Canad. J. Math. Vol. **65** (1), 2013 pp. 222–240
http://dx.doi.org/10.4153/CJM-2012-022-4

Distance Sets of Urysohn Metric Spaces

N. W. Sauer

Abstract. A metric space $\mathrm{M} = (M; \mathrm{d})$ is *homogeneous* if for every isometry f of a finite subspace of M to a subspace of M there exists an isometry of M onto M extending f. The space M is *universal* if it isometrically embeds every finite metric space F with $\mathrm{dist}(\mathrm{F}) \subseteq \mathrm{dist}(\mathrm{M})$ (with dist(M) being the set of distances between points in M).

A metric space U is a *Urysohn* metric space if it is homogeneous, universal, separable, and complete. (We deduce as a corollary that a Urysohn metric space U isometrically embeds every separable metric space M with $\mathrm{dist}(\mathrm{M}) \subseteq \mathrm{dist}(U)$.)

The main results are: (1) A characterization of the sets $\mathrm{dist}(U)$ for Urysohn metric spaces U. (2) If R is the distance set of a Urysohn metric space and M and N are two metric spaces, of any cardinality with distances in R, then they amalgamate disjointly to a metric space with distances in R. (3) The completion of every homogeneous, universal, separable metric space M is homogeneous.

1 Introduction

The classical *Urysohn metric space* $U_{\Re_{\geq 0}}$ is a homogeneous separable complete metric space that embeds every separable metric space; see [19]. In this paper we will discuss the following question. For which subsets R of the reals $\Re$ and for which subsets of the set of properties of being homogeneous, separable, complete, universal or embedding every separable metric space with distances in R, does there exist a metric space M with $\mathrm{dist}(\mathrm{M}) = R$? It is, for example, well known that there exists a unique homogeneous, separable, complete metric space that isometrically embeds every separable metric space with set of distances a subset of the interval $[0, 1]$. This space is the *Urysohn sphere* $U_{[0,1]}$.

A note as to terminology: countable homogeneous universal metric spaces are separable and so is their completion. In earlier papers, for example [12, 14, 15], such countable metric spaces are called Urysohn spaces. The reason for this derived from the fact that in the case of a countable homogeneous universal metric space with set of distances the non negative rationals, its completion is the classical Urysohn space. The completion was then thought of as an easy final step in the construction. In this paper we realized that a sharper distinction is necessary and hence suggest the definition given above. Note that a countable, homogeneous, universal, metric space with a finite set of distances or for which 0 is not a limit of the set of distances is a Urysohn metric space in both terminologies. The notion of homogeneous in model theory and Fraïssé theory specializes in the case of metric spaces to the one

Received by the editors August 3, 2011.
Published electronically July 3, 2012.
Supported by NSERC of Canada Grant # 691325
AMS subject classification: **03E02**, 22F05, 05C55, 05D10, 22A05, 51F99.
Keywords: partitions of metric spaces, Ramsey theory, metric geometry, Urysohn metric space, oscillation stability.

given here. But in some areas the notion homogeneous means vertex transitive, and ultrahomogeneous is used for what is defined to be homogeneous here.

In [6], Kechris, Pestov, and Todorcevic established a connection between structural Ramsey theory and automorphism groups of homogeneous relational structures. There, and in particular also in [17], the notion of oscillation stability of such groups is defined. It is shown that this notion is equivalent to a partition problem in the case of homogeneous metric spaces. A metric space $\mathrm{M} = (M; \mathrm{d})$ is *oscillation stable* if for every $\epsilon > 0$ and $f \colon M \to \Re$ bounded and uniformly continuous there exists a copy $\mathrm{M}^* = (M^*; \mathrm{d})$ of M in M such that

$$\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon.$$

Prompted by results of V. Milman [11] it is shown in [16] that the Hilbert sphere ℓ_2 is not oscillation stable and in [14] that the Urysohn sphere $U_{[0,1]}$ is oscillation stable. In particular it was shown in [14] that the Urysohn metric spaces U_n are indivisible, which, due to the main result in [8], implies the oscillation stability of $U_{[0,1]}$. (U_n is the Urysohn metric space with $\mathrm{dist}(U_n) = \{0, 1, 2, \dots, n\}$.) It is shown in [18] that all Urysohn metric spaces with a finite distance set are indivisible. This result together with the characterization result of Urysohn metric spaces given in this paper will be used (just as in the case of the results in [8] and [14] for the classical Urysohn metric space), to further investigate the oscillation stability question for Urysohn metric spaces. See [12] for details and additional references on oscillation stability.

J. Clemens in [2] proved that given a set of non-negative reals, $R \subseteq \Re_{\geq 0}$, the set R is the set of distances for some complete and separable metric space if and only if R is an analytic set containing 0 and either R is countable or 0 is a limit point of A. Clemens then seeks to determine distance sets of metric spaces that are homogeneous. The following three definitions describe the metric spaces under consideration and define the basic tool for their characterization.

Definition 1.1 A metric space M is *universal* if it isometrically embeds every finite metric space F with $\mathrm{dist}(\mathrm{F}) \subseteq \mathrm{dist}(\mathrm{M})$.

Definition 1.2 A metric space U is a *Urysohn* metric space if it is homogeneous, universal, separable, and complete.

Definition 1.3 A triple (a, b, c) of non negative numbers is *metric* if $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$.

A set $R \subseteq \Re_{\geq 0}$ satisfies the 4-*values condition* if for all pairs of metric triples of numbers in R of the form (a, b, x) and (c, d, x) exists a number $y \in R$ so that the triples (b, c, y) and (a, d, y) are metric.

The notion of the 4-values condition was first formulated in [3] and will be discussed and used at length in this paper.

We will prove the following theorem.

Theorem 1.4 (See Theorems 4.13 and 2.11) *Let $0 \in R \subseteq \Re_{\geq 0}$ with 0 as a limit. Then there exists a Urysohn metric space U_R if and only if R is a closed subset of $\Re$ that satisfies the* 4-*values condition.*

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Let $0 \in R \subseteq \Re_{\geq 0}$ that does not have 0 as a limit. Then there exists a Urysohn metric space U_R if and only if R is a countable subset of $\Re$ that satisfies the 4-values condition.

Any two Urysohn metric spaces having the same set of distances are isometric.

It follows from [3, Theorem 1.4] and is stated in this paper as Theorem 3.11 that if $0 \in R \subseteq \Re_{\geq 0}$ is a countable set of numbers that satisfies the 4-values condition, then there exists a unique countable homogeneous universal metric space U_R with $\operatorname{dist}(U_R) = R$. Note that this space U_R is not a Urysohn metric space if 0 is a limit of R and R is not closed in $\Re$.

Theorem 1.5 (See Theorem 3.9) *The set of distances of a homogeneous universal metric space satisfies the 4-values condition.*

Proposition 10 of [12] provides an example of a countable homogeneous metric space whose completion is not homogeneous. But, in the case of homogeneous universal metric spaces, we have the following theorem.

Theorem 1.6 (See Theorem 4.5) *The completion of a homogeneous universal separable metric space* M *is homogeneous.*

On the other hand, according to Example 5.3, the completion of a homogeneous, universal, separable, metric space U need not be universal. That is, the completion M might not embed every finite metric space with distances in $\operatorname{dist}(\mathrm{M})$. The next theorem characterizes the finite metric spaces that have an embedding into M. In particular it follows from the next theorem that $\operatorname{dist}(\mathrm{M})$ is the closure of $\operatorname{dist}(U)$ in $\Re$ (Corollary 4.7). See Example 5.2 for a metric space for which the distance set of its completion is not closed.

Theorem 1.7 (See Theorem 4.6) *Let $0 \in R \subseteq \Re_{\geq 0}$ be countable, satisfy the 4-values condition, and have 0 as a limit. Let $\mathrm{M} = (M; \mathrm{d})$ be the completion of U_R, the countable homogeneous universal metric space given by Theorem 3.11, with $\operatorname{dist}(U_R) = R$.*

A finite metric space $\mathrm{A} = (A; \mathrm{d_A})$ with $A = \{a_i \mid i \in m\}$ has an isometric embedding into M if and only if for every $\epsilon > 0$ there exists a metric space B with $B = \{b_i \mid i \in m\}$ and distances in R so that $|\,\mathrm{d}(a_i, a_j) - \mathrm{d}(b_i, b_j)| < \epsilon$ for all $i, j \in m$.

Using "Katětov functions", M. Katětov in [5] generalized Urysohn's construction to metric spaces that are "κ-homogeneous" and have "weight" κ for κ an inaccessible cardinal number, thus extending Urysohn's original result. The distance sets of the such constructed Urysohn type spaces are either $\Re_{\geq 0}$ or the unit interval. More recently those Urysohn spaces attracted attention because of interesting properties of their isometry group, $\operatorname{Iso}(\mathbf{U})$. For example, Uspenskij's result [20] that the isometry group of the Urysohn space is a universal Polish group and the connection of $\operatorname{Iso}(U_{[0,1]})$ to minimal topological groups, [21]. See also Mbombo and Pestov [9] and Melleray [10] for further discussion.

We do not follow Katětov's method but lean instead on the general Fraïssé theory (see [4]) for our results. Fraïssé theory is particularly well suited for investigating partition problems of separable metric spaces, our main interest. Nevertheless it turned out to be easy to extend the arguments to obtain a general amalgamation result, Theorem 3.8, for metric spaces whose distance sets are subsets of a closed set of reals satisfying the 4-values condition. This then implies, by extending the Fraïssé constructions in an obvious way (see for example [7] or more recently [1] or many other recent generalizations), the existence of Urysohn type metric spaces U that are "κ-homogeneous" and have weight κ for κ an inaccessible cardinal. The distance sets of those spaces U are closed subsets of $\Re$ satisfying the 4-values condition. Providing another construction for the "Katětov type metric spaces" with sets of distances all of $\Re$ or the unit interval.

2 Notation and Fraïssé Theory

For another and more detailed introduction to Fraïssé theory in the context of metric spaces, see [12]. The exposition here is complete and self contained but might require some, indeed very limited, familiarity with simple model theoretic constructions.

A pair $\mathrm{H} = (H, \mathrm{d})$ is a *premetric space* if $\mathrm{d}\colon H^2 \to \Re_{\geq 0}$, the *distance function of* H, is a function with $\mathrm{d}(x, y) = 0$ if and only if $x = y$ and $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all $x, y \in H$. For $A \subseteq H$ we denote by $\mathrm{H} \restriction A$ the *substructure of* H *generated by* A, that is, the premetric space on A with distance function the restriction of d to A^2. The *skeleton* of H is the set of finite induced subspaces of H, and the *age* of H is the class of finite premetric spaces isometric to some element of the skeleton of H. Let $\mathrm{dist}(\mathrm{H}) = \{\mathrm{d}(x, y) \mid x, y \in H\}$.

A function $\mathfrak{t}\colon F \to \Re_{>0}$ with F a finite subset of H is a *type function* of H. For $\mathfrak{t}$ a type function let $\mathrm{Sp}(\mathfrak{t})$ be the premetric space on $F \cup \{\mathfrak{t}\}$ for which:

(i) $\mathrm{Sp}(\mathfrak{t}) \restriction \mathrm{dom}(\mathfrak{t}) = \mathrm{H} \restriction \mathrm{dom}(\mathfrak{t})$.
(ii) $\forall x \in F \ \big(\mathrm{d}(\mathfrak{t}, x) = \mathfrak{t}(x)\big)$.

Note that $\mathrm{Sp}(\mathfrak{t})$ is a metric space if and only if $\mathrm{H} \restriction \mathrm{dom}(\mathfrak{t})$ is a metric space and if for all $x, y \in \mathrm{dom}(\mathfrak{t})$:

$$|\mathfrak{t}(x) - \mathfrak{t}(y)| \leq d(x, y) \leq \mathfrak{t}(x) + \mathfrak{t}(y).$$

For $\mathfrak{t}$ a type function let

$$\mathrm{tset}(\mathfrak{t}) = \big\{\, y \in H \setminus \mathrm{dom}(\mathfrak{t}) : \forall\, x \in \mathrm{dom}(\mathfrak{t}) \,\big(\mathrm{d}(y, x) = \mathfrak{t}(x)\big)\,\big\},$$

the *typeset* of $\mathfrak{t}$. Every element $y \in \mathrm{tset}(\mathfrak{t})$ is a *realization* of $\mathfrak{t}$ in H. Let $\mathrm{dist}(\mathfrak{t}) = \{\mathfrak{t}(x) \mid x \in \mathrm{dom}(\mathfrak{t})\}$.

Definition 2.1 Let $\mathrm{M} = (M; \mathrm{d})$ be a metric space. A type function $\mathfrak{k}$ of M is *metric* if $\mathrm{Sp}(\mathfrak{k})$ is a metric space and it is a *Katětov function* of M if $\mathrm{Sp}(\mathfrak{k})$ is an element of the age of M.

A type function of M is *restricted* if it is metric and if $\mathrm{dist}(\mathfrak{k}) \subseteq \mathrm{dist}(\mathrm{M})$.

Note that a type function $\mathfrak{k}$ of a universal metric space M is a Katětov function if and only if it is restricted.

Lemma 2.2 *If every restricted type function of a metric space* $\mathrm{M} = (M; \mathrm{d})$ *has a realization in* M*, then every countable metric space* $\mathrm{N} = (N; \mathrm{d})$ *with* $\mathrm{dist}(\mathrm{N}) \subseteq \mathrm{dist}(\mathrm{M})$ *has an isometric embedding into* M.

https://doi.org/10.4153/CJM-2012-022-4 Published online by Cambridge University Press

If every Katětov function of a metric space $\mathrm{M} = (M; \mathrm{d})$ *has a realization in* M*, then every countable metric space* $\mathrm{N} = (N; \mathrm{d})$ *whose age is a subset of the age of* M *has an isometric embedding into* M*.*

Proof Enumerate N into an ω sequence $(v_i; i \in \omega)$ and let $N_n = \{v_i \mid i \in n\}$ for $n \in \omega$. If f_n is an isometry of $\mathrm{N} \restriction N_n$ into M, let f_{n+1} be the extension of f_n to an isometry of $\mathrm{N} \restriction N_{n+1}$ into M constructed as follows: Let $\mathfrak{k}$ be the type function of M with $\mathrm{dom}(\mathfrak{k}) = f_n[N_n]$ and $\mathfrak{k}(f_n(x)) = \mathrm{d}(x, v_n)$. Then $\mathfrak{k}$ is a restricted type function of M and hence has a realization, say a, in M. Let $f_{n+1}(v_n) = a$.

Then $f = \bigcup_{n\in\omega} f_n$ with f_0 the empty function is an isometry of N into M.

The proof of the second part of the lemma is analogous. ■

Lemma 2.3 *Let* M *and* N *be two countable metric spaces with* $\mathrm{dist}(\mathrm{M}) = \mathrm{dist}(\mathrm{N})$ *and so that every restricted type function of* M *has a realization in* M *and every restricted type function of* N *has a realization in* N*.*

Or, let M *and* N *be two countable metric spaces with equal ages and so that every Katětov function of* M *has a realization in* M *and every Katětov function of* N *has a realization in* N*.*

Then every isometry of a finite subspace of M *into* N *has an extension to an isometry of* M *onto* N*.*

Proof Extend the proof of Lemma 2.2 to a back and forth argument by alternating the extension of finite isometries between M and N. (As in the standard proof that every countable dense and unbounded linear order is order isomorphic to the rationals.) ■

Corollary 2.4 *Let* M *be a countable metric space so that every Katětov function of* M *has a realization in* M*. Then* M *is homogeneous. If every restricted type function of* M *has a realization in* M*, then* M *is homogeneous universal.*

Lemma 2.5 *Every Katětov function of a homogeneous metric space* M *has a realization in* M*. Every restricted type function of a homogeneous universal metric space* M *has a realization in* M*.*

Proof If M is homogeneous and $\mathfrak{k}$ is a Katětov function of M, there exists an isometry f of $\mathrm{Sp}(\mathfrak{k})$ into M. Let g be the restriction of f to $\mathrm{dom}(\mathfrak{k})$. Then g^{-1} is an isometry of a finite subspace of M to a finite subspace of M, which has, because M is homogeneous, an extension, say h, to an isometry of M onto M. The point $h(f(\mathfrak{k}))$ is a realization of $\mathfrak{k}$. ■

Lemma 2.6 *Let* $\mathrm{M} = (M; \mathrm{d})$ *be a homogeneous metric space and let* $\mathrm{A} = (A; \mathrm{d}_\mathrm{A})$ *be a countable metric space with* $A \cap M$ *finite whose age is a subset of the age of* M *and for which* $\mathrm{d}(x, y) = \mathrm{d}_\mathrm{A}(x, y)$ *for all* $x, y \in A \cap M$*. Then there exists a realization of* A *in* M*, that is, a subset* $B \subseteq M \setminus (A \cap M)$ *for which there is an isometry of* A *onto* $\mathrm{M} \restriction (B \cup (M \cap A))$ *that fixes* $A \cap M$ *pointwise.*

Proof The proof is by induction on $A \setminus M$ or a recursive construction realizing Katětov functions step by step. ■

Corollary 2.7 *Every complete and homogeneous universal metric space* V *embeds isometrically every separable metric space* $\mathrm{M} = (M; \mathrm{d})$ *with* $\mathrm{dist}(\mathrm{M}) \subseteq \mathrm{dist}(\mathrm{V})$. *A Urysohn metric space* $U = (U; \mathrm{d})$ *embeds isometrically every separable metric space* M *with* $\mathrm{dist}(\mathrm{M}) \subseteq \mathrm{dist}(U)$.

Proof Let A be a dense subset of M and $\mathrm{A} = \mathrm{M} \restriction A$. Because $\mathrm{dist}(\mathrm{A}) \subseteq \mathrm{dist}(\mathrm{M}) \subseteq \mathrm{dist}(\mathrm{V})$ and V is universal, the age of A is a subset of the age of V. Lemma 2.6 supplies an isometric embedding of A into V, which, because V is complete and A is dense in M, has an extension to an isometric embedding of M into V. The second assertion follows because every Urysohn metric space is homogeneous and complete. ■

Lemma 2.8 *Let* $\mathrm{M} = (M; \mathrm{d})$ *be a separable metric space and* T *a countable subset of* M. *If* M *realizes all of its restricted type functions, then it contains a countable dense subspace* S *with* $T \subseteq S$, *which realizes all of its restricted type functions and for which* $\mathrm{dist}(\mathrm{S})$ *is a dense subset of* $\mathrm{dist}(\mathrm{M})$. *If* M *realizes all of its Katětov functions, then it contains a countable dense subspace* S *with* $T \subseteq S$, *which realizes all of its Katětov functions and for which* $\mathrm{dist}(\mathrm{S})$ *is a dense subset of* $\mathrm{dist}(\mathrm{M})$.

Proof Let M be separable and realize all of its restricted type functions. For $A \subseteq M$ let $\mathrm{spec}(A)$ be the set of distances between points of A and $\mathcal{K}(A)$ the set of restricted type functions $\mathfrak{k}$ of M with $\mathrm{dom}(\mathfrak{k}) \subseteq A$.

Let S_0 be a countable dense subset of M with $T \subseteq S_0$ and so that the set $\mathrm{dist}(\mathrm{M} \restriction S_0)$ is a dense subset of $\mathrm{dist}(\mathrm{M})$. If for $n \in \omega$ a countable set S_n has been determined, choose a realization $\bar{\mathfrak{k}}$ for every restricted type function $\mathfrak{k} \in \mathcal{K}(S_n)$. Let $S_{n+1} = S_n \cup \{\bar{\mathfrak{k}} \mid \mathfrak{k} \in \mathcal{K}(S_n)\}$. The set S_{n+1} is countable, because $\mathcal{K}(S_n)$ is countable. Then $S = \bigcup_{n \in \omega} S_n$ is countable and every restricted type function $\mathfrak{k} \in \mathcal{K}(S)$ has a realization in S. ■

Hence we obtain the following from Corollary 2.4.

Corollary 2.9 *Every separable, homogeneous, universal, metric space* $\mathrm{M} = (M; \mathrm{d})$ *contains a countable, dense, homogeneous, universal subspace* N *for which* $\mathrm{dist}(\mathrm{N})$ *is a dense subset of* $\mathrm{dist}(\mathrm{M})$. *Every separable, homogeneous, metric space* $\mathrm{M} = (M; \mathrm{d})$ *contains a countable, dense, homogeneous subspace* N *for which* $\mathrm{dist}(\mathrm{N})$ *is a dense subset of* $\mathrm{dist}(\mathrm{M})$.

Theorem 2.10 *Every Katětov function of a homogeneous metric space* M *has a realization in* M. *Every restricted type function of a homogeneous universal metric space* M *has a realization in* M.

If a metric space M *is countable and every Katětov function has a realization in* M, *then* M *is homogeneous. If a metric space* M *is countable and every restricted type function has a realization in* M, *then* M *is homogeneous universal.*

If a metric space M *is complete and separable and every Katětov function has a realization in* M, *then* M *is homogeneous. If a metric space* M *is complete and separable and every restricted type function has a realization in* M, *then* M *is a Urysohn metric space.*

Proof On account of Corollary 2.4 and Lemma 2.5, it remains to consider the case that M is complete and separable.

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Let F be a finite subset of M and f an isometry of $\mathrm{M}\restriction F$ into M. Lemma 2.8 yields a dense countable subspace $\mathrm{S} = (S; \mathrm{d})$ of M, with $F \subseteq S$, which in the case of Katětov functions realizes all of its Katětov functions and hence is homogeneous on account of Corollary 2.4. It follows that there is an extension g of f to an isometry of S onto S. Because M is complete, the isometry g has an extension to an isometry of M to M. It follows that M is homogeneous.

In the case of reduced type functions the metric space M is homogeneous as well, because every reduced type function is a Katětov function. Let $\mathrm{N} = (N; \mathrm{d})$ be a separable metric space with $\mathrm{dist}(\mathrm{N}) \subseteq \mathrm{dist}(\mathrm{M})$. Let T be a countable dense subset of N. According to Lemma 2.2 there exists an isometry f of $\mathrm{N}\restriction T$ into M, which because M is complete, has an extension to an isometry of N into M. Hence M is Urysohn. ■

Theorem 2.11 *Any two homogeneous, separable, and complete metric spaces* M *and* N *with the same age are isometric. Any two Urysohn metric spaces* M *and* N *with* $\mathrm{dist}(\mathrm{M}) = \mathrm{dist}(\mathrm{N})$ *are isometric.*

Proof Let S_0 be a countable dense subset of M and let T_0 be a countable dense subset of N. There exists an isometry f of $\mathrm{M}\restriction S_0$ into N and then a dense countable homogeneous subspace $\mathrm{T} = (T; \mathrm{d})$ of N with $f[S_0] \cup T_0 \subseteq N$. There exists an isometry g of T into M with $g(f(x)) = x$ for all $x \in S_0$. Then $g[T]$ is dense in M and because M, and N are complete, there exists an extension of g to an isometry of N onto M. ■

Definition 2.12 A pair (A, B) of metric spaces is an *amalgamation instance* if $\mathrm{d_A}(x, y) = \mathrm{d_B}(x, y)$ for all $x, y \in A \cap B$. Then $\amalg(\mathrm{A}, \mathrm{B})$ is the set of metric spaces with

$$\amalg(\mathrm{A}, \mathrm{B}) = \big\{ \mathrm{C} = (A \cup B; \mathrm{d_C}) \mid \mathrm{C}\restriction A = \mathrm{A} \text{ and } \mathrm{C}\restriction B = \mathrm{B} \big\}.$$

For $R \subseteq \Re_{\geq 0}$ let

$$\amalg_R(\mathrm{A}, \mathrm{B}) = \{\mathrm{C} \in \amalg(\mathrm{A}, \mathrm{B}) \mid \mathrm{dist}(\mathrm{C}) \subseteq R\}.$$

Definition 2.13 An *age of metric spaces* is a class of finite metric spaces closed under subspaces and isometric copies and which is updirected; that is, for all metric spaces A and B in the class there exists a metric space C in the class that isometrically embeds both spaces A and B. An age is *countable* if it has countably many isometry classes.

Definition 2.14 A *Fraïssé class* $\mathcal{A}$ of metric spaces is a countable age of finite metric spaces that is closed under amalgamation. That is, for all amalgamation instances (A, B) with $\mathrm{A}, \mathrm{B} \in \mathcal{A}$, there exists a metric space $\mathrm{C} \in \mathcal{A} \cap \amalg(\mathrm{A}, \mathrm{B})$.

Theorem 2.15 (Fraïssé) *For every Fraïssé class* $\mathcal{A}$ *of metric spaces there exists a unique countable homogeneous metric space* $U_{\mathcal{A}}$, *the* Fraïssé limit *of* $\mathcal{A}$, *whose age is equal to* $\mathcal{A}$.

Note that Theorem 2.15 implies that two countable homogeneous universal metric spaces with the same set of distances are isometric and, together with Theorem 2.11, that any two homogeneous and countable or separable and complete metric spaces with the same age are isometric. Hence we can give the following definition.

Definition 2.16 A Urysohn metric space or countable homogenous universal metric space with set of distances equals to R will be denoted by U_R. A homogenous metric space with age $\mathcal{A}$ which is countable or separable and complete will be denoted by $U_{\mathcal{A}}$.

3 The 4-values Condition

Note that a triple (a, b, c) of numbers is metric if and only if $|a - b| \leq c \leq a + b$. For (a, b, c, d) a quadruple of numbers and x a number, write $x \rightsquigarrow (a, b, c, d)$ to mean that the triples (x, a, b) and (x, c, d) are metric. Then Definition 1.3 can be reworded to: a set $R \subseteq \Re_{\geq 0}$ satisfies the 4-values condition if and only if for all quadruples (a, b, c, d) of numbers in R for which there exists an $x \in R$ with $x \rightsquigarrow (a, b, c, d)$ there exists also a $y \in R$ with $y \rightsquigarrow (a, d, c, b)$.

Note that if $x \rightsquigarrow (a, b, c, d)$, then $x \rightsquigarrow (b, a, c, d)$, $x \rightsquigarrow (a, b, d, c)$, and $x \rightsquigarrow (c, d, a, b)$, but not necessarily $x \rightsquigarrow (a, c, b, d)$. It follows that in order to verify the 4-values condition for R it is sufficient to consider quadruples (a, b, c, d) for which $a \geq \max\{b, c, d\}$.

Definition 3.1 For $R \subseteq \Re_{\geq 0}$ let $Q(R)$ be the set of quadruples (a, b, c, d) of numbers in R for which there exists a number $x \in R$ with $x \rightsquigarrow (a, b, c, d)$ and for which $a \geq \max\{b, c, d\}$.

It follows that a set of numbers $R \subseteq \Re_{\geq 0}$ satisfies the 4-values condition if and only if $(a, b, c, d) \in Q(R)$ implies that $(a, d, c, b) \in Q(R)$.

Lemma 3.2 *Let $x, a, b, c, d \in \Re_{\geq 0}$. Then $x \rightsquigarrow (a, b, c, d)$ implies that $|a - d| \leq b + c$ and $|b - c| \leq a + d$.*

Proof If $a \geq b$, then $|b - c| \leq a + d$ and $a - b \leq x \leq c + d$ implying $|a - d| \leq b + c$. If $b \geq a$, then $|a - d| \leq b + c$ and $b - a \leq x \leq c + d$ implying $|b - c| \leq a + d$. ∎

The next lemma appeared first in [3], and for completeness the statement and proof are given in the present notation.

Lemma 3.3 *A set $R \subseteq \Re_{\geq 0}$ satisfies the* 4-values condition *if and only if for any two metric spaces of the form* $\mathrm{A} = (\{p, v, w\}; \mathrm{d}_{\mathrm{A}})$ *and* $\mathrm{B} = (\{q, v, w\}; \mathrm{d}_{\mathrm{B}})$ *with* $\mathrm{dist}(\mathrm{A}) \subseteq R$ *and* $\mathrm{dist}(\mathrm{B}) \subseteq R$ *and* $R \ni x = \mathrm{d}_{\mathrm{A}}(v, w) = \mathrm{d}_{\mathrm{B}}(v, w)$ *the set* $\amalg_R(\mathrm{A}, \mathrm{B}) \neq \varnothing$.

Proof Let R satisfy the 4-values condition and assume that the spaces A and B with $x = \mathrm{d}_{\mathrm{A}}(v, w) = \mathrm{d}_{\mathrm{B}}(v, w)$ are given. Let

$$a = \mathrm{d}_{\mathrm{A}}(p, v), \quad b = \mathrm{d}_{\mathrm{A}}(p, w), \quad c = \mathrm{d}_{\mathrm{B}}(q, w), \quad d = \mathrm{d}_{\mathrm{B}}(q, v). \tag{3.1}$$

Then $x \rightsquigarrow (a, b, c, d)$. Hence there is a number $y \in R$ with $y \rightsquigarrow (a, d, c, b)$ implying that the space $\mathrm{C} = (\{p, q, v, w\}; \mathrm{d}_{\mathrm{C}}) \in \amalg_R(\mathrm{A}, \mathrm{B})$ with $\mathrm{d}_{\mathrm{C}}(p, q) = y$.

For the other direction of the proof let $x, a, b, c, d \in R$ and $x \rightsquigarrow (a, b, c, d)$. Then let $\mathrm{A} = (\{p, v, w\}; \mathrm{d}_{\mathrm{A}})$ and $\mathrm{B} = (\{q, v, w\}; \mathrm{d}_{\mathrm{B}})$ be metric spaces with distances as in (3.1) and with $x = \mathrm{d}_{\mathrm{A}}(v, w) = \mathrm{d}_{\mathrm{B}}(v, w)$. Let $\mathrm{C} = (\{p, q, v, w\}; \mathrm{d}_{\mathrm{C}}) \in \amalg_R(\mathrm{A}, \mathrm{B})$. Then $R \ni y \rightsquigarrow (a, d, c, b)$ for $y = \mathrm{d}_{\mathrm{C}}(p, q)$. ∎

Definition 3.4 For two metric spaces $\mathrm{A} = (\{p, v, w\}; \mathrm{d_A})$ and $\mathrm{B} = (\{q, v, w\}; \mathrm{d_B})$ with distances as in (3.1) let

$$\mathfrak{u}(\mathrm{A}, \mathrm{B}) = \max\{|a-d|, |b-c|\}, \quad \mathfrak{l}(\mathrm{A}, \mathrm{B}) = \min\{a+d, b+c\}$$

Lemma 3.5 *Let* $\mathrm{A} = (\{p, v, w\}; \mathrm{d_A})$ *and* $\mathrm{B} = (\{q, v, w\}; \mathrm{d_B})$ *be two metric spaces with* $\mathrm{d_A}(v, w) = \mathrm{d_B}(v, w)$*, then* $\mathfrak{u}(\mathrm{A}, \mathrm{B}) \leq \mathfrak{l}(\mathrm{A}, \mathrm{B})$ *and*

$$\big[\mathfrak{u}(\mathrm{A}, \mathrm{B}), \mathfrak{l}(\mathrm{A}, \mathrm{B})\big] = \big\{ \mathrm{d_C}(p, q) \mid \mathrm{C} \in \amalg(\mathrm{A}, \mathrm{B}) \big\}.$$

Let $R \subseteq \Re_{\geq 0}$ *satisfy the* 4*-values condition and* $\mathrm{dist}(\mathrm{A}) \cup \mathrm{dist}(\mathrm{B}) \subseteq R$*. Then there exists a number* $y \in R \cap [\mathfrak{u}(\mathrm{A}, \mathrm{B}), \mathfrak{l}(\mathrm{A}, \mathrm{B})]$.

Proof Let the numbers (a, b, c, d) be given by condition (3.1).

That $\mathfrak{u}(\mathrm{A}, \mathrm{B}) \leq \mathfrak{l}(\mathrm{A}, \mathrm{B})$ follows from Lemma 3.2. If $y \in [\mathfrak{u}(\mathrm{A}, \mathrm{B}), \mathfrak{l}(\mathrm{A}, \mathrm{B})]$, then $|a-d| \leq y \leq a+d$ and $|b-c| \leq y \leq b+c$, and hence the triples (y, a, d) and (y, b, c) are metric. If $y = \mathrm{d_C}(p, q)$ for $\mathrm{C} \in \amalg(\mathrm{A}, \mathrm{B})$, then the triples (y, a, d) and (y, b, c) are metric, and hence $|a-d| \leq y \leq a+d$ and $|b-c| \leq y \leq b+c$.

If R satisfies the 4-values condition and $\mathrm{dist}(\mathrm{A}) \cup \mathrm{dist}(\mathrm{B}) \subseteq R$, it follows from Lemma 3.3 that $\amalg_R(\mathrm{A}, \mathrm{B}) \neq \varnothing$, and hence

$$\begin{aligned} \varnothing \neq \{\mathrm{d_C}(p, q) \mid \mathrm{C} \in \amalg_R(\mathrm{A}, \mathrm{B})\} &= R \cap \{\mathrm{d_C}(p, q) \mid \mathrm{C} \in \amalg(\mathrm{A}, \mathrm{B})\} \\ &= R \cap [\mathfrak{u}(\mathrm{A}, \mathrm{B}), \mathfrak{l}(\mathrm{A}, \mathrm{B})]. \end{aligned}$$

■

Lemma 3.6 *Let the set* $R \subseteq \Re_{\geq 0}$ *satisfy the* 4*-values condition and let* $(\mathrm{A} = (A; \mathrm{d_A}), \mathrm{B} = (B; \mathrm{d_B}))$ *with* $\mathrm{dist}(\mathrm{A}) \cup \mathrm{dist}(\mathrm{B}) \subseteq R$ *be an amalgamation instance. Let* $A \setminus B = \{p\}$ *and* $B \setminus A = \{q\}$.

Then $\amalg_R(\mathrm{A}, \mathrm{B}) \neq \varnothing$ *if* $A \cup B$ *is finite or if* R *is closed.*

Proof For $v, w \in A \cap B$ let $\mathrm{A}_{v,w} = \mathrm{A} \restriction \{p, v, w\}$ and $\mathrm{B}_{v,w} = \mathrm{B} \restriction \{q, v, w\}$. Note that $(\mathrm{A}_{v,w}, \mathrm{B}_{v,w})$ is an amalgamation instance. We have to prove that there is a number $y \in R$ so that the premetric space $\mathrm{C} = (A \cup B; \mathrm{d})$ with $\mathrm{d}(p, q) = y$ and $\mathrm{d}(p, v) = \mathrm{d_A}(p, v)$ and $\mathrm{d}(q, v) = \mathrm{d_B}(q, v)$ and $\mathrm{d}(v, w) = \mathrm{d_A}(v, w) = \mathrm{d_B}(v, w)$ for all $v, w \in A \cap B$ is a metric space. That is, for

$$\mathcal{S} := \bigcap_{v,w \in A \cap B} \big[\mathfrak{u}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w}), \mathfrak{l}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w})\big],$$

we have to prove, according to Lemma 3.5, that $\mathcal{S} \cap R \neq \varnothing$. Let

$$\widehat{\mathfrak{u}} := \sup\big\{ |\, \mathrm{d_A}(p, v) - \mathrm{d_B}(q, v)| \mid v \in A \cap B \big\}, \quad \widehat{\mathfrak{l}} := \inf\big\{ \mathrm{d_A}(p, v) + \mathrm{d_B}(q, v) \big\}.$$

Then $\widehat{\mathfrak{u}} \leq \widehat{\mathfrak{l}}$ according to Lemma 3.2 and $[\widehat{\mathfrak{u}}, \widehat{\mathfrak{l}}] \subseteq \mathcal{S}$ according to Definition 3.4.

Let R be closed. There exists for every $\epsilon > 0$ a point $v \in A \cap B$ with $\widehat{\mathfrak{u}} - \epsilon < |\, \mathrm{d_A}(p, v) - \mathrm{d_B}(q, v)| \leq \widehat{\mathfrak{u}}$ and a point $w \in A \cap B$ with $\widehat{\mathfrak{l}} \leq \mathrm{d_A}(p, w) + \mathrm{d_B}(q, w) \leq \widehat{\mathfrak{l}} + \epsilon$. Then

$$\begin{aligned} \widehat{\mathfrak{u}} - \epsilon < |\, \mathrm{d_A}(p, v) - \mathrm{d_B}(q, v)| &\leq \mathfrak{u}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w}) \leq \widehat{\mathfrak{u}} \\ &\leq \widehat{\mathfrak{l}} \leq \mathrm{d_A}(p, w) + \mathrm{d_B}(q, w) \leq \mathfrak{l}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w}) < \widehat{\mathfrak{l}} + \epsilon. \end{aligned}$$

https://doi.org/10.4153/CJM-2012-022-4 Published online by Cambridge University Press

Because R satisfies the 4-values condition, it follows that for every $\epsilon > 0$ there exists a number $y_\epsilon \in R$ with

$$\widehat{\mathfrak{u}} - \epsilon < \mathfrak{u}(\mathrm{A}_{v,w}) \leq y_\epsilon \leq \mathfrak{l}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w}) < \widehat{\mathfrak{l}} + \epsilon.$$

Hence, because R is closed, there exists a number $y \in R \cap [\widehat{\mathfrak{u}}, \widehat{\mathfrak{l}}] \subseteq R \cap \mathcal{S}$.

If $A \cup B$ is finite, let $v \in A \cap B$ such that $\widehat{\mathfrak{u}} = |\, \mathrm{d_A}(p, v) - \mathrm{d_B}(q, v)|$ and $w \in A \cap B$ such that $\widehat{\mathfrak{l}} = \mathrm{d_A}(p, w) = \mathrm{d_B}(q, w)$. Then

$$\widehat{\mathfrak{u}} = \mathfrak{u}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w}) \leq \mathfrak{l}(\mathrm{A}_{v,w}, \mathrm{B}_{v,w}) = \widehat{\mathfrak{l}}.$$

Because R satisfies the 4-values condition, there exists a number $y \in R \cap [\widehat{\mathfrak{u}}, \widehat{\mathfrak{l}}] \subseteq R \cap \mathcal{S}$. ■

Lemma 3.7 *Let the set* $R \subseteq \Re_{\geq 0}$ *satisfy the* 4*-values condition and let* $(\mathrm{A} = (A; \mathrm{d_A}), \mathrm{B} = (B; \mathrm{d_B}))$ *with* $\mathrm{dist}(\mathrm{A}) \cup \mathrm{dist}(\mathrm{B}) \subseteq R$ *be an amalgamation instance. Let* $A \setminus B = \{p\}$.

Then $\amalg_R(\mathrm{A}, \mathrm{B}) \neq \varnothing$ *if* $A \cup B$ *is finite or if* R *is closed.*

Proof Note that for $A \cap B \subseteq C \subseteq B$ and $\mathrm{C} = \mathrm{B}\restriction C$, the pair (A, C) is an amalgamation instance. Let

$$\mathcal{M} = \bigcup_{A \cap B \subseteq C \subseteq B} \amalg_R(\mathrm{A}, \mathrm{B} \restriction C).$$

Then $(\mathcal{M}; \preceq)$ is a partial order for $\mathrm{L} = (L; \mathrm{d}) \preceq \mathrm{N} = (N; \mathrm{d})$ if $L \subseteq N$ and $\mathrm{N} \restriction L = \mathrm{L}$. Every chain in the partial order $(\mathcal{M}; \preceq)$ has an upper bound, and hence using Zorn's lemma, the partial order $(\mathcal{M}; \preceq)$ has a maximal element $\mathrm{M} = (M; \mathrm{d_M})$. If $M = A \cup B$, then $\mathrm{M} \in \amalg_R(\mathrm{A}, \mathrm{B})$. Otherwise let $b \in (A \cup B) \setminus M$ and let $D = (M \setminus \{p\}) \cup \{b\}$ and $\mathrm{D} = \mathrm{B} \restriction D$. Lemma 3.6 applied to the amalgamation instance (M, D) results in a metric space contradicting the maximality of M. ■

Theorem 3.8 *Let the set* $R \subseteq \Re_{\geq 0}$ *satisfy the* 4*-values condition and let* $(\mathrm{A} = (A; \mathrm{d_A}), \mathrm{B} = (B; \mathrm{d_B}))$ *with* $\mathrm{dist}(\mathrm{A}) \cup \mathrm{dist}(\mathrm{B}) \subseteq R$ *be an amalgamation instance.*

Then $\amalg_R(\mathrm{A}, \mathrm{B}) \neq \varnothing$ *if* $A \cup B$ *is finite or if* R *is closed.*

Proof Let

$$\mathcal{M} = \bigcup_{A \cap B \subseteq C \subseteq B} \amalg_R(\mathrm{A}, \mathrm{B} \restriction C).$$

Then $(\mathcal{M}; \preceq)$ is a partial order for $\mathrm{L} = (L; \mathrm{d}) \preceq \mathrm{N} = (N; \mathrm{d})$ if $L \subseteq N$ and $\mathrm{N} \restriction L = \mathrm{L}$. Every chain in the partial order $(\mathcal{M}; \preceq)$ has an upper bound, and hence using Zorn's lemma the partial order $(\mathcal{M}; \preceq)$ has a maximal element $\mathrm{M} = (M; \mathrm{d_M})$. If $M = A \cup B$ then $\mathrm{M} \in \amalg_R(\mathrm{A}, \mathrm{B})$. Otherwise let $b \in (A \cup B) \setminus M$ and let $D = (M \cap B) \cup \{b\}$ and $\mathrm{D} = \mathrm{B} \restriction D$. Lemma 3.7 applied to the amalgamation instance (M, D) results in a metric space contradicting the maximality of M. ■

Theorem 3.9 *The set of distances of a homogeneous universal metric space satisfies the* 4*-values condition.*

Proof Let $\mathrm{M} = (M; \mathrm{d})$ be a homogeneous universal metric space with $R = \mathrm{dist}(\mathrm{M})$. Let $\mathrm{A} = (\{p, v, w\}; \mathrm{d_A})$ and $\mathrm{B} = (\{q, v, w\}; \mathrm{d_B})$ with $\mathrm{dist}(\mathrm{A}) \subseteq R$ and $\mathrm{dist}(\mathrm{B}) \subseteq R$ and let $R \ni x = \mathrm{d_A}(v, w) = \mathrm{d_B}(v, w)$. There exists an isometric copy with points $\{p', v', w'\} \subseteq M$ in M. Let $\mathfrak{t}$ be the restricted type function with $\mathrm{dom}(\mathfrak{t}) = \{v', w'\}$ and with $\mathfrak{t}(v') = \mathrm{d_B}(q, v)$ and $\mathfrak{t}(w') = \mathrm{d_B}(q, w)$. Let q' be a realization of $\mathfrak{t}$. Then the metric space $\mathrm{C} = (\{p, v, w, q\}; \mathrm{d_C})$ with $\mathrm{C} \restriction \{p, v, w\} = \mathrm{A}$ and $\mathrm{C} \restriction \{q, v, w\} = \mathrm{B}$, and $\mathrm{d_C}(p, q) = \mathrm{d}(p', q')$ is a metric space in $\amalg_R(\mathrm{A}, \mathrm{B})$. Hence the theorem follows from Lemma 3.3. ∎

Definition 3.10 Let $R \subseteq \Re_{\geq 0}$, then $\mathcal{F}_R$ is the class of finite metric spaces M with $\mathrm{dist}(\mathrm{M}) \subseteq R$.

Theorem 3.11 *Let $0 \in R \subseteq \Re_{\geq 0}$ be a countable set of numbers satisfying the 4-values condition. Then there exists a countable homogeneous universal metric space U_R.*

If there exists a countable homogeneous universal metric space U_R, then R satisfies the 4-values condition.

Proof The class $\mathcal{F}_R$ of finite metric spaces is closed under isometric copies and substructures, and it follows from Theorem 3.8 and Definition 2.12 that $\varnothing \neq \amalg_R(\mathrm{A}, \mathrm{B}) \subseteq \mathcal{F}_R$ for all $\mathrm{A}, \mathrm{B} \in \mathcal{F}_R$. Hence $\mathcal{F}_R$ is updirected and closed under amalgamation and hence a Fraïssé class. According to Theorem 2.15 there exists a countable homogeneous metric space $U_{\mathcal{F}_R}$ whose age is equal to $\mathcal{F}_R$. It follows that $U_{\mathcal{F}_R}$ is the countable homogeneous universal metric space U_R. ∎

Lemma 3.12 *Let $R \subseteq \Re_{\geq 0}$ be a set of numbers that satisfies the 4-values condition. For every countable subset T of R, there exists a dense countable subset $C \supseteq T$ of R that satisfies the 4-values condition.*

Proof Let $S \supseteq T$ be a countable dense subset of R. There are countably many instances of the form $x \rightsquigarrow (a, b, c, d)$ with numbers in S. Because R does satisfy the 4-values condition there is a countably set $S' \subseteq R$ so that for all those quadruples there is a $y \in S'$ with $y \rightsquigarrow (a, d, c, b)$. Repeating this process countably often leads to a countable subset $C \subseteq R$ that satisfies the 4-values condition. ∎

In order to verify the 4-values condition the following lemma is often useful.

Lemma 3.13 *If $a \leq b + c$ or $a \leq b + d$ or $a \leq c + d$ and $(a, b, c, d) \in Q(R)$, then there exists $y \in \{a, b, c, d\}$ with $y \rightsquigarrow (a, d, c, b)$.*

Proof If $a \leq b + c$, then $a \rightsquigarrow (a, d, c, b)$. If $a \leq b + d$ then $b \rightsquigarrow (a, d, c, b)$ unless $2b < c$ in which case $c \rightsquigarrow (a, d, c, b)$. If $a \leq c + d$, then $c \rightsquigarrow (a, d, c, b)$ unless $2c < b$ in which case $b \rightsquigarrow (a, d, c, b)$. ∎

Hence, in order to verify that R satisfies the 4-values condition, it suffices to consider quadruples for which a is larger than the sum of any two of the other three numbers.

4 Completion of Homogeneous Universal Metric Spaces

Definition 4.1 Let $\mathrm{A} = (A; \mathrm{d_A})$ and $\mathrm{B} = (B; \mathrm{d_B})$ be two metric spaces with $A = \{a_i \mid i \in m \in \omega\}$ and $B = \{b_i \mid i \in m \in \omega\}$ and $A \cap B = \varnothing$. A metric space $\mathrm{P} = (A \cup B; \mathrm{d_P})$ with $\mathrm{P} \restriction A = \mathrm{A}$ and $\mathrm{P} \restriction B = \mathrm{B}$ is an h-join of A and B if $\mathrm{d_P}(a_i, b_i) < h$ for all $i \in m$. (The h-join P depends explicitly on the enumeration of A and B.)

Let $\mathcal{F}_R$ be the class of finite metric spaces F with $\mathrm{dist}(\mathrm{F}) \subseteq R$. For r a positive real, let $\mathcal{F}_{R|r}$ be the metric spaces in $\mathrm{F} \in \mathcal{F}_R$ for which the smallest postive number in $\mathrm{dist}(\mathrm{F})$ is larger than r.

Lemma 4.2 *Let $R \subseteq \Re_{\geq 0}$ satisfy the* 4*-values condition and have* 0 *as a limit.*
Let $\mathrm{A} = (A; \mathrm{d_A})$ and $\mathrm{B} = (B; \mathrm{d_B})$ be two metric spaces in $\mathcal{F}_R$ for which $A = \{a_0, a_1, a_2, \dots, a_m\}$ and $B = \{b_0, b_1, b_2, \dots, b_m\}$ and $A \cap B = \varnothing$. Let

$$A' = \{a_i \mid i \in m\} \quad \textit{and} \quad B' = \{b_i \mid i \in m\}$$

and A', B' metric spaces with $\mathrm{A}' = \mathrm{A} \restriction A'$ and $\mathrm{B}' = \mathrm{B} \restriction B'$. Let

$$k \geq \max\{|\,\mathrm{d_A}(a_m, a_i) - \mathrm{d_B}(b_m, b_i)| \mid i \in m\}$$

and $h \in R$ and $l \in R$ with

$$l + k \leq h \leq \min\big(\{\mathrm{d_A}(a_m, a_i) \mid i \in m\} \cup \{\mathrm{d_B}(b_m, b_i) \mid i \in m\}\big).$$

Then if there exists an l-join $\mathrm{Q} = (A' \cup B'); \mathrm{d_Q}) \in \mathcal{F}_R$ of A' and B', there exists an h-join $\mathrm{P} \in \mathcal{F}_R$ of A *and* B.

Proof There exist, according to Theorem 3.8, a metric space $\mathrm{A}^* = (A^*; \mathrm{d_{A^*}}) \in \amalg_R(\mathrm{A}, \mathrm{Q})$ and a metric space $\mathrm{B}^* = (B^*; \mathrm{d_{B^*}}) \in \amalg_R(\mathrm{B}, \mathrm{Q})$. Note that

$$|\,\mathrm{d_A}(a_m, a_i) - \mathrm{d_{A^*}}(a_m, b_i)| < l \quad \text{and} \quad |\,\mathrm{d_B}(b_m, b_i) - \mathrm{d_{B^*}}(b_m, a_i)| < l$$

for all $i \in m$. Let $\mathrm{P} = (A \cup B; \mathrm{d})$ be the premetric space with $\mathrm{P} \restriction (A \cup B') = \mathrm{A}^*$ and $\mathrm{P} \restriction (B \cup A') = \mathrm{B}^*$ and with $\mathrm{d}(a_m, b_m) = h$.

In order to see that P is a metric space we have to check the triples of the form (a_m, b_m, a_i) and (a_m, b_m, b_i) for all $i \in m$. Indeed,

$$\begin{aligned} |\,\mathrm{d}(a_m, a_i) - \mathrm{d}(b_m, a_i)| &\leq |\,\mathrm{d}(a_m, a_i) - \mathrm{d}(b_m, b_i)| + |\,\mathrm{d}(b_m, b_i) - \mathrm{d}(b_m, a_i)| \\ &\leq k + l \leq h = \mathrm{d}(a_m, b_m). \end{aligned}$$

This verifies that the triple (a_m, b_m, a_i) is metric, because $h \leq \mathrm{d}(a_m, a_i)$, and hence the distance $\mathrm{d}(a_m, b_m)$ is not larger than the other two distances in the triple (a_m, b_m, a_i). Similarly, the triangles of the form (a_m, b_m, b_i) are metric. ■

Lemma 4.3 *Let $R \subseteq \Re_{\geq 0}$ satisfy the* 4*-values condition and have* 0 *as a limit. Then, for every $m \in \omega$ and $r > 0$ and $h \in \Re_{>0}$ there exists a number $\gamma(h) < h$ such that for all metric spaces*

$$\mathrm{A} = \big(\{a_i \mid i \in m\}; \mathrm{d_A}\big) \in \mathcal{F}_{R|r} \quad \textit{and} \quad \mathrm{B} = \big(\{b_i \mid i \in m\}; \mathrm{d_B}\big) \in \mathcal{F}_{R|r}$$

with $|\,\mathrm{d_A}(a_i, a_j) - \mathrm{d_B}(b_i, b_j)| < \gamma(h)$ for all $i, j \in m$ and $A \cap B = \varnothing$ exists an h-join of A *and* B *in $\mathcal{F}_R$.*

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Proof For every $0 < x \in R$ let $0 < x^* \in R$ with $2 \cdot x^* < x$. Given h and r let $0 < h_{m-1} < \min\{h, r\}$ and for all integers $m - 1 > i \geq 0$ let $h_i = h^*_{i+1}$ and $\gamma(h) = h_0$. The lemma follows from Lemma 4.2 via induction on i. ■

Note: Let a, b, a', b' be four points in a metric space, then

$$|\,\mathrm{d}(a,b) - \mathrm{d}(a',b')| \leq |\,\mathrm{d}(a,b) - \mathrm{d}(a,b')| + |\,\mathrm{d}(a,b') - \mathrm{d}(a',b')| \leq \mathrm{d}(b,b') + \mathrm{d}(a,a'). \tag{4.1}$$

Theorem 4.4 *Let $0 \in R \subseteq \Re_{\geq 0}$ be countable, satisfy the 4-values condition, and have 0 as a limit. Let U_R be the countable homogenous universal metric space with R as set of distances given by Theorem 3.11. Then $\mathrm{M} = (M; \mathrm{d})$, the completion of U_R, is homogeneous, separable, and complete.*

Proof The homogeneous universal space $U_R = (U; \mathrm{d})$ is dense in M. It follows from Theorem 2.10 that M is homogeneous if every Katětov function of M has a realization in M.

Let $\mathfrak{k}$ be a Katětov function of M with $\mathrm{dom}(\mathfrak{k}) = \{a_i \mid i \in m \in \omega\} := A$. There exists a subset $B = \{b_i \mid i \in m\} \cup \{b_m\} \subseteq M$ with $\mathrm{d}(a_i, a_j) = \mathrm{d}(b_i, b_j)$ and $\mathfrak{k}(a_i) = \mathrm{d}(b_m, b_i)$ for all $i, j \in m$. Let $k = \min(\mathrm{dist}(\mathrm{Sp}(\mathfrak{k})))$.

Let $0 < e \in \Re$ and $B' = \{b'_i \mid 0 \leq i \leq m\} \subseteq U$ with $\mathrm{d}(b_i, b'_i) < e$. Then from inequality (4.1): $|\,\mathrm{d}(b_i, b_j) - \mathrm{d}(b'_i, b'_j)| < 2e$ for all $0 \leq i, j \leq m$. Note that if $e < \frac{k}{4}$ then $\min\{\mathrm{d}(b'_i, b'_j) \mid 0 \leq i, j \leq m\} > \frac{k}{2}$. If there is a set of points $A' = \{a'_i \mid i \in m\} \subseteq U$ with $\mathrm{d}(a_i, a'_i) < e$, then $|\,\mathrm{d}(a_i, a_j) - \mathrm{d}(a'_i, a'_j)| < 2e$ for all $i, j \in m$. Hence, because $\mathrm{d}(a_i, a_j) = \mathrm{d}(b_i, b_j)$,

$$|\,\mathrm{d}(a'_i, a'_j) - \mathrm{d}(b'_i, b'_j)| < 4e.$$

It follows from Lemma 4.3 that if $4e < \min(\gamma(h), k)$ then there exists an h-join $\mathrm{C}' = (\{B' \cup A'\}; \mathrm{d}_{\mathrm{C}'}) \in \mathfrak{F}_R$ of $U_R \restriction B'$ and $U_R \restriction A'$. It follows from Lemma 2.5 that there exists a realization $\mathrm{C} = (\{c_i \mid i \in m\}; \mathrm{d})$ of C' in U_R with $\mathrm{d}(a'_i, c_i) < h$ and $\mathrm{d}(c_i, c_j) = \mathrm{d}(b'_i, b'_j)$ for all $i, j \in m$. Also,

$$\mathrm{d}(c_i, a_i) \leq \mathrm{d}(c_i, a'_i) + \mathrm{d}(a'_i, a_i) < h + e < 2h \text{ for all } i \in m. \tag{4.2}$$

Let $(h_n; n \in \omega)$ and $(e_n; n \in \omega)$ be a sequence of numbers in R with $h_n > 2 \cdot h_{n+1}$ and $4e_n < \min(\gamma(\frac{1}{2}h_n), k)$. Then there exist sets of points:

$$\begin{gathered} B'_n = \{b'_{n,i} \mid 0 \leq i \leq m\} \text{ with } \mathrm{d}(b'_{n,i}, b_i) < e_n < \frac{1}{2}h_n \text{ and} \\ C_n = \{c_{n,i} \mid i \in m\} \text{ with } \mathrm{d}(c_{n,i}, a_i) < h_n \text{ and with} \\ \mathrm{d}(b'_{n,i}, b'_{n,j}) = \mathrm{d}(c_{n,i}, c_{n,j}) \text{ and } \mathrm{d}(c_i, a_i) < h_n \text{ for all } i, j \in m \text{ and} \\ \mathrm{d}(b'_{n,m}, b'_{n+1,m}) \leq \mathrm{d}(b'_{n,m}, b_{n,m}) + \mathrm{d}(b_{n,m}, b'_{n+1,m}) < h_n. \end{gathered} \tag{4.3}$$

https://doi.org/10.4153/CJM-2012-022-4 Published online by Cambridge University Press

Using the fact that Katětov functions of U_R have realizations in U construct recursively points $c_{n,m} \in U$ so that for all $i \in m$ and $n \in \omega$:

$$\mathrm{d}(c_{n,m}, c_{n,i}) = \mathrm{d}(b'_{n,m}, b'_{n,i}), \quad \mathrm{d}(c_{n,m}, c_{n+1,m}) = \mathrm{d}(b'_{n,m}, b'_{n+1,m}) < h_n.$$

That is, the function f with $f(b'_{n,i}) = c_{n,i}$ for $0 \le i \le m$ and $n \in \omega$ is an isometry of a subset of U to a subset of U.

It follows from inequality (4.2) that for every $i \in m$, the sequence $(c_{n,i})$ converges to a_i and from inequality (4.3) that the sequence $(c_{n,m})$ is Cauchy converging to, say c_m. For every $0 \le i \le m$ the sequence $(b'_{n,i})$ converges to b_i, and hence for every $i \in m$,

$$\lim_{n\to\infty} \mathrm{d}(b'_{n,m}, b_{n,i}) = \mathrm{d}(b_m, b_i) = \mathfrak{k}(a_i).$$

It follows that $\mathrm{d}(c_m, a_i) = \lim_{n\to\infty} \mathrm{d}(c_{n,m}, c_{n,i}) = \mathfrak{k}(a_i)$ implying that c_m is a realization of the Katětov function $\mathfrak{k}$. ■

Theorem 4.5 *The completion of a homogeneous universal separable metric space* M *is homogeneous.*

Proof The space M contains a countable dense homogenous universal subspace N according to Corollary 2.9. Let $R = \mathrm{dist}(\mathrm{N})$. It follows from Theorem 2.11 that we can take N to be the homogeneous universal metric space U_R. The theorem follows from Theorem 4.4, because the completion of U_R is equal to the completion of M. ■

Theorem 4.6 *Let $0 \in R \subseteq \Re_{\ge 0}$ be countable, satisfy the 4-values condition, and have 0 as a limit and let $\mathrm{M} = (M; \mathrm{d})$ be the completion of $U_R = (U; \mathrm{d})$.*

A finite metric space $\mathrm{A} = (A; \mathrm{d_A})$ with $A = \{a_i \mid i \in m\}$ has an isometric embedding into M if and only if for every $\epsilon > 0$, there exists a metric space $\mathrm{B} = (B; \mathrm{d_B}) \in \mathcal{F}_R$ with $B = \{b_i \mid i \in m\}$ so that $|\,\mathrm{d}(a_i, a_j) - \mathrm{d}(b_i, b_j)| < \epsilon$ for all $i, j \in m$.

Proof The condition is clearly necessary.

Let $k = \min(\mathrm{dist}(\mathrm{A}))$. Let (h_n) be a sequence of positive numbers in R so that $h_0 < \frac{k}{4}$ and $2h_{n+1} < h_n$ for all $n \in \omega$. Let (e_n) be a sequence of positive numbers in R so that $\gamma(h_n) < 2e_n$ and $e_{n+1} < e_n$, with γ given by Lemma 4.3. For $n \in \omega$ let $\mathrm{B}_n = (B_n; \mathrm{d}_{\mathrm{B}_n})$ be a metric space with $B_n = \{b_{n,i} \mid i \in m\} \in \mathcal{F}_R$ and with

$$|\,\mathrm{d_A}(a_i, a_j) - \mathrm{d}_{\mathrm{B}_n}(b_{n,i}, b_{n,j})| < e_n < h_n. \tag{4.4}$$

Then

$$|\,\mathrm{d}_{\mathrm{B}_{n+1}}(b_{n+1,i}, b_{n+1,j}) - \mathrm{d}_{\mathrm{B}_n}(b_{n,i}, b_{n,j})| < e_{n+1} + e_n < 2e_n.$$

It follows from Lemma 4.3 that there exists, for every $n \in \omega$, an h_n-join $\mathrm{P}_n = (B_n \cup B_{n+1}; \mathrm{d_P}) \in \mathcal{F}_R$ of B_n with B_{n+1}.

The space U_R is homogeneous universal, and hence each of the finite metric spaces P_n has an isometric embedding into U_R. It follows from Lemma 2.6 via a recursive construction that there exist isometric copies $B'_n = \{b'_{n,i} \mid i \in m\}$ of the sets B_n in U so that, for all $n \in \omega$,

$$\mathrm{d}(b'_{n,i}, b'_{n+1,i}) = \mathrm{d}_{\mathrm{P}_n}(b_{n,i}, b_{n+1,i}) < h_n.$$

It follows that for every $i \in m$ the sequence $(b_{n,i})$ is Cauchy and hence has a limit, say $b_i \in M$. Also for all $i, j \in m$,

$$\lim_{n\to\infty} \mathrm{d}(b'_{n,i}, b'_{n,j}) = \lim_{n\to\infty} \mathrm{d}(b_{n,i}, b_{n,j}) = \mathrm{d}_{\mathrm{A}}(a_i, a_j),$$

with the last equality implied by inequality (4.4). ∎

Corollary 4.7 *Let $0 \in R \subseteq \Re_{\geq 0}$ be countable, satisfy the 4-values condition, and have 0 as a limit, and let $\mathrm{M} = (M; \mathrm{d})$ be the completion of $U_R = (U; \mathrm{d})$. Then $\mathrm{dist}(\mathrm{M})$ is the closure of R. The set of distances of the completion N of a homogeneous universal separable metric space is a closed subset of $\Re$.*

Proof Let $\epsilon > 0$ be given and let a be in the closure of $\mathrm{dist}(\mathrm{M})$. There exists a number $b \in \mathrm{dist}(\mathrm{M})$ with $|a - b| < \frac{\epsilon}{2}$. There exists a number $c \in R$ with $|b - c| < \frac{\epsilon}{2}$. That $\mathrm{dist}(\mathrm{N})$ is closed follows as in the proof of Theorem 4.5. ∎

Note that in general the distance set of the completion of a metric space need not be closed. (See Example 5.2.)

Definition 4.8 For $R \subseteq \Re$ let

$$R^{()} = \big\{ x \in R \mid \exists \epsilon > 0 \ \big(\ (x, x + \epsilon) \cap R = \varnothing \big) \big\}.$$

Lemma 4.9 *Let $R \subseteq \Re_{\geq 0}$, satisfy the 4-values condition and have 0 as a limit. If $\{x, y, z\} \subseteq R$ with $z = y + x$ and $x \in R^{()}$, then $z \in R^{()}$.*

Proof Let $\{x, y, z\} \subseteq R$ with $z = y+x$ and $x \in R^{()}$ and $\epsilon > 0$ so that $(x, x+\epsilon) \cap R = \varnothing$ and let $0 < \delta < \min\{\epsilon, x\}$. If $z \notin R^{()}$, there exists $z < z' \in R$ with $z' - z < \delta$. Then $z \rightsquigarrow (z', \delta, x, y)$. If $R \ni u \rightsquigarrow (z', y, x, \delta)$, then the triple (δ, x, u) is metric and hence $u \leq x + \delta$, which implies, because $(x, \delta] \cap R = \varnothing$, that $u \leq x$. It follows that $u + y \leq x + y = z < z'$, and hence that the triple (u, y, z') is not metric, which contradicts R satisfying the 4-values condition. ∎

Lemma 4.10 *Let $R \subseteq \Re_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit. If $\{x, y, z\} \subseteq R$ with $z = y + x$ and $\{x, y\} \subseteq R^{()}$, then both x and y are isolated points of R and $z \in R^{()}$.*

Proof It follows from Lemma 4.9 that $z \in R^{()}$. Let $z = y + x$ and $\{x, y\} \subseteq R^{()}$. If, say x, is not isolated in R, let $\epsilon > 0$ be such that $(y, y + \epsilon) \cap R = \varnothing$. Let $R \ni \delta < x$ with $0 < \delta < \epsilon$ and let $0 < u < x$ with $u \in R$ such that $x - u \leq \delta$. Note that $x \rightsquigarrow (z, y, \delta, u)$.

If $R \ni r \rightsquigarrow (z, u, \delta, y)$, then $r \leq y + \delta$ because the triple (r, y, δ) is metric and hence $r \leq y$ because of the choice of δ. Then $r + u \leq y + u < y + x = z$ and hence the triple (r, u, z) is not metric in contradiction to $r \rightsquigarrow (z, u, \delta, y)$. ∎

Lemma 4.11 *Let $0 \in R \subseteq \Re_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit. Let S be a dense subset of R.*

Then there exists, for every metric space $\mathrm{A} = (A; \mathrm{d}_{\mathrm{A}}) \in \mathfrak{F}_R$ with $A = \{a_i \mid i \in m\}$ and every $\epsilon > 0$, a metric space $\mathrm{B} = (B; \mathrm{d}_{\mathrm{B}}) \in \mathfrak{F}_S$ with $B = \{b_i \mid i \in m\}$ so that $|\,\mathrm{d}_{\mathrm{A}}(a_i, a_j) - \mathrm{d}_{\mathrm{B}}(b_i, b_j)| < \epsilon$ for all $i, j \in m$.

Proof Let

$$\Delta = \frac{1}{3}\min\{\, y + x - z \mid\; z < y + x \text{ and } \{x, y, z\} \subseteq \operatorname{dist}(\mathrm{A})\}\,.$$

Let $I \subseteq \operatorname{dist}(\mathrm{A})$ be the set of isolated points of R that are elements of $\operatorname{dist}(\mathrm{A})$. Note that $I \subseteq S$.

Let E be the set of positive numbers in $\operatorname{dist}(\mathrm{A}) \setminus R^{()}$ with $e_0 < e_1 < e_2 < \cdots < e_{n-1} < e_n$ an enumeration of E. Let $e_0 < \widehat{e_0} \in S$ so that $\widehat{e_0} - e_0 < \min\{\Delta, \epsilon\}$. The numbers $\widehat{e_i} \in S$ are determined recursively so that $\widehat{e_i} > e_i$ and $\frac{1}{3}(\widehat{e_i} - e_i) > \widehat{e}_{i+1} - e_{i+1}$ for all indices $i \in n$.

Let K be the set of positive numbers in $\operatorname{dist}(\mathrm{A}) \cap R^{()}$ that are not isolated and let $k_0 > k_1 > k_2 > \cdots > k_{r-1} > k_r$ be an enumeration of K. Let $k_0 > \widehat{k_0} \in S$ so that $k_0 - \widehat{k_0} < \frac{1}{3}(\widehat{e_n} - e_n)$. The numbers $\widehat{k_i} \in S$ are determined recursively so that $k_i > \widehat{k_i}$ and $\frac{1}{3}(k_i - \widehat{k_i}) > k_{i+1} - \widehat{k}_{i+1}$ for all indices $i \in r$.

For every $x \in I$ let $\widehat{x} = x$ and let $\widehat{0} = 0$. Note that the inequalities above imply for $\{x, y, z\} \subseteq \operatorname{dist}(\mathrm{A})$ with $x, y \neq 0$ and $z = x + y$ and $x \in E$ or $y \in E$ that $\widehat{z} \leq \widehat{x} + \widehat{y}$.

Claim *If (x, y, z) is a metric triple of numbers with entries in* $\operatorname{dist}(\mathrm{A})$, *then $(\widehat{x}, \widehat{y}, \widehat{z})$ is a metric triples of numbers with entries in S.*

Proof Let $z \geq \max\{y, x\}$. It follows from the choice of Δ and the definition of the function $\widehat{\ }$ that $\widehat{z} \geq \max\{\widehat{y}, \widehat{x}\}$, and if $z < y + x$, then $\widehat{z} < \widehat{y} + \widehat{x}$, and hence that the triple $(\widehat{z}, \widehat{y}, \widehat{x})$ is metric.

Let $z = y + x$ with $x, y \neq 0$. If at least one of x and y are in E, then $\widehat{z} \leq \widehat{x} + \widehat{y}$. If both are not in E, then they are both in $R^{()}$ and it follows from Lemma 4.10 that both x and y are isolated in R and $z \in R^{()}$. If z is isolated, then $\widehat{z} = z = x + y = \widehat{x} + \widehat{y}$. If z is not isolated, then $\widehat{z} < z = x + y = \widehat{x} + \widehat{y}$. ∎

It follows that the premetric space $\mathrm{B} = (B; \mathrm{d_B})$ with $B = \{b_i \mid i \in m\}$ and $\mathrm{d_B}(b_i, b_j) = \widehat{x}_{i,j}$ for $x_{i,j} = \mathrm{d_A}(a_i, a_j)$ is a metric space with $|\,\mathrm{d_A}(a_i, a_j) - \mathrm{d_B}(b_i, b_j)| < \epsilon$ for all $i, j \in m$. ∎

Corollary 4.12 *Let $0 \in R \subseteq \Re_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit. Let S be a dense subset of R.*

Then there exists, for every metric space $\mathrm{A} = (A; \mathrm{d_A}) \in \mathcal{F}_R$ with $A = \{a_i \mid i \in m\}$ and every $\epsilon > 0$, a metric space $\mathrm{B} = (B; \mathrm{d_B}) \in \mathcal{F}_S$ with $B = \{b_i \mid i \in m\}$ so that $\mathrm{d_A}(a_i, b_i) < \epsilon$ for all $i \in m$.

Proof The proof follows from Lemmas 4.11 and 4.3. ∎

Theorem 4.13 *Let $0 \in R \subseteq \Re_{\geq 0}$ with 0 as a limit. Then there exists a Urysohn metric space U_R if and only if R is a closed subset of $\Re$ that satisfies the 4-values condition.*

Let $0 \in R \subseteq \Re_{\geq 0}$ that does not have 0 as a limit. Then there exists a Urysohn metric space U_R if and only if R is a countable subset of $\Re$ satisfying the 4-values condition.

Proof Let $0 \in R \subseteq \Re_{\geq 0}$ with 0 as a limit.

If there exists a Urysohn metric space U_R, it follows from Theorem 3.9 that U_R satisfies the 4-values condition, because Urysohn metric spaces are homogeneous.

The space U_R contains, according to Corollary 2.9, a dense, countable, homogeneous, universal subspace U_T with T a dense subset of $\text{dist}(U_R)$. Then U_R is the completion of U_T, and by Corollary 4.7 the set R is closed.

Let R be closed and satisfy the 4-values condition. It follows from Lemma 3.12 that R has a countable dense subset T that satisfies the 4-values condition. Let the countable homogeneous universal metric space U_T be given by Theorem 3.11. The completion M of U_T is homogeneous and separable and complete according to Theorem 4.4. It follows from Corollary 4.7 that dist(M) is the closure of T that implies because T is dense in R and R is closed that $\text{dist}(\mathrm{M}) = R$. It remains to prove that M is universal, that is, that every finite metric space $\mathrm{A} \in \mathcal{F}_R$ has an isometric embedding into M. This then indeed follows from Lemma 4.11 and Theorem 4.6

Let $0 \in R \subseteq \Re_{\geq 0}$ that does not have 0 as a limit.

If R is uncountable, then there does not exist a Urysohn metric space U_R, because Urysohn metric spaces are separable. If R is countable, then there exists a homogeneous universal metric space U_R according to Theorem 3.11. The space U_R is a Urysohn metric space, because the completion of U_R is equal to U_R. ■

5 Examples

Example 5.1 It is not difficult to check that the set of reals in the intervals $[0, \infty)$ and $[0, 1]$ satisfy the 4-values condition. Hence there exist, according to Theorem 4.13, a Urysohn space $U_{[0,\infty)}$, the classical *Urysohn space* and $U_{[0,1]}$, the *Urysohn sphere*.

Example 5.2 The set of distances of the completion of a metric space need not be closed:

Let R be the set of rationals in the interval $[0, 1]$ and

$$V = \{a_i \mid i \in R\} \cup \{b_i \mid i \in R\}.$$

Let $\mathrm{V} = (V; \mathrm{d})$ be the metric space with $\mathrm{d}(a_i, b_i) = i$ and $\mathrm{d}(a_i, a_j) = \mathrm{d}(a_i, b_j) = 1$ for all $i, j \in R$ with $i \neq j$. The completion of V is V.

Example 5.3 The completion of a homogeneous universal separable metric space U_R is homogeneous according to Theorem 4.5, but as the example below shows, the completion need not be universal. The age of the completion consists of all finite metric spaces that can be "approximated" by metric spaces with distances in R, Theorem 4.6 and Lemma 4.11.

Let R be the set of rationals in the interval $[0, 1)$ together with the number 2. Then R satisfies the 4-values condition. To see this, let $x \rightsquigarrow (a, b, c, d)$ with $x, a, b, c, d \in R$. According to Lemma 3.13 it suffices to assume $a > \max\{b + c, b + d, c + d\}$. If $a \in [0, 1)$, then $b + c \rightsquigarrow (a, d, c, b)$. If $a = 2$, then $b = 2$ or $x = 2$. If $b = 2$, then $2 \rightsquigarrow (a, d, c, b)$. If $x = 2$, then $c = 2$ or $d = 2$. If $c = 2$, then $2 \rightsquigarrow (a, d, c, b)$ and if $d = 2$, then $b + c \rightsquigarrow (a, d, c, b)$.

Hence, according to Theorem 3.11, there exists a homogeneous universal countable metric space U_R. Let M be the completion of U_R. According to Corollary 4.7

dist(M) $=[0,1]\cup\{2\}:=T$, which does not satisfy the 4-values condition, because $1\rightsquigarrow(2,1,\frac{1}{2},\frac{1}{2})$, but there is no number $y\in T$ with $y\rightsquigarrow(2,\frac{1}{2},\frac{1}{2},1)$. The class $\mathcal{F}_T$ contains a triangle with distance set $\{2,1,1\}$. The class $\mathcal{R}$ does not contain any triangle with distance set of the form $\{a,b,c\}$ with $2-\frac{1}{4}<2<2+\frac{1}{4}$ and $1-\frac{1}{4}<b,c<1+\frac{1}{4}$. Hence M does not contain a triangle with distance set $\{2,1,1\}$ and is therefore not universal. Theorem 4.6 characterizes the finite metric spaces in the age of M.

Indeed, it is not difficult to check that M consists of countably many copies of the Urysohn space $U_{[0,1]}$ for which any two points in different copies have distance 2.

Example 5.4 Some additional examples of subsets R of the reals satisfying the 4-values condition.

To decide wether a finite set of numbers satisfies the 4-values condition is only of polynomial complexity. On the other hand, there does not seem to be an easy way to see directly if a finite set satisfies the 4-values condition. For examples of such finite sets R see [13]. In the case of infinite sets R it can be quite challenging to determine whether R satisfies the 4-values condition. It is not difficult to see that the sets

$$R_1=0\cup\left\{\frac{1}{2^{2n}}\mid n\in\omega\right\},\quad R_2=\left\{x+3n\mid x\in[0,1],n\in\omega\right\}.$$

satisfy the 4-values condition, while the set

$$R_3=\left[0,\frac{1}{2}\right]\cup\left\{x+3n\mid x\in[0,1],1\leq n\in\omega\right\}$$

does not. Note that if R satisfies the 4-values condition and $a>0$, then the sets $[0,a]\cap R$ and $\{ax\mid x\in R\}$ satisfy the 4-values condition. Every sum closed set $R\subseteq\Re_{\geq 0}$ containing 0 satisfies the 4-values condition. The set ω and every initial interval of ω satisfies the 4-values condition. The sets $R=[0,1]\cup[3,4]\cup[9,10]$ and $R=[0,1]\cup[3,4]\cup(8,9]$ satisfy the 4-values condition. The set $R=[0,1]\cup[3,4]\cup[8,9]$ does not satisfy the 4-values condition.

It is a bit more challenging to prove that the set F of Cantor numbers with finitely many digits 2 in the ternary expansion satisfy the 4-values condition. Hence there exists a unique countable homogeneous universal metric space U_F according to Theorem 3.11. The set of all Cantor numbers does not satisfy the 4-values condition, but it follows from Theorem 4.5 that there exists separable complete homogeneous metric space $U_{\bar{C}}$ whose set of distances is the set of Cantor numbers. It follows from Theorem 2.11 that $U_{\bar{C}}$ is the unique, separable, complete, metric space whose age is the set of finite metric spaces that can be approximated by finite metric spaces with distances in C as described in Theorem 4.6.

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Department of Mathematics and Statistics, University of Calgary, Calgary, AB
e-mail: nsauer@ucalgary.ca