) , \tag{56.94}$$

where we have used the relation:

$$B_{B_q}(\nu) = \hat{B}_{B_q}\alpha_s^{-\frac{\gamma_0}{\beta_1}}\left\{1 - \left(\frac{5165}{12696}\right)\left(\frac{\alpha_s}{\pi}\right)\right\} , \tag{56.95}$$

with $\gamma_0 = 1$ as the anomalous dimension of the operator $\mathcal{O}_q$ and $\beta_1 = -23/6$ for five flavours. The NLO corrections have been obtained in the $\overline{MS}$ scheme [665]. We have also used, to this order, the value [148,149,3]:

$$\bar{m}_b(m_b) = (4.24 \pm 0.06)\ \text{GeV} , \tag{56.96}$$

and $\Lambda_5 = (250 \pm 50)$ MeV [139]. In a forthcoming paper [817], we study (*for the first time*), from the QSSR method, the $SU(3)$ breaking effects on the ratio: ξ defined previously

[2] For detailed criticisms, see [3].

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in Eq. (56.86), where a similar analysis of the ratios of the decay constants has given the values [716]:

$$\frac{f_{D_s}}{f_D} \simeq 1.15 \pm 0.04 \,, \qquad \frac{f_{B_s}}{f_B} \simeq 1.16 \pm 0.04 \,. \tag{56.97}$$

56.2.3 Results and implications on $|V_{ts}|^2/|V_{td}|^2$ and ΔM_s

We deduce by taking the average from the moments and Laplace sum rules results [817]:

$$\xi \equiv \frac{f_{B_s}\sqrt{B_{B_s}}}{f_B\sqrt{B_B}} \simeq 1.18 \pm 0.03 \,, \qquad f_B\sqrt{\hat{B}_B} \simeq (247 \pm 59)\ \text{GeV}, \tag{56.98}$$

in units where $f_\pi = 130.7$ MeV. For the ratio, one expects small errors due to the cancellation of the systematics, while for $f_B\sqrt{\hat{B}_B}$, the error estimate comes mainly from the one of m_b and the estimate of higher-order terms of the QCD series. These results can be compared with different lattice determinations compiled in [823,723]. By comparing the ratio with the one of f_{B_s}/f_{B_d} in Eq. (56.97),[3] one can conclude (to a good approximation) that:

$$\hat{B}_{B_s} \approx \hat{B}_{B_d} \simeq (1.65 \pm 0.38) \Longrightarrow B_{B_{d,s}}\left(4m_b^2\right) \simeq (1.10 \pm 0.25) \,, \tag{56.99}$$

indicating a negligible $SU(3)$ breaking for the bag parameter. For a consistency, we have used the estimate to order α_s [698]:

$$f_B \simeq (1.47 \pm 0.10) f_\pi \,, \tag{56.100}$$

and we have assumed that the error from f_B compensates the one in Eq. (56.98). The result is in excellent agreement with the previous result of [816] in Eqs. (56.93) and (56.94). Using the experimental values:

$$\Delta M_d = 0.472(17)\ ps^{-1} \,, \qquad \Delta M_s \geq 13.1\ ps^{-1}\ \ (95\%\ \text{CL}) \,, \tag{56.101}$$

one can deduce from Eq. (56.85):

$$\rho_{sd} \equiv \left|\frac{V_{ts}}{V_{td}}\right|^2 \geq 20.0(1.1) \,. \tag{56.102}$$

Alternatively, using:

$$\rho_{sd} \simeq \frac{1}{\lambda^2[(1-\bar{\rho})^2 + \bar{\eta}^2]} \simeq 28.4(2.9) \,, \tag{56.103}$$

with [723]:

$$\lambda \simeq 0.2237(33) \,, \qquad \bar{\rho} \equiv \rho\left(1 - \frac{\lambda^2}{2}\right) \simeq 0.223(38) \,,$$
$$\bar{\eta} \equiv \eta\left(1 - \frac{\lambda^2}{2}\right) \simeq 0.316(40) \,, \tag{56.104}$$

[3] One can notice that similar strengths of the $SU(3)$ breakings are obtained for the $B \to K^*\gamma$ and $B \to K l \nu$ form factors [818].

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$\lambda,\ \rho,\ \eta$ being the Wolfenstein parameters, we deduce:

$$\Delta M_s \simeq 18.6(2.2)\text{ps}^{-1}\ , \tag{56.105}$$

in good agreement with the present experimental lower bound.

56.2.4 Conclusions

We have applied QCD spectral sum rules for extracting *(for the first time)* the $SU(3)$ breaking parameter in Eq. (56.98). The phenomenological consequences of our results for the $B^0_{d,s}$-$\bar{B}^0_{d,s}$ mass differences and CKM mixing angle have been discussed. An extension of this work to the study of the $B^0_{s,d}$-$\bar{B}^0_{s,d}$ width difference is in progress.

56.3 The $\Delta S = 2$ transition of the K^0-$\bar{K}^0$ mixing

56.3.1 Estimate of the bag constant B_K

This parameter plays an important rôle for the CP violation parameter in connection with the previous quantities f_B and B_B. The B_K-parameter is associated to the K^0-$\bar{K}^0$ mixing matrix as:

$$\langle \bar{K}^0 | \bar{b}\gamma^L_\mu d\bar{d}\gamma^L_\mu b | K^0 \rangle = \frac{4}{3} f_K^2 M_K^2 B_K(\nu)\ , \tag{56.106}$$

where as before, one has also introduced the RGI parameter $\hat{B}_K$. We estimate this quantity using the four-quark two-point correlator as in [814,815]. Using the Laplace sum rule (LSR) and adopting the parametrization of the spectral function in [814], we have obtained the *conservative* estimate [815]:

$$\hat{B}_K \simeq (0.58 \pm 0.22)\ , \tag{56.107}$$

where the central value is slightly higher than the one from FESR [814]: $\hat{B}_K \simeq (0.39 \pm 0.10)$. This difference might be attributed to the fact that FESR is strongly affected by the higher radial excitation contributions that are not under good control. LSR has the advantage is less sensitive to these effects due to the exponential factor which suppresses their relative contributions. One can also notice that this result from the two-point function sum rule is more accurate than the one from the three-point function [3], which ranges from 0.2 to 1.3, although the result of [3] is in good agreement with ours. This inaccuracy can be intuitively understood from the relative complexity of the three-point function sum rule analysis for parametrizing the higher-states contributions to the spectral function.

56.3.2 Estimate of the CP violation parameters $(\bar{\rho}, \bar{\eta})$

We are now ready to discuss the implications of the previous results for the estimate of the CKM parameters $(\bar{\rho}, \bar{\eta})$ defined in the standard way within the Wolfenstein parametrization [16,665,500,820].

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Within this parametrization, one can express the CP violation of the kaon system as:

$$|\epsilon| = C_\epsilon A^2\lambda^6\bar{\eta}[-\eta_1 S(x_c) + \eta_2 S(x_t)(A^2\lambda^4(1-\bar{\rho})) + \eta_3 S(x_c, x_t)]\hat{B}_K\ , \quad (56.108)$$

where:

$$C_\epsilon = \frac{G_F^2 f_K^2 M_K M_W^2}{3\sqrt{2}\pi^2 \Delta M_K}\ ; \quad (56.109)$$

$S(x_i)$, $S(x_i, x_j)$, $\eta_1 \simeq 1.38$, $\eta_2 \simeq 0.574$, $\eta_3 \simeq 0.47$ are short-distance functions calculable perturbatively [811,475,665,812] with $x_q \equiv m_q^2/M_W^2$; $(A, \lambda, \bar{\rho}, \bar{\eta})$ are set of CKM parameters within the Wolfenstein parametrization. For a self-consistent analysis, it is essential to use the previous values of f_B, B_B and B_K, which are all obtained from a unique method. Using the phenomenological analysis in [723,820], one can approximately obtain:

$$|\epsilon| \simeq \frac{4}{3}\hat{B}_K \mathrm{Im}(V_{ts}^* V_{td})(18.9 - 14.4\bar{\rho})\ , \quad (56.110)$$

where $\mathrm{Im}(V_{ts}^* V_{td}) \simeq (1.2 \pm 0.2) \times 10^{-4}$ and $\bar{\rho} \simeq 0.2 \pm 0.1$. With such values, one can, for example, deduce:

$$|\epsilon| \simeq (14.8 \pm 5.6) \times 10^{-4}\ , \quad (56.111)$$

which agrees within about 1σ with the experimental value in Eq. (56.81).

56.4 Kaon penguin matrix elements and ϵ'/ϵ

56.4.1 SM theory of ϵ'/ϵ

In the SM, it is customary to study the $\Delta S = 1$ process from the weak Hamiltonian:

$$\mathcal{H}_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu) Q_i(\mu)\ , \quad (56.112)$$

where $C_i(\mu)$ are known perturbative Wilson coefficients including complete NLO QCD corrections [665], which read in the notation of [665]:

$$C_i(\mu) \equiv z_i(\mu) - \frac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*} y_i(\mu)\ , \quad (56.113)$$

where V_{ij} are elements of the CKM-matrix; $Q_i(\mu)$ are non-perturbative hadronic matrix elements which need to be estimated from different non-perturbative methods of QCD (chiral perturbation theory, lattice, QCD spectral sum rules, ...). In the choice of basis of [665], the dominant contributions come from the four-quark operators which are classified as:

- **Current-current:**

$$\mathcal{Q}_1 \equiv (\bar{s}_\alpha u_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A}\ , \qquad \mathcal{Q}_2 \equiv (\bar{s}u)_{V-A}(\bar{u}d)_{V-A}\ . \quad (56.114)$$

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- **QCD penguins:**

$$\mathcal{Q}_3 \equiv (\bar{s}d)_{V-A} \sum_{u,d,s} (\bar{\psi}\psi)_{V-A} , \qquad \mathcal{Q}_4 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} (\bar{\psi}_\beta \psi_\alpha)_{V-A} ,$$
$$\mathcal{Q}_5 \equiv (\bar{s}d)_{V-A} \sum_{u,d,s} (\bar{\psi}\psi)_{V+A} , \qquad \mathcal{Q}_6 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} (\bar{\psi}_\beta \psi_\alpha)_{V+A} . \qquad (56.115)$$

- **Electroweak penguins:**

$$\mathcal{Q}_7 \equiv \frac{3}{2}(\bar{s}d)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}\psi)_{V+A} , \qquad \mathcal{Q}_8 \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}_\beta \psi_\alpha)_{V+A} ,$$
$$\mathcal{Q}_9 \equiv \frac{3}{2}(\bar{s}d)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}\psi)_{V-A} , \qquad \mathcal{Q}_{10} \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{u,d,s} e_\psi (\bar{\psi}_\beta \psi_\alpha)_{V-A}, \qquad (56.116)$$

where α, β are colour indices; e_ψ denotes the electric charges[4] reflecting the electroweak nature of $\mathcal{Q}_{7,\ldots10}$, while $V-(+)A \equiv (1-(+)\gamma_5)\,\gamma_\mu$. Using an OPE of the amplitudes, one obtains:

$$\frac{\epsilon'}{\epsilon} \simeq \mathrm{Im}\lambda_t \big[P^{(1/2)} - P^{(3/2)}\big] e^{i\Phi} , \qquad (56.117)$$

where $\Phi \equiv \Phi_{\epsilon'} - \Phi_\epsilon \approx 0$ (see previous section); $\lambda_t \equiv V_{td}V_{ts}^*$ can be expressed in terms of the CKM matrix elements as (δ being the CKM phase) [665,820]:

$$\mathrm{Im}\lambda_t \approx |V_{ub}||V_{cb}|\sin\delta \simeq (1.33 \pm 0.14) \times 10^{-4} , \qquad (56.118)$$

from B-decays and ϵ. The QCD quantities $P^{(I)}$ read:

$$P^{(1/2)} = \frac{G_F|\omega|}{2|\epsilon|\mathrm{Re}A_0} \sum_i C_i(\mu)\langle(\pi\pi)_{I=0}|\mathcal{Q}_i|K^0\rangle_0 \,(1-\Omega_{IB}) \;,$$
$$P^{(3/2)} = \frac{G_F}{2|\epsilon|\mathrm{Re}A_2} \sum_i C_i(\mu)\langle(\pi\pi)_{I=2}|\mathcal{Q}_i|K^0\rangle_2 \,. \qquad (56.119)$$

$\Omega_{IB} \simeq (0.16 \pm 0.03)$ quantifies the $SU(2)$-isospin breaking effect, which includes the one of the π^0-η mixing [821], and which reduces the usual value of (0.25 ± 0.08) [665] due to η'-η mixing. It is also expected that the QCD- and electroweak-penguin operators:

$$\mathcal{Q}_8^{3/2} \approx B_8^{3/2}/m_s^2 + \mathcal{O}(1/N_c) , \qquad \mathcal{Q}_6^{1/2} \approx B_6^{1/2}/m_s^2 + \mathcal{O}(1/N_c) , \qquad (56.120)$$

give the dominant contributions to the ratio ϵ'/ϵ [822]; B are the bag factors which are expected to be 1 in the large N_c-limit. Therefore, a simplified approximate but very informative

[4] Though apparently suppressed, the effect of the electroweak penguins are enhanced by $1/\omega$ as we shall see later on in Eq. (56.119).

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expression of the theoretical predictions can be derived [665]:

$$\frac{\epsilon'}{\epsilon} \approx 13\ \mathrm{Im}\lambda_t \left(\frac{110}{\bar{m}_s(2)\ [\mathrm{MeV}]}\right)^2 \times \left[B_6^{1/2}(1-\Omega_{IB}) - 0.4 B_8^{3/2}\left(\frac{m_t}{165\ \mathrm{GeV}}\right)\right]\left(\frac{\Lambda^{(4)}_{\overline{MS}}}{340\ \mathrm{MeV}}\right), \quad (56.121)$$

where the average value $\hat{B}_K = 0.80 \pm 0.15$ of the $\Delta S = 2$ process has been used. This value includes the conservative value 0.58 ± 0.22 from Laplace sum rules [815]. The values of the top quark mass and the QCD scale $\Lambda^{(4)}_{\overline{MS}}$ [16,139] are under quite good control and have small effects. A recent review of the light quark mass determinations [54] also indicates that the strange quark mass is also under control and a low value advocated in the previous literature to explain the present data on ϵ'/ϵ is unlikely to be due to the lower bound constraints from the positivity of the QCD spectral function or from the positivity of the m^2 corrections to the GMOR PCAC relation. For a consistency with the approach used in this paper, we shall use the average value of the light quark masses from QCD spectral sum rules(QSSR), e^+e^- and τ-decays given in [54] (previous chapter):

$$\bar{m}_s(2) \simeq (117 \pm 23)\ \mathrm{MeV}, \quad \bar{m}_d(2) \simeq (6.5 \pm 1.2)\ \mathrm{MeV}, \quad \bar{m}_u(2) \simeq (3.6 \pm 0.6)\ \mathrm{MeV}. \quad (56.122)$$

Using the previous experimental values, one can deduce the constraint in [54] updated:

$$\mathcal{B}_{68} \equiv B_6^{1/2} - 0.48 B_8^{3/2} \simeq 1.4 \pm 0.6\ (\text{resp.} \geq 0.5), \quad (56.123)$$

if one uses the value of m_s in Eq. (56.122) (resp. the lower bound of 71 MeV reported in [54]). This result shows a possible violation of more than 2σ for the leading $1/N_c$ vacuum saturation prediction ≈ 0.52 corresponding to $B_6^{1/2} \approx B_8^{3/2} \approx 1$. Consulting the available predictions reviewed in [665], which we will summarize and update in Table 56.1, one can notice that the values of the B parameters have large errors. One can also see that results from QCD first principles (lattice and $1/N_c$) fail to explain the data, which however can be accomodated by various QCD-like models. We shall come back to this discussion when we shall compare our results with presently available predictions. It is, therefore, clear that the present estimate of the four-quark operators, and in particular the estimates of the dominant penguin ones given previously in Eq. (56.120), need to be re-investigated. Due to the complex structures and large size of these operators, they should be difficult to extract unambiguously from different approaches. In this paper, we present alternative theoretical approaches based also on first principles of QCD (τ–decay data, analyticity), for predicting the size of the QCD- and electroweak-penguin operators given in Eq. (56.120). In performing this analysis, we shall also encounter the electroweak penguin operator:

$$\mathcal{Q}_7^{3/2} \approx B_7^{3/2}/m_s^2 + \mathcal{O}(1/N_c). \quad (56.124)$$

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Table 56.1. *Penguin B parameters for the $\Delta S = 1$ process from different approaches at $\mu = 2$ GeV. We use the value $m_s(2) = (117 \pm 23)$ MeV from [54], and predictions based on dispersion relations [833,832] have been rescaled according to it. We also use for our results $f_\pi = 92.4$ MeV [16], but we give in the text their m_s and f_π dependences. Results without any comments on the scheme have been obtained in the $\overline{MS} - NDR-$scheme (see discussions on γ_5 in Appendix D). However, at the present accuracy, one cannot differentiate these results from the ones of $\overline{MS} - HV-$scheme. More recent results can also found in [838].*

Methods	$B_6^{1/2}$	$B_8^{3/2}$	$B_7^{3/2}$	Comments
Lattice [823,824,825]	$0.6 \sim 0.8$	$0.7 \sim 1.1$	$0.5 \sim 0.8$	Huge NLO
	unreliable			at matching [826]
Large N_c [827]	$0.7 \sim 1.3$	$0.4 \sim 0.7$	$-0.10 \sim 0.04$	$\mathcal{O}(p^0/N_c,\ p^2)$
				scheme?
	$1.5 \sim 1.7$	–	–	$\mathcal{O}(p^2/N_c)$; $m_q = 0$
				scheme?
Models				
Chiral QM [828]	$1.2 \sim 1.7$	~ 0.9	$\approx B_8^{3/2}$	$\mu = 0.8$ GeV
				rel. with $\overline{MS}$?
ENJL + IVB [829]	2.5 ± 0.4	1.4 ± 0.2	0.8 ± 0.1	NLO in $1/N_c$
				$m_q = 0$
Lσ-model [830]	~ 2	~ 1.2	–	Not unique
				$\mu \approx 1$ GeV; scheme?
NL σ-model [831]	$1.6 \sim 3.0$	$0.7 \sim 0.9$	–	M_σ: free; $SU(3)_F$ trunc.
				$\mu \approx 1$ GeV; scheme?
Dispersive				
Large N_c+ LMD	–	–	0.9	NLO in $1/N_c$,
+ LSD–match. [832]			strong μ-dep.	
DMO-like SR [833]	–	1.6 ± 0.4	0.8 ± 0.2	$m_q = 0$
		huge NLO		Strong s, μ–dep.
FSI [834]	1.4 ± 0.3	0.7 ± 0.2	–	Debate for fixing
				the Slope [835]
This work [836,34]				
DMO-like SR:	–	2.2 ± 1.5	0.7 ± 0.2	$m_q = 0$
[833] revisited		inaccurate		Strong s, μ–dep.
τ-like SR	–	–	inaccurate	t_c–changes
R_τ^{V-A}	–	1.7 ± 0.4	–	$m_q = 0$
$S_2 \equiv (\bar{u}u + \bar{d}d)$	1.0 ± 0.4	–	–	$\overline{MS}$ scheme
from QSSR	$\leq 1.5 \pm 0.4$			$m_s(2) \geq 90$ MeV

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56.4.2 Soft pion and kaon reductions of $\langle(\pi\pi)_{I=2}|\mathcal{Q}^{3/2}_{7,8}|K^0\rangle$ to vacuum condensates

We shall consider here the kaon electroweak penguin matrix elements:

$$\left\langle \mathcal{Q}^{3/2}_{7,8}\right\rangle_{2\pi} \equiv \left\langle (\pi\pi)_{I=2}\middle|\mathcal{Q}^{3/2}_{7,8}\middle|K^0\right\rangle , \tag{56.125}$$

defined as:

$$\begin{aligned}
\mathcal{Q}^{3/2}_7 &\equiv \frac{3}{2}\,(\bar{s}d)_{V-A}\sum_{u,d,s} e_\psi(\bar\psi\psi)_{V+A}\ ,\\
\mathcal{Q}^{3/2}_8 &\equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A}\sum_{u,d,s} e_\psi(\bar\psi_\beta\psi_\alpha)_{V+A}\ ,
\end{aligned} \tag{56.126}$$

where α, β are colour indices; e_ψ denotes the electric charges. In the chiral limit $m_{u,d,s} \sim m^2_\pi \simeq m^2_K = 0$, one can use soft pion and kaon techniques in order to relate the previous amplitude to the four-quark vacuum condensates [833] (see also [832]):[5]

$$\begin{aligned}
\left\langle \mathcal{Q}^{3/2}_7\right\rangle_{2\pi} &\simeq -\frac{2}{f^3_\pi}\left\langle \mathcal{O}^{3/2}_7\right\rangle ,\\
\left\langle \mathcal{Q}^{3/2}_8\right\rangle_{2\pi} &\simeq -\frac{2}{f^3_\pi}\left\{\frac{1}{3}\left\langle \mathcal{O}^{3/2}_7\right\rangle + \frac{1}{2}\left\langle \mathcal{O}^{3/2}_8\right\rangle\right\}\ ,
\end{aligned} \tag{56.127}$$

where we use the shorthand notations: $\langle 0|\mathcal{O}^{3/2}_{7,8}|0\rangle \equiv \langle \mathcal{O}^{3/2}_{7,8}\rangle$, and $f_\pi = (92.42 \pm 0.26)$ MeV.[6] Here:

$$\begin{aligned}
\mathcal{O}^{3/2}_7 &= \sum_{u,d,s} \bar\psi\gamma_\mu\frac{\tau_3}{2}\psi\bar\psi\gamma_\mu\frac{\tau_3}{2}\psi - \bar\psi\gamma_\mu\gamma_5\frac{\tau_3}{2}\psi\bar\psi\gamma_\mu\gamma_5\frac{\tau_3}{2}\psi\ ,\\
\mathcal{O}^{3/2}_8 &= \sum_{u,d,s} \bar\psi\gamma_\mu\lambda_a\frac{\tau_3}{2}\psi\bar\psi\gamma_\mu\lambda_a\frac{\tau_3}{2}\psi - \bar\psi\gamma_\mu\gamma_5\lambda_a\frac{\tau_3}{2}\psi\bar\psi\gamma_\mu\gamma_5\lambda_a\frac{\tau_3}{2}\psi\ ,
\end{aligned} \tag{56.128}$$

where τ_3 and λ_a are flavour and colour matrices. Using further pion and kaon reductions in the chiral limit, one can relate this matrix element to the B-parameters [833]:

$$\begin{aligned}
B^{3/2}_7 &\simeq \frac{3}{2}\frac{(m_u+m_d)}{m^2_\pi}\frac{(m_u+m_s)}{m^2_K}\frac{1}{f_\pi}\left\langle \mathcal{Q}^{3/2}_7\right\rangle_{2\pi}\\
B^{3/2}_8 &\simeq \frac{1}{2}\frac{(m_u+m_d)}{m^2_\pi}\frac{(m_u+m_s)}{m^2_K}\frac{1}{f_\pi}\left\langle \mathcal{Q}^{3/2}_8\right\rangle_{2\pi}
\end{aligned} \tag{56.129}$$

where all QCD quantities will be evaluated in the NDR-$\overline{MS}$ scheme and at the scale M_τ.

[5] In the following discussion, we shall use a normalization of the matrix elements which differ by a factor 2 from the one used in [833,836]. This is due to the uses of the operator $\mathcal{Q}^{3/2}_8$ in Eq. 56.126 currently used in the literature rather the one: $(\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} - (\bar{d}_\beta d_\alpha)_{V+A} + (\bar{s}_\beta s_\alpha)_{V+A}]$ used in [833] and [836].

[6] In the chiral limit f_π would be about 87 MeV. However, it is not clear to us what value of f_π should be used here because we shall use real data from τ-decay. Therefore, we shall leave it as a free parameter which the reader can fix at his convenience.

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56.4.3 The $\langle\mathcal{O}_{7,8}^{3/2}\rangle$ condensates from DMO-like sum rules in the chiral limit

In previous papers [833,832], the vacuum condensates $\langle\mathcal{O}_{7,8}^{3/2}\rangle$ have been extracted using Das–Mathur–Okubo(DMO)- and Weinberg-like sum rules based on the difference of the vector and axial-vector spectral functions $\rho_{V,A}$ of the $I=1$ component of the neutral current:

$$2\pi\langle\alpha_s\mathcal{O}_8^{3/2}\rangle = \int_0^\infty ds\, s^2\frac{\mu^2}{s+\mu^2}(\rho_V-\rho_A)\ ,$$
$$\frac{16\pi^2}{3}\langle\mathcal{O}_7^{3/2}\rangle = \int_0^\infty ds\, s^2 log\left(\frac{s+\mu^2}{s}\right)(\rho_V-\rho_A)\ , \tag{56.130}$$

where μ is the subtraction point. In this normalization, the first Weinberg sum rule gives in the chiral limit:

$$\int_0^\infty ds\,(\rho_V-\rho_A) = f_\pi^2\ . \tag{56.131}$$

Due to the quadratic divergence of the integrand, the previous sum rules are expected to be sensitive to the high energy tails of the spectral functions where the present ALEPH/OPAL data from τ-decay [193,199] are inaccurate. This inaccuracy can a priori affect the estimate of the four-quark vacuum condensates. On the other hand, the explicit μ–dependence of the analysis can also induce another uncertainty. En passant, we check below the effects of these two parameters t_c and μ. After evaluating the spectral integrals, we obtain at $\mu=2$ GeV and for our previous values of $t_c \simeq (1.48\pm 0.02)$ GeV^2 (see Chapter on Weinberg sum rules), the values (in units of 10^{-3} GeV^6) using the cut-off momentum scheme (c.o):

$$\alpha_s\langle\mathcal{O}_8^{3/2}\rangle_{c.o} \simeq -(0.69\pm0.06)\ ,\qquad \langle\mathcal{O}_7^{3/2}\rangle_{c.o}\simeq -(0.11\pm0.01)\ , \tag{56.132}$$

where the errors come mainly from the small changes of t_c-values. If instead, we use the second set of values of $t_c \simeq (2.4\sim 2.6)$ GeV^2 (see Chapter on Weinberg sum rules), we obtain by setting $\mu = 2$ GeV:

$$\alpha_s\langle\mathcal{O}_8^{3/2}\rangle_{c.o} \simeq -(0.6\pm0.3)\ ,\qquad \langle\mathcal{O}_7^{3/2}\rangle_{c.o}\simeq -(0.10\pm0.03)\ , \tag{56.133}$$

which is consistent with the one in Eq. (56.132), but with larger errors as expected. We have also checked that both $\langle\mathcal{O}_8^{3/2}\rangle$ and $\langle\mathcal{O}_7^{3/2}\rangle$ increase in absolute value when μ increases where a stronger change is obtained for $\langle\mathcal{O}_7^{3/2}\rangle$, a feature which has been already noticed in [832]. In order to give a more conservative estimate, we consider as our final value the largest range spanned by our results from the two different sets of t_c-values. This corresponds to the one in Eq. (56.133) which is the less accurate prediction. We shall use the relation between the momentum cut-off (c.o) and $\overline{MS}$ schemes given in [833]:

$$\langle\mathcal{O}_7^{3/2}\rangle_{\overline{MS}} \simeq \langle\mathcal{O}_7^{3/2}\rangle_{c.o} + \frac{3}{8}a_s\left(\frac{3}{2}+2d_s\right)\langle\mathcal{O}_8^{3/2}\rangle$$
$$\langle\mathcal{O}_8^{3/2}\rangle_{\overline{MS}} \simeq \left(1-\frac{119}{24}a_s \pm\left(\frac{119}{24}a_s\right)^2\right)\langle\mathcal{O}_8^{3/2}\rangle_{c.o} - a_s\langle\mathcal{O}_7^{3/2}\rangle\ , \tag{56.134}$$

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where $d_s = -5/6$ (resp 1/6) in the so-called Naïve Dimensional Regularization NDR (resp. t'Hooft-Veltmann HV) schemes;[7] $a_s \equiv \alpha_s/\pi$. One can notice that the a_s coefficient is large in the second relation (50% correction), and the situation is worse because of the relative minus sign between the two contributions. Therefore, we have added a rough estimate of the a_s^2 corrections based on the naïve growth of the PT series, which here gives 50% corrections of the sum of the two first terms. For a consistency of the whole approach, we shall use the value of α_s obtained from τ-decay, which is [193,199]:

$$\alpha_s(M_\tau)|_{\rm exp} = 0.341 \pm 0.05 \quad \Longrightarrow \quad \alpha_s(2\ {\rm GeV}) \simeq 0.321 \pm 0.05\ . \tag{56.135}$$

Then, we deduce (in units of 10^{-4} GeV^6) at 2 GeV:

$$\langle \mathcal{O}_7^{3/2} \rangle_{\overline{MS}} \simeq -(0.7 \pm 0.2)\ , \qquad \langle \mathcal{O}_8^{3/2} \rangle_{\overline{MS}} \simeq -(9.1 \pm 6.4)\ , \tag{56.136}$$

where the large error in $\langle \mathcal{O}_8^{3/2} \rangle$ comes from the estimate of the a_s^2 corrections appearing in Eq. (56.134). In terms of the B factor and with the mean value of the light quark masses quoted in [54], this result, at $\mu = 2$ GeV, can be translated into:

$$B_7^{3/2} \simeq (0.7 \pm 0.2) \left(\frac{m_s(2)\ [{\rm MeV}]}{119} \right)^2 k^4\ ,$$
$$B_8^{3/2} \simeq (2.5 \pm 1.3) \left(\frac{m_s(2)\ [{\rm MeV}]}{119} \right)^2 k^4\ , \tag{56.137}$$

where:

$$k \equiv \frac{92.4}{f_\pi\ [{\rm MeV}]}\ . \tag{56.138}$$

- Our results in Eqs. (56.136) compare quite well with the ones obtained by [833] in the $\overline{MS}$ scheme (in units of 10^{-4} GeV^6) at 2 GeV:

$$\langle \mathcal{O}_8^{3/2} \rangle_{\overline{MS}} \simeq -(6.7 \pm 0.9)\ , \qquad \langle \mathcal{O}_7^{3/2} \rangle_{\overline{MS}} \simeq -(0.70 \pm 0.10)\ , \tag{56.139}$$

  using the same sum rules but presumably a slightly different method for the uses of the data and for the choice of the cut-off in the evaluation of the spectral integral.
- Our errors in the evaluation of the spectral integrals, leading to the values in Eqs. (56.132) and (56.133), are mainly due to the slight change of the cut-off value t_c.[8]
- The error due to the passage into the $\overline{MS}$ scheme is due mainly to the truncation of the QCD series, and is important (50%) for $\langle \mathcal{O}_8^{3/2} \rangle$ and $B_8^{3/2}$, which is the main source of errors in our estimate.
- As noticed earlier, in the analysis of the pion mass difference, it looks more natural to do the subtraction at t_c. We also found that moving the value of μ can affects the value of $B_{7,8}^{3/2}$.

For the above reasons, we expect that the results given in [833] for $\langle \mathcal{O}_8^{3/2} \rangle$ although interesting are quite fragile, while the errors quoted there have been presumably underestimated.

[7] The two schemes differ by the treatment of the γ_5 matrix (see Section 8.2).

[8] A slight deviation from such a value affects notably previous predictions as the t_c-stability of the results ($t_c \approx 2$ GeV^2) does not coincide with the one required by the second Weinberg sum rules. At the stability point the predictions are about a factor 3 higher than the one obtained previously.

Therefore, we think that a reconsideration of these results using alternative methods are mandatory.[9]

56.4.4 The $\langle\mathcal{O}_{7,8}^{3/2}\rangle$ condensates from hadronic tau inclusive decays

In the following, we shall not introduce any new sum rule, but, instead, we shall exploit known informations from the total τ-decay rate and available results from it, which have not the previous drawbacks. The V-A total τ-decay rate, for the $I = 1$ hadronic component, can be deduced from BNP [325], and reads:[10]

$$R_{\tau,V-A} = \frac{3}{2}|V_{ud}|^2 S_{EW} \sum_{D=2,4,\ldots} \delta_{V-A}^{(D)} . \tag{56.140}$$

$|V_{ud}| = 0.9753 \pm 0.0006$ is the CKM-mixing angle, while $S_{EW} = 1.0194$ is the electroweak corrections [326]. In the following, we shall use the BNP results for $\mathcal{R}_{\tau,V/A}$ in order to deduce $R_{\tau,V-A}$:

- The chiral invariant $D = 2$ term due to a short distance tachyonic gluon mass [162,161] cancels in the V-A combination. Therefore, the $D = 2$ contributions come only from the quark mass terms:

$$M_\tau^2 \delta_{V-A}^{(2)} \simeq 8\left[1 + \frac{25}{3}a_s(M_\tau)\right] m_u m_d , \tag{56.141}$$

  as can be obtained from the first calculation [28], where $m_u \equiv m_u(M_\tau) \simeq (3.6 \pm 0.6)$ MeV, $m_d \equiv m_d(M_\tau) \simeq (6.5 \pm 1.2)$ MeV [54] (previous chapter) are respectively the running coupling and quark masses evaluated at the scale M_τ.
- The dimension-four condensate contribution reads:

$$M_\tau^4 \delta_{V-A}^{(4)} \simeq 32\pi^2 \left(1 + \frac{9}{2}a_s^2\right) m_\pi^2 f_\pi^2 + \mathcal{O}\left(m_{u,d}^4\right) , \tag{56.142}$$

  where we have used the $SU(2)$ relation $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle$ and the Gell-Mann–Oakes–Renner PCAC relation:

$$(m_u + m_d)\langle\bar{u}u + \bar{d}d\rangle = -2m_\pi^2 f_\pi^2 . \tag{56.143}$$

- By inspecting the structure of the combination of dimension-six condensates entering in $R_{\tau,V/A}$ given by BNP [325], which are renormalizaton group invariants, and using a $SU(2)$ isospin rotation which relates the charged and neutral (axial)-vector currents, the $D = 6$ contribution reads:

$$M_\tau^6 \delta_{V-A}^{(6)} = -2 \times 48\pi^4 a_s \left[\left[1 + \frac{235}{48}a_s \pm \left(\frac{235}{48}a_s\right)^2 - \frac{\lambda^2}{M_\tau^2}\right]\langle\mathcal{O}_8^{3/2}\rangle + a_s\langle\mathcal{O}_7^{3/2}\rangle\right], \tag{56.144}$$

  where the overall factor 2 in front expresses the different normalization between the neutral isovector and charged currents used respectively in [833] and [325], whilst all quantities are evaluated at the scale $\mu = M_\tau$. The last two terms in the Wilson coefficients of $\langle\mathcal{O}_8^{3/2}\rangle$ are new: the first term is an estimate of the NNLO term by assuming a naïve geometric growth of the a_s series; the second one is the effect of a tachyonic gluon mass introduced in [161], which takes into account

[9] In recent works [838], these results have been also reconsidered.
[10] Hereafter we shall work in the $\overline{MS}-$ *NDR* scheme.

the re-summation of the QCD asymptotic series, with: $a_s\lambda^2 \simeq -0.06$ GeV^2.[11] Using the values of $\alpha_s(M_\tau)$ given previously, the corresponding QCD series behaves quite well as:

$$\text{Coef.}\,\langle\mathcal{O}_8^{3/2}\rangle \simeq 1 + (0.53 \pm 0.08) \pm 0.28 + 0.18\,, \tag{56.145}$$

where the first error comes from the one of α_s, while the second one is due to the unknown a_s^2-term, which introduces an uncertainty of 16% for the whole series. The last term is due to the tachyonic gluon mass. This leads to the numerical value:

$$M_\tau^6\delta^{(6)}_{V-A} \simeq -(1.015 \pm 0.149) \times 10^3\left[(1.71 \pm 0.29)\langle\mathcal{O}_8^{3/2}\rangle + a_s\langle\mathcal{O}_7^{3/2}\rangle\right], \tag{56.146}$$

- If, one estimates the $D = 8$ contribution using a vacuum saturation assumption, the relevant V-A combination vanishes to leading order of the chiral symmetry breaking terms. Instead, we shall use the combined ALEPH/OPAL [193,199] fit for $\delta^{(8)}_{V/A}$, and deduce:

$$\delta^{(8)}_{V-A}\big|_{\text{exp}} = -(1.58 \pm 0.12) \times 10^{-2}\,. \tag{56.147}$$

We shall also use the combined ALEPH/OPAL data for $R_{\tau,V/A}$, in order to obtain:

$$R_{\tau,V-A}|_{\text{exp}} = (5.0 \pm 1.7) \times 10^{-2}\,, \tag{56.148}$$

Using the previous information in the expression of the rate given in Eq. (56.140), one can deduce:

$$\delta^{(6)}_{V-A} \simeq (4.49 \pm 1.18) \times 10^{-2}\,. \tag{56.149}$$

This result is in good agreement with the result obtained by using the ALEPH/OPAL fitted mean value for $\delta^{(6)}_{V/A}$:

$$\delta^{(6)}_{V-A}|_{\text{fit}} \simeq (4.80 \pm 0.29) \times 10^{-2}\,. \tag{56.150}$$

We shall use as a final result the average of these two determinations, which coincides with the most precise one in Eq. (56.150). We shall also use the result:

$$\frac{\langle\mathcal{O}_7^{3/2}\rangle}{\langle\mathcal{O}_8^{3/2}\rangle} \simeq \frac{1}{8.3}\left(\text{resp. } \frac{3}{16}\right), \tag{56.151}$$

where, for the first number we use the value of the ratio of $B_7^{3/2}/B_8^{3/2}$ which is about $0.7 \sim 0.8$ from, for example, lattice calculations quoted in Table 56.1, and the formulae in Eqs. (56.127) to (56.129); for the second number we use the vacuum saturation for the four-quark vacuum condensates [1]. The result in Eq. (56.151) is also comparable with the estimate of [833] from the sum rules given in Eq. (56.130). Therefore, at the scale $\mu = M_\tau$, Eqs. (56.144), (56.150) and (56.151) lead, in the $\overline{MS}$ scheme, to:

$$\langle\mathcal{O}_8^{3/2}\rangle(M_\tau) \simeq -(0.94 \pm 0.21) \times 10^{-3}\ \text{GeV}^6\,, \tag{56.152}$$

where the main errors come from the estimate of the unknown higher-order radiative corrections. It is instructive to compare this result with the one using the vacuum saturation

[11] This contribution may compete with the dimension-eight operators discussed in [837].

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assumption for the four-quark condensate (see e.g. BNP):

$$\langle \mathcal{O}_8^{3/2} \rangle|_{v.s} \simeq -\frac{32}{18} \langle \bar{u}u \rangle^2 (M_\tau) \simeq -0.65 \times 10^{-3} \text{ GeV}^6 , \quad (56.153)$$

which shows about 1σ violation of this assumption. We have used for the estimate of $\langle \bar{\psi}\psi \rangle$ the value of $(m_u + m_d)(M_\tau) \simeq 10$ MeV [54] and the GMOR pion PCAC relation. However, this violation of the vacuum saturation is not quite surprising, as a similar fact has also been observed in other channels [3,193,199], although it also appears that the vacuum saturation gives a quite good approximate value of the ratio of the condensates [3,193,199]. The result in Eq. (56.152) is comparable with the value $-(0.98 \pm 0.26) \times 10^{-3}$ GeV^6 at $\mu = 2$ GeV $\approx M_\tau$ obtained by [833] using a DMO-like sum rule, but, as discussed previously, the DMO-like sum rule result is very sensitive to the value of μ if one fixes t_c as 1.48 GeV^2 (see chapter on Weinberg sum rules) according to the criterion discussed above. Here, the choice $\mu = M_\tau$ is well-defined, and then the result becomes more accurate (as mentioned previously our errors come mainly from the estimated unknown α_s^3 term of the QCD series). Using Eqs. (56.127) and (56.151), our previous results in Eq. (56.136) for $\mathcal{O}_7^{3/2}$ and in Eq. (56.152) for $\mathcal{O}_8^{3/2}$ can be translated into the prediction on the weak matrix elements in the chiral limit and at the scale 2 GeV for the NDR scheme ($k \equiv 92.4/f_\pi$[MeV] is defined in Eq. (56.138)):[12]

$$\begin{aligned} \langle (\pi\pi)_{I=2} | \mathcal{Q}_7^{3/2} | K^0 \rangle (2) &\simeq (0.18 \pm 0.05) \text{ GeV}^3 \, k^3 \\ \langle (\pi\pi)_{I=2} | \mathcal{Q}_8^{3/2} | K^0 \rangle (2) &\simeq (1.35 \pm 0.30) \text{ GeV}^3 \, k^3 , \end{aligned} \quad (56.154)$$

normalized to f_π, which avoids the ambiguity on the real value of f_π to be used in such an expression. Our result is in agreement with different determinations from dispersive approaches [832,833,838]. Our result is higher by about a factor of 2 than the quenched lattice result [823]. A resolution of this discrepancy can only be found after the inclusion of chiral corrections in Eqs. (56.127) to (56.129), and after the use of dynamic fermions on the lattice. However, some parts of the chiral corrections in the estimate of the vacuum condensates are already included into the QCD expression of the τ-decay rate and these corrections are negligibly small. We might expect that chiral corrections, which are smooth functions of m_π^2 will not strongly affect the relation in Eqs. (56.127) to (56.129), although an evaluation of their exact size is mandatory. Using the previous mean values of the light quark running masses [54], we deduce in the chiral limit and at the scale M_τ:

$$B_8^{3/2} \simeq (1.70 \pm 0.39) \left(\frac{m_s(M_\tau) \text{ [MeV]}}{119} \right)^2 k^4 , \quad (56.155)$$

where k is defined in Eq. (56.138). One should notice that, contrary to the B-factor, the result in Eq. (56.154) is independent to leading order of the value of the light quark masses.

[12] As already mentioned, this normalization differs by a factor 2 than the one used in [833,836].

56.4.5 Impact of the results on the CP violation parameter ϵ'/ϵ

One can combine the previous result of B_8 with the value of the B_6 parameter of the QCD penguin diagram [665]:

$$\begin{aligned}\langle \mathcal{Q}_6^{1/2}\rangle_{2\pi} &\equiv \langle (\pi^+\pi^-)_{I=0}|\mathcal{Q}_6^{1/2}|K^0\rangle \\ &\simeq -[2\langle\pi^+|\bar{u}\gamma_5 d|0\rangle\langle\pi^-|\bar{s}u|K^0\rangle \\ &\quad + \langle\pi^+\pi^-|\bar{d}d+\bar{u}u|0\rangle\langle 0|\bar{s}\gamma_5 d|K^0\rangle\ ] \\ &\simeq -4\sqrt{\frac{3}{2}}\left(\frac{m_K^2}{m_s+m_d}\right)^2 \\ &\quad \times\sqrt{2}\,(f_K-f_\pi)\,B_6^{1/2}(m_c)\,. \end{aligned} \tag{56.156}$$

We have estimated the $\langle \mathcal{Q}_6^{1/2}\rangle_{2\pi}$ matrix element by relating its first term to the $K\to\pi\, l\nu_l$ semi-leptonic form factors as usually done (see e.g. [822]), while the second term has been obtained from the contribution of the $S_2 \equiv (\bar{u}u+\bar{d}d)$ scalar meson having its mass and coupling fixed by QCD spectral sum rules [3,688] and in the scheme where the observed low mass σ meson results from a maximal mixing between the S_2 and the σ_B associated to the gluon component of the trace of the anomaly [686,680,688]:[13]

$$\theta_\mu^\mu = \frac{1}{4}\beta(\alpha_s)G^2 + (1+\gamma_m(\alpha_s))\sum_{u,d,s} m_i\bar{\psi}_i\psi_i\ , \tag{56.157}$$

where β and γ_m are the β function and mass anomalous dimension. In this way, one obtains at the scale m_c:

$$B_6^{1/2}(m_c)\ \simeq\ 3.7\left(\frac{m_s+m_d}{m_s-m_u}\right)^2\times\left[(0.65\pm0.09)-(0.53\pm0.13)\left(\frac{(m_s-m_u)\ [\text{MeV}]}{142.6}\right)\right], \tag{56.158}$$

which satisfies the double chiral constraint. We have used the running charm quark mass $m_c(m_c) = 1.2\pm0.05$ GeV [54]. Evaluating the running quark masses at 2 GeV, with the values given in [54], one deduces:

$$\begin{aligned} B_6^{1/2}(2) &\simeq (1.1\pm0.4) \quad \text{for } m_s(2) = 117 \text{ MeV}\ , \\ &\leq (2.1\pm0.4) \quad \text{for } m_s(2)\geq 71\text{MeV}\ . \end{aligned} \tag{56.159}$$

The errors added quadratically have been relatively enhanced by the partial cancellations of the two contributions. Therefore, we deduce the combination:

$$\begin{aligned} \mathcal{B}_{68} &\equiv B_6^{3/2} - 0.48B_8^{3/2} \\ &\simeq (0.3\pm0.4) \quad \text{for } m_s(2) = 117 \text{ MeV}\ , \\ &\leq (1.3\pm0.4) \quad \text{for } m_s(2)\geq 71 \text{ MeV}\ , \end{aligned} \tag{56.160}$$

[13] Present data appear to favour this scheme [690].

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where we have added the errors quadratically. Using the approximate simplified expresssion [665]:

$$\frac{\epsilon'}{\epsilon} \approx 14.5 \times 10^{-4} \left(\frac{110}{\bar{m}_s(2)\ [\text{MeV}]} \right)^2 \mathcal{B}_{68}\ , \tag{56.161}$$

one can deduce the result in units of 10^{-4}:

$$\begin{aligned} \frac{\epsilon'}{\epsilon} &\simeq (4 \pm 5) \quad \text{for } m_s(2) = 117 \text{ MeV}\ , \\ &\leq (45 \pm 14)\ , \quad \text{for } m_s(2) \geq 71 \text{ MeV}\ , \end{aligned} \tag{56.162}$$

where the errors come mainly from $\mathcal{B}_{68}$ (40%). The upper bound, though rather weak, agrees quite well with the experimental world average data [599]:

$$\frac{\epsilon'}{\epsilon} \simeq (17.2 \pm 1.8) \times 10^{-4}\ . \tag{56.163}$$

We expect that the failure of the inaccurate estimate for reproducing the data is not naïvely due to the value of the quark mass, but may indicate the need for other important contributions than the single $\bar{q}q$ scalar meson S_2 (not the observed σ)-meson which have not been considered so far in the analysis. Among others, a much better understanding of the effects of the gluonium (expected large component of the σ-meson [686,688,687]) in the amplitude, through presumably a new operator, needs to be studied. This effect might be signalled by the success of the final state interaction approach within an effective approach (quark and gluon content blind) for reproducing the previous data [835].

56.4.6 Summary and conclusions

We have explored the V-A component of the hadronic tau decays for predicting non-perturbative QCD parameters. Our main results are summarized as:

- Electroweak penguins:
  - Eq. (56.137): $B_7^{3/2}$,
  - Eq. (56.155): $B_8^{3/2}$
  - Eq. (56.154): $\langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{3/2}|K^0\rangle$.
- QCD penguin: Eq. (56.159).
- ϵ'/ϵ: Eq. (56.162) .

Our results are compared with some other predictions in Table 56.1 (see also [838]). However, as mentioned in the table caption, a direct comparison of these results is not straightforward due to the different schemes and values of the scale where the results have been obtained. In most of the approaches, the values of $B_7^{3/2}$ are in agreement within the errors and are safely in the range $0.5 \sim 1.0$. For $B_8^{3/2}$ the predictions can differ by a factor 2 and cover the range $0.7 \sim 2.1$. There are strong disagreements by a factor 4 for the values of $B_6^{1/2}$ which range from $0.6 \sim 3.0$. We are still far from having good control of these non-perturbative parameters. This weak point does not permit us to give a reliable prediction of the CP violation parameter ϵ'/ϵ. Therefore, no definite bound for new physics effects can be derived at present, before improvements of these SM predictions.