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Sub-actions for geodesic flows on locally CAT(−1) spaces

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Abstract. We extend a result of Lopes and Thieullen [Sub-actions for Anosov flows. *Ergod. Th. & Dynam. Sys.* **25**(2) (2005), 605–628] on sub-actions for smooth Anosov flows to the setting of geodesic flow on locally CAT(−1) spaces. This allows us to use arguments originally due to Croke and Dairbekov to prove a volume rigidity theorem for some interesting locally CAT(−1) spaces, including quotients of Fuchsian buildings and surface amalgams.

Key words: subactions, Livšic theorem, geodesic flow, CAT(− 1) spaces
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Contents

1. *Introduction*

The classical Livšic theorem for a smooth, transitive Anosov flow $\{\phi_t\}$ states the following [**Liv71**]: any Hölder function A that integrates to zero over each closed orbit for the flow is itself a derivative. That is, there is a function V, smooth in the flow direction and still Hölder, such that $A(x) = d/dt|_{t=0} V(\phi_t x)$. (See, e.g., [**KH95**] for a proof and discussion.) Equivalently, but also providing a formulation for this statement which does not require V to be smooth in the flow direction, $\int_0^T A(\phi_t x)\, dx = V(\phi^T x) - V(x)$.

This theorem has far-reaching consequences for these flows. One with a particular connection to the present paper is that if two negatively curved Riemannian metrics on the same compact manifold have the same marked length spectrum, then their geodesic flows are conjugate. This is the starting point for Croke's proof of marked length spectrum rigidity for surfaces in [**Cro90**]. This rigidity result was proved independently by Otal in [**Ota90**] using tools which will play an important role in the current paper.

A natural generalization of the Livšic theorem asks whether assuming that the periodic integrals of A are all non-negative (or, with trivial modifications, non-positive) guarantees a V whose derivative bounds A below. Lopes and Thieullen term such a V a 'sub-action' for A and prove that whenever A is Hölder, a sub-action which is smooth in the flow direction and still Hölder exists [**LT05**]. Independently and concurrently, Pollicott and Sharp proved a similar theorem in [**PS04**]. The proof in [**PS04**] is simpler, but establishes less—they do not obtain that the sub-action V is smooth in the flow direction or some of the more detailed results on regularity provided by [**LT05**]. See [**PS04**] for a good survey of related results in this area.

1.1. *Statement of results.*

In this paper, we follow the approach of [**LT05**], applied specifically to geodesic flow on a CAT(-1) space. This is a *metric Anosov flow* (or *Smale flow*)—it satisfies the essential properties of an Anosov flow, abstracted from the smooth setting to the general metric setting by Pollicott [**Pol87**] (see §2.3). This was proved in [**CLT20a**], where the authors found a coding for the geodesic flow using carefully chosen Poincaré sections. These sections turn out to be perfect candidates for the sections used in the arguments of [**LT05**] (see §3). Our main theorem is the following.

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MAIN THEOREM. *Let (X, d_X) be a locally CAT(−1) space with geodesic flow $\{g_t\}$. Let $A : GX \to \mathbb{R}$ be Hölder. Then, there exists a map $V : GX \to \mathbb{R}$, called a sub-action, that is Hölder, smooth in the flow direction, such that for any geodesic $\gamma \in GX$ and every $T > 0$,*

$$\int_0^T A \circ g_t(\gamma)\, dt \geq V \circ g_T(\gamma) - V(\gamma) + mT$$

for some constant $m = m(A)$. Equivalently, for any $\gamma \in GX$,

$$A(\gamma) = m + \left(\frac{d}{dt}\right)\bigg|_{t=0} V(g_t(\gamma)) + H(\gamma)$$

for some non-negative function $H : GX \to \mathbb{R}_{\geq 0}$ that is smooth along the flow direction and Hölder.

Remark 1.1. Theorem 1.1 can be generalized to the setting of metric Anosov flows that satisfy an analog of Proposition 3.4. (See [**Pol87**] for the definition of these flows.) Geodesic flows on CAT(−1) spaces are the natural first example of such flows; geodesic flows for projective Anosov representations are another example of much recent interest (see, for example, [**BCLS15**]). Proposition 3.4 guarantees the existence of *Markov proper families* for the flow with, importantly, Lipschitz return time (see §3.1 for more details). Reference [**CLT20a**] establishes these properties for both of these geodesic flows.

We turn to an application of the Main Theorem to the marked length spectrum in §6. We prove a *volume rigidity* result for *surface amalgams*, which, roughly speaking, are constructed by identifying finitely many compact surfaces with boundary along their boundary components. For a more precise definition, we refer the reader to §6.1. Before stating the volume rigidity result for surface amalgams, we introduce some terminology.

Definition 1.2. (Marked length spectrum) The *marked length spectrum of a metric space* (X, g) is the class function

$$\mathscr{L}_g : \pi_1(X) \to \mathbb{R}^+, \quad [\alpha] \mapsto \inf_{\gamma\in[\alpha]} \ell_g(\gamma)$$

which assigns to each free homotopy class $[\alpha] \in \pi_1(X)$ the infimum of lengths in the g-metric of curves in the class.

Note that in the case of CAT(−1) spaces, the marked length spectrum is simply a length assignment to every closed geodesic in (X, g), as each homotopy class has a unique geodesic representative.

Otal [**Ota90**] and Croke [**Cro90**] proved that for compact, negatively curved surfaces, if $\mathscr{L}_{g_0} = \mathscr{L}_{g_1}$, then g_0 and g_1 are isometric, with the classical Livšic theorem playing a role in Croke's proof. Suppose instead that $\mathscr{L}_{g_0} \leq \mathscr{L}_{g_1}$, that is, for every free homotopy class $[\alpha] \in \pi_1(X)$, $\inf_{\gamma\in[\alpha]} \ell_{g_0}(\gamma) \leq \inf_{\gamma'\in[\alpha]} \ell_{g_1}(\gamma')$. In the setting of negatively curved surfaces, Croke and Dairbekov proved that such a marked length spectrum inequality implies a corresponding inequality in volumes of the surface with respect to g_0 and g_1. Furthermore, if the volumes are equal, then g_0 and g_1 are isometric [**CD04**]. The proof of rigidity in the equality case crucially uses Lopes and Thieullen's sub-action result. Having

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extended that result to the CAT(-1) setting, we use Croke and Dairbekov's idea as well as some recent results on marked length spectrum rigidity for surface amalgams and similar spaces to prove the following theorem.

VOLUME RIGIDITY COROLLARY. *Let* (X, g_0) *and* (X, g_1) *be two simple, thick, negatively curved surface amalgams satisfying certain smoothness conditions around the gluing curves. Suppose* $\mathscr{L}_{g_0} \leq \mathscr{L}_{g_1}$. *Then,* $\mathrm{Vol}_{g_0}(X) \leq \mathrm{Vol}_{g_1}(X)$. *Furthermore, if* $\mathrm{Vol}_{g_0}(X) = \mathrm{Vol}_{g_1}(X)$*, then* (X, g_0) *and* (X, g_1) *are isometric.*

The fact that the sub-action V is smooth in the flow direction is used in the proof of the Volume Rigidity Corollary. This provides some justification for our adoption of the more complicated but slightly stronger proof strategy in [**LT05**].

1.2. *An outline of the paper.* In §2, we collect definitions and basic results about CAT(-1) spaces, geodesic flow on such spaces, and it's properties. We also prove some basic geometric facts which will be used later in the paper.

In §3, we describe the construction of Poincaré sections for the geodesic flow which allow us to 'discretize' the flow. We describe and prove some key properties of these sections that will be useful for subsequent arguments.

The proof of the Main Theorem begins in earnest with §4. In this section, we follow the arguments of [**LT05**] and solve the discretized version of the sub-action problem provided by the Poincaré sections.

Section 5 continues in the steps of [**LT05**] to extend the solution of the discretized problem to a sub-action for the flow. A careful inductive scheme allows one to make this extension while ensuring the desired regularity of the sub-action.

Finally, in §6, we use the Main Theorem to prove the Volume Rigidity Corollary in its full generality.

2. *Preliminaries*

2.1. *Geodesics and geodesic flow.* Let $(\tilde{X}, d_{\tilde{X}})$ be a CAT(-1) space and Γ be a discrete group of isometries of $\tilde{X}$ acting freely, properly discontinuously, and cocompactly. (Recall that a group Γ acting on X by isometries is *discrete* if for every ball $B = B(x, r) \subset X$, $\{g \in G : gB \cap B \neq \varnothing\}$ is finite. See [**BH99**] for background on CAT(-1) spaces.) The resulting quotient $X = \tilde{X}/\Gamma$, with the metric d_X induced by $d_{\tilde{X}}$, is a compact, locally CAT(-1) space.

To study the geodesic flow on X, we need an analog of the unit tangent bundle appropriate for this non-smooth setting which admits a unit-speed geodesic flow. This is provided by the following definition.

Definition 2.1. The space of (unit-speed) geodesics of $\tilde{X}$ is

$$G\tilde{X} := \{\tilde{\gamma} : \mathbb{R} \to \tilde{X} : d_{\tilde{X}}(\tilde{\gamma}(s), \tilde{\gamma}(t)) = |s - t|\}.$$

The geodesic flow g_t on $G\tilde{X}$ is given by $g_t\tilde{\gamma}(s) = \tilde{\gamma}(s + t)$ for any $t \in \mathbb{R}$. Additionally, $GX = G\tilde{X}/\Gamma$ is the space of geodesics of X.

There are natural metrics on these spaces.

Definition 2.2. We equip $G\tilde{X}$ with the metric

$$d_{G\tilde{X}}(\tilde{\gamma}_1, \tilde{\gamma}_2) := \int_{-\infty}^{\infty} d_{\tilde{X}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))e^{-2|t|}\, dt,$$

and GX with the metric

$$d_{GX}(\gamma_1, \gamma_2) = \min_{\widetilde{\gamma_1}, \widetilde{\gamma_2}} \int_{-\infty}^{\infty} d_{\tilde{X}}(\widetilde{\gamma_1}(t), \widetilde{\gamma_2}(t))e^{-2|t|}\, dt,$$

where the minimum is taken over all lifts $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ of γ_1 and γ_2, respectively.

A straightforward computation shows that the geodesic flow is unit-speed with respect to d_{GX}. (This is the reason for the normalizing factor 2 in the exponent.) We record here a few basic facts about d_{GX} and its interaction with the geodesic flow.

LEMMA 2.3. [**CLT20b**, Lemma 2.8] *For all $\gamma_1, \gamma_2 \in GX$, $d_X(\gamma_1(0), \gamma_2(0)) \leq 2d_{GX}(\gamma_1, \gamma_2)$.*

LEMMA 2.4. [**CLT20a**, Lemma 2.5] *For any t and any $\gamma_1, \gamma_2 \in GX$, $d_{GX}(g_t\gamma_1, g_t\gamma_2) \leq e^{2|t|}d_{GX}(\gamma_1, \gamma_2)$.*

A central tool in the geometry of CAT(−1) spaces is the boundary at infinity.

Definition 2.5. The *boundary at infinity* $\partial^\infty \tilde{X}$ is the space of equivalence classes of geodesic rays in $\tilde{X}$, where two rays $c_1, c_2 : [0, \infty) \to \tilde{X}$ are equivalent if there exists $M \geq 0$ such that $d_{\tilde{X}}(c_1(t), c_2(t)) \leq M$ for all $t \geq 0$. For a geodesic $\tilde{\gamma}$, we denote by $\tilde{\gamma}(-\infty)$ and $\tilde{\gamma}(+\infty)$ its backward and forward endpoints at $\partial^\infty \tilde{X}$.

In CAT(−1) spaces, a pair $\{\xi, \eta\}$ of distinct points on $\partial^\infty \tilde{X}$ uniquely determines an unparameterized, unoriented geodesic. Once we specify an orientation and the time-0 point on the geodesic, we have a geodesic in $G\tilde{X}$. Therefore, $G\tilde{X}$ can be identified with $[(\partial^\infty \tilde{X} \times \partial^\infty \tilde{X})\backslash\Delta] \times \mathbb{R}$, where Δ is the diagonal. For ease of notation, let $\partial^{(2)}_\infty \tilde{X} = (\partial^\infty \tilde{X} \times \partial^\infty \tilde{X})\backslash\Delta$.

2.2. *Stable and unstable sets.* Under the geodesic flow on a negatively curved manifold, the unit tangent bundle admits foliations by stable and unstable manifolds which are essential tools for studying the dynamics of these flows. In the CAT(−1) setting, we have a purely geometric description of analogous stable and unstable sets. Below, we will note that these are also analogous in their dynamical role.

Definition 2.6. Let $p \in \tilde{X}$, $\xi \in \partial^\infty \tilde{X}$, and $\tilde{\gamma}$ be the geodesic ray from p to ξ. The *Busemann function centered at ξ with basepoint p* is defined as

$$\begin{aligned} B_p(-, \xi) : \tilde{X} &\to \mathbb{R} \\ q &\mapsto \lim_{t\to\infty} (d_{\tilde{X}}(q, \tilde{\gamma}(t)) - t). \end{aligned}$$

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For convenience, given geodesic ray $\tilde{\gamma}(t)$, the *Busemann function determined by* $\tilde{\gamma}$ is the Busemann function centered at $\tilde{\gamma}(+\infty)$ with basepoint $\tilde{\gamma}(0)$:

$$B_{\tilde{\gamma}}(-) := B_{\tilde{\gamma}(0)}(-, \tilde{\gamma}(+\infty)).$$

The level sets for $B_p(-, \xi)$ are *horospheres*.

Definition 2.7. Let $\tilde{\gamma} \in G\tilde{X}$.

- The *strong stable set through* $\tilde{\gamma}$ is
$$W^{ss}(\tilde{\gamma}) = \{\tilde{\gamma}' \in G\tilde{X} : \tilde{\gamma}'(+\infty) = \tilde{\gamma}(+\infty) \text{ and } B_{\tilde{\gamma}}(\tilde{\gamma}'(0)) = 0\}.$$
- For $\delta > 0$,
$$W^{ss}_{\delta}(\tilde{\gamma}) = \{\tilde{\gamma}' \in G\tilde{X} : \tilde{\gamma}'(+\infty) = \tilde{\gamma}(+\infty), \\ B_{\tilde{\gamma}}(\tilde{\gamma}'(0)) = 0, \text{ and } d_{G\tilde{X}}(\tilde{\gamma}, \tilde{\gamma}') < \delta\}.$$
- The *weak stable set through* $\tilde{\gamma}$ is
$$W^{cs}(\tilde{\gamma}) = \bigcup_{t \in \mathbb{R}} g_t(W^{ss}(\tilde{\gamma})).$$
- Similarly, the *strong and weak unstable sets through* $\tilde{\gamma}$ are
$$W^{uu}(\tilde{\gamma}) = \{\tilde{\gamma}' \in G\tilde{X} : \tilde{\gamma}'(-\infty) = \tilde{\gamma}(-\infty) \text{ and } B_{-\tilde{\gamma}}(\tilde{\gamma}'(0)) = 0\},$$

$$W^{uu}_{\delta}(\tilde{\gamma}) = \{\tilde{\gamma}' \in G\tilde{X} : \tilde{\gamma}'(+\infty) = \tilde{\gamma}(+\infty), \\ B_{-\tilde{\gamma}}(\tilde{\gamma}'(0)) = 0, \text{ and } d_{G\tilde{X}}(\tilde{\gamma}, \tilde{\gamma}') < \delta\},$$

  and
$$W^{cu}(\tilde{\gamma}) = \bigcup_{t \in \mathbb{R}} g_t(W^{uu}(\tilde{\gamma})).$$

(See Figure 1.)

The next lemma outlines some useful, standard properties of Busemann functions.

LEMMA 2.8. *Let* $\tilde{\gamma} \in G\tilde{X}$.

(1) *For any* $s \in \mathbb{R}$, $B_{g_s\tilde{\gamma}}(-) = B_{\tilde{\gamma}}(-) + s$, *so the* 0*-level set for* $B_{g_s\tilde{\gamma}}(-)$ *is the* $(-s)$*-level set for* $B_{\tilde{\gamma}}(-)$.
(2) *If* $\tilde{\eta} \in W^{cs}(\tilde{\gamma})$, *then for* $s_1, s_2 \in \mathbb{R}$, $B_{\tilde{\gamma}}(\tilde{\eta}(s_1)) - B_{\tilde{\gamma}}(\tilde{\eta}(s_2)) = s_2 - s_1$.
(3) $B_{\tilde{\gamma}}(-) : \tilde{X} \to \mathbb{R}$ *is* 1*-Lipschitz.*

Proof. Let q be any point in $\tilde{X}$. Then,

$$\begin{aligned} B_{g_s\tilde{\gamma}}(q) &= \lim_{t\to\infty}(d_{\tilde{X}}(q, \tilde{\gamma}(s+t)) - t) \\ &= \lim_{t'\to\infty}(d_{\tilde{X}}(q, \tilde{\gamma}(t')) - (t' - s)) \\ &= B_{\tilde{\gamma}}(q) + s, \end{aligned}$$

which proves part (1).

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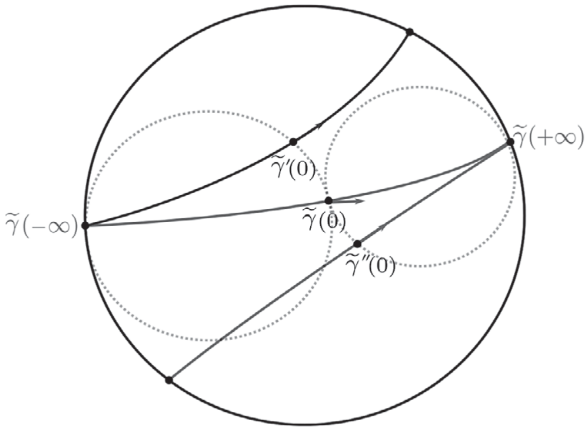

FIGURE 1. Here, $\widetilde{\gamma'} \in W^{uu}(\widetilde{\gamma})$ and $\widetilde{\gamma''} \in W^{ss}(\widetilde{\gamma})$. The 0-level sets of the Busemann functions are the dashed circles.

Since $\tilde{\eta} \in W^{cs}(\tilde{\gamma})$, $g_r\tilde{\eta} \in W^{ss}(\tilde{\gamma})$ for some $r \in \mathbb{R}$ and for $i \in \{1, 2\}$, $g_{s_i}\tilde{\eta} \in W^{ss}(g_{s_i-r}\tilde{\gamma})$. Therefore, $B_{g_{s_i-r}\tilde{\gamma}}(\tilde{\eta}(s_i)) = 0$. By part (1), this implies $B_{\tilde{\gamma}}(\tilde{\eta}(s_i)) = r - s_i$, from which part (2) follows immediately.

Part (3) holds as $B_{\tilde{\gamma}}$ is a limit of 1-Lipschitz functions. See also [**BH99**, II.8] for how this fact follows naturally from a slightly different presentation of Busemann functions. □

2.3. *Metric Anosov flows.* In [**Pol87**], Pollicott defines metric Anosov flows, generalizing the essential properties of Anosov flows to the non-smooth setting.

A continuous flow on a compact metric space is Anosov if, roughly speaking, it has a topological local product structure which coheres with the dynamics of the flow in a way which mimics the analogous structure for Anosov flows. Locally, Y looks like a product—two nearby points can always be connected by a small step along a 'stable' set, then a small step along an 'unstable' set, then a small move along the flow. This coheres with the dynamics of the flow in that the 'stable' set is truly stable in the dynamical sense: there are constants $C, \lambda > 0$ such that if x and y are nearby points on the same stable set, then

$$d(\phi_t x, \phi_t y) \leq Ce^{-\lambda t} d(x, y) \quad \text{for } t \geq 0.$$

The corresponding expression for exponential contraction in backwards time holds for pairs of nearby points on a common unstable set. (See [**Pol87**] or [**CLT20a**] for more detail.)

Metric Anosov flows have many of the properties of Anosov flows. As a first example (which we will use below), we have the following proposition.

PROPOSITION 2.9. [**Bow73**, Corollary 1.6] and [**Pol87**, Proposition 1] *A metric Anosov flow is expansive.*

The flows we consider in this paper are metric Anosov.

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THEOREM 2.10. [**CLT20a**, Theorem 3.4] *For a compact, locally CAT(−1) space X, the geodesic flow on GX is metric Anosov with* $\lambda = 1$. *Specifically, the strong stable and unstable sets of Definition 2.7 are the stable and unstable sets for the Anosov flow. That is, there exist small* $\delta > 0$ *and* $C > 0$ *such that*

$$d_{G\tilde{X}}(g_t\tilde{\gamma}, g_t\tilde{\gamma}') \leq Ce^{-t} d_{G\tilde{X}}(\tilde{\gamma}, \tilde{\gamma}') \quad \textit{for } t \geq 0 \textit{ and } \tilde{\gamma}' \in W^{ss}_\delta(\tilde{\gamma});$$

$$d_{G\tilde{X}}(g_{-t}\tilde{\gamma}, g_{-t}\tilde{\gamma}') \leq Ce^{-t} d_{G\tilde{X}}(\tilde{\gamma}, \tilde{\gamma}') \quad \textit{for } t \geq 0 \textit{ and } \tilde{\gamma}' \in W^{uu}_\delta(\tilde{\gamma}).$$

2.4. *Doubling and covering properties.* In the proof of Theorem 1.1, we will need a technical fact about the geometry of GX. We collect the necessary arguments leading to that fact here.

Let (X, d) be a metric space. We introduce two useful properties of metric spaces.

Definition 2.11. An open set $\mathcal{U} \subset X$ with compact closure satisfies the *doubling* property if there exists $N_1 \in \mathbb{N}$ such that for all $r > 0$ and $x \in \mathcal{U}$, the closed ball $\overline{B(x, 2r)}$ can be covered by at most N_1 balls of radius r.

Definition 2.12. An open set $\mathcal{U} \subset X$ with compact closure satisfies the *packing* property if there exists $N_2 \in \mathbb{N}$ such that for all $r > 0$ and $x \in \mathcal{U}$, the maximum cardinality of an r-separated subset of $B(x, 2r)$ is less than or equal to N_2.

One can prove that doubling implies packing (although the converse is not true).

LEMMA 2.13. *If* $\mathcal{U}$ *satisfies the doubling property, then it also satisfies the packing property.*

Proof. Take an arbitrary $x \in \mathcal{U}$ and $r > 0$, and suppose $\overline{B(x, 2r)}$ can be covered by at most N_1 balls of radius r, and these balls can in turn be covered by N_1 balls of radius $r/2$. We thus have a covering $\{B(x_i, r/2)\}_{i \in I}$ of $\mathcal{U}$ of cardinality at most N_1^2. Consider a $2r$-separated set $S(2r) = \{x_i \in \overline{B(x, 2r)} : d(x_i, x_j) \geq 2r \text{ for all } i \neq j\}$. Note that each $B(x_i, r/2)$ can contain at most one element of $S(2r)$; otherwise, if $y_1, y_2 \in B(x_i, r/2)$, then $d(y_1, y_2) \leq r$. Hence, $\mathcal{U}$ satisfies the packing property with $N_2 = N_1^2$. □

Now, let X be locally CAT(−1). Provided the double boundary $\partial^{(2)}_\infty \tilde{X}$ of $\widetilde{X}$ satisfies the doubling property, one can show an analogue of a technical lemma [**LT05**, Lemma 22].

LEMMA 2.14. *Suppose that* $\partial^{(2)}_\infty \tilde{X}$ *satisfies the doubling property for some metric. Then there exists a constant C depending only on X such that for every* $\epsilon > 0$ *and every open* $\Sigma \subset \partial^{(2)}_\infty \tilde{X}$ *with* $\overline{\Sigma}$ *compact, there exists a finite cover* $\{B_{jk}\}_{j \in J, 1 \leq k \leq C}$ *of* Σ *by* ϵ*-balls such that for every fixed* $1 \leq k^* \leq C$, *the balls in* $\{B_{jk^*}\}_{j \in J}$ *are disjoint.*

Proof. Let $\{x_l\}_{l \in L}$ be a maximal set in Σ such that $d(x_{l_1}, x_{l_2}) \geq 2\epsilon$. By maximality, $\bigcup_{l \in L} B(x_l, 2\epsilon)$ covers Σ. By the doubling condition, we can cover each $B(x_l, 2\epsilon)$ by at most N_1 ϵ-balls for some N_1 depending only on Σ. For each $l \in L$, we choose this collection minimally and use $\{B(y_{lm}, \epsilon)\}_{m \leq M(l)}$ to denote the collection covering $B(x_l, 2\epsilon)$.

Note that if $B(y_{l_1m_1}, \epsilon) \cap B(y_{l_2m_2}, \epsilon) \neq \varnothing$, then $d(x_{l_1}, x_{l_2}) < 6\epsilon$. Indeed, if $x \in B(y_{l_1m_1}, \epsilon) \cap B(y_{l_2m_2}, \epsilon)$, then for $i = 1, 2$, we have that $d(y_{l_im_i}, x_{l_im_i}) \le 2\epsilon$ and $d(x, y_{l_im_i}) < \epsilon$, so by the triangle inequality, $d(x_{l_1}, x_{l_2}) < 6\epsilon$.

We claim that for each l_1,

$$\begin{aligned}\#\{y_{l_2m_2} : B(y_{l_1m_1}, \epsilon) \cap B(y_{l_2m_2}, \epsilon) \neq \varnothing\} &\le N_1(\#\{x_{l_2} : d(x_{l_1}, x_{l_2}) < 6\epsilon\}) \\ &\le N_1^2 N_2 = N_1^4. \end{aligned} \tag{2.1}$$

The first inequality follows from the bound obtained above on $d(x_{l_1}, x_{l_2})$ and the fact that at most N_1 of the $y_{l_im_i}$ terms are associated to each x_{l_i}. For the second inequality, we can cover $B(x_{l_1}, 6\epsilon)$ by at most N_1 balls of radius 3ϵ by doubling. Since the $\{x_{l_i}\}$ are 2ϵ-separated, by packing, at most N_2 of the x_l terms are in each of these balls.

We now use $\{B(y_{l_m})\}_{l\in L}$ to construct a finite-valence dual graph Γ as follows:

- $V(\Gamma) = \{y_{l_m}\}_{l\in L, 1\le m\le M(l)}$ (where $M(l) \le N_1$);
- $E(\Gamma) = \{[y_{l_1m_2}, y_{l_2m_2}] : B(y_{l_1m_1}, \epsilon) \cap B(y_{l_2m_2}, \epsilon) \neq \varnothing\}$.

By equation (2.1), each vertex in $V(\Gamma)$ has valence at most N_1^4. By Brooks' theorem ([**Bro41**] or [**Lov75**]), Γ can be colored with at most $C := N_1^4 + 1$ colors. Relabel the set $\{y_{lm}\}$ to $\{y_{jk}\}$, where $k \in [1, C]$ indexes the colors in the coloring of $V(\Gamma)$, and $j \in [1, J(k)]$ indexes the vertices colored with the kth color.

By definition of a coloring of a graph, for a fixed k^*, the set $\{y_{jk^*}\}_{1\le j\le J(k^*)}$ is a totally disconnected subgraph of Γ. Then, the balls in $\{B(y_{jk^*}, \epsilon)\}_{1\le j\le J(k^*)}$ are pairwise disjoint, as desired. □

To use Lemma 2.14, we need a metric on $\partial_\infty^{(2)} \tilde{X}$ which is doubling. Recall that the ℓ^∞*product metric* on the product $X \times Y$ of two metric spaces (X, d_X) and (Y, d_Y) is defined by

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

The proof of the following is relatively straightforward.

LEMMA 2.15. *The ℓ^∞ product metric space $(X \times Y, d_{X\times Y})$ is doubling if (X, d_X) and (Y, d_Y) are doubling. More precisely, if M and N are the doubling constants of (X, d_X) and (Y, d_Y), respectively, then MN is the doubling constant of $(X \times Y, d_{X\times Y})$.*

Recall that we can equip the boundary of a CAT(−1) space with a *visual metric*, which we now define. First, if X is CAT(−1), given a choice of basepoint $x_0 \in X$, we define the *Gromov product* of $\xi, \eta \in \partial^\infty X$, which is denoted by $(\xi.\eta)_{x_0}$:

$$(\xi.\eta)_{x_0} = \lim_{\substack{x_n\to\xi \\ y_n\to\eta}} \tfrac{1}{2}(d(x_0, x_n) + d(x_0, y_n) - d(x_n, y_n)).$$

Definition 2.16. (Visual metric for CAT(−1) spaces, [**Bou95**]) If X is a *proper* CAT(-1) space, then we can define the *visual metric* to be

$$d_{x_0}(\eta, \xi) = \begin{cases} e^{-(\xi.\eta)_{x_0}} & \text{if } \xi \neq \eta, \\ 0 & \text{otherwise.} \end{cases}$$

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Although we do not delve into the definition of visual metrics for Gromov hyperbolic spaces, in general, we remark that they exist (see [BH99, Ch. III.H]).

Recall that a metric space is *proper* if every closed ball is compact. We now show that for a proper geodesic Gromov hyperbolic metric space X equipped with a discrete, cocompact action (e.g., the CAT(-1) space $\tilde{X}$), $\partial_\infty^{(2)}\tilde{X}$ equipped with a (ℓ^∞) product of visual metrics is doubling. To do so, we must define some stronger properties of metric spaces.

Definition 2.17. A metric space X is *Ahlfors Q-regular* if there exists a positive Borel measure μ on X such that

$$C^{-1}R^Q \leq \mu(B(x, R)) \leq CR^Q,$$

where $R \leq \mathrm{diam}(X)$ and $C \geq 1$ is a constant independent of x and R.

While we defined doubling for metric spaces, we can also say a *measure* admitted by a metric space is doubling.

Definition 2.18. A positive Borel measure μ on a metric space X is *doubling* if there exists a constant $C = C(\mu)$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

We now briefly prove a lemma that is usually assumed in the literature due to the simplicity of the proof.

LEMMA 2.19. *Any Ahlfors Q-regular metric space satisfies the doubling property.*

Proof. We first note that if a metric space is Ahlfors Q-regular, then it admits a doubling measure. Indeed, μ itself is a doubling measure; by the Ahlfors Q-regularity of X, we have

$$C^{-1}R^Q \leq \mu(B(x, R)) \leq CR^Q \quad \text{and} \quad C^{-1}2^Q R^Q \leq \mu(B(x, 2R)) \leq C2^Q R^Q.$$

Thus, $R^Q \leq C\mu(B(x, R))$, so

$$\mu(B(x, 2R)) \leq C2^Q R^Q \leq 2^Q C^2 \mu(B(x, R)).$$

Thus, μ is doubling with constant $2^Q C^2$. Finally, by [LS98], we have that a metric space is doubling if and only if it admits a doubling measure. □

This allows us to state the following theorem.

THEOREM 2.20. [Kle06, Theorem 3.3] *If X is a proper, geodesic Gromov hyperbolic metric space which admits a discrete, cocompact isometric action, then $\partial^\infty X$ equipped with a visual metric is Ahlfors Q-regular for some Q.*

The original proof of Theorem 2.20 is actually due to Coornaert (see [Coo93, Proposition 7.4]), who shows that any Γ-quasiconformal measure on $\partial^\infty X$ with support contained in the limit set of Γ is Ahlfors Q-regular, where Γ is a discrete, cocompact isometric group action on X. In particular, if μ is any non-zero measure on $\partial^\infty X$, then μ is Γ-quasiconformal (see [Coo93, Proposition 4.3]). As a consequence of Lemma 2.19, we thus have the following corollary.

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COROLLARY 2.21. *If X is a proper, geodesic Gromov hyperbolic metric space which admits a discrete, cocompact isometric action, then $\partial^\infty X$ equipped with a visual metric is doubling.*

Finally, we reach a key corollary.

COROLLARY 2.22. *If X is a locally CAT(-1) space, then $\partial^{(2)}_\infty(\widetilde{X})$ equipped with the ℓ^∞ product of two visual metrics is doubling.*

Proof. Note that by properties of locally CAT(-1) spaces, $\widetilde{X}$ is a proper geodesic Gromov hyperbolic space that admits a discrete, cocompact isometric group action, so by Theorem 2.20, $\partial^\infty\widetilde{X}$ equipped with the visual metric is doubling. By Lemma 2.15,

$$(\partial^\infty\widetilde{X} \times \partial^\infty\widetilde{X}) \setminus \Delta = \partial^{(2)}_\infty\widetilde{X}$$

equipped with the ℓ^∞ product of two visual metrics is doubling. □

3. *Poincaré sections and symbolic codings for the geodesic flow*

A key tool in [**LT05**]'s approach to the sub-actions problem is to introduce carefully chosen Poincaré sections which discretize the geodesic flow, encoding it with a shift space. We follow the same approach, using the work already done in [**CLT20a**] to produce a particularly nice collection of sections which satisfies the main conditions needed in [**LT05**]'s argument.

We begin by recalling the constructions and properties of these sections from [**CLT20a**]. Then we check that the key properties needed in the main argument of this paper hold for these sections.

3.1. *Sections and Markov proper families.*

Again, let (Y, d_Y) be a compact metric space with a continuous metric Anosov flow $\{\phi_t\}$. Assume that ϕ_t has no fixed points.

Definition 3.1. A *Poincaré section* (or simply *section*) is a closed subset $D \subset Y$ such that the map $(y, t) \mapsto \phi_t y$ is a homeomorphism between $D \times [-\tau^*, \tau^*]$ and $\phi_{[-\tau^*,\tau^*]}D$ for some time $\tau^* > 0$.

We will use the following notation related to sections.

Definition 3.2. Given a section D, we denote by $\mathrm{Int}_\phi D$ the interior of D transverse to the flow; that is,

$$\mathrm{Int}_\phi D = D \cap \bigcap_{\epsilon>0} (\phi_{(-\epsilon,\epsilon)} D)^\circ.$$

Let $\mathrm{Proj}_D : \phi_{[-\tau,\tau]}D \to D$ be the projection map defined by $\mathrm{Proj}_D(\phi_t y) = y$.

Given a collection $\mathcal{D} = \{D_1, \ldots, D_n\}$ of disjoint sections, if $\bigcup_{i=1}^n \phi_{(-\alpha,0)}\mathrm{Int}_\phi D_i = Y$, then any orbit of ϕ_t crosses an infinite sequence of sections. We let ψ be the first-return map for this collection, and let τ be the first-return time for the collection, defined by $\psi(x) = \phi_{\tau(x)}x$ for all $x \in \bigcup_{i=1}^n D_i$.

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Any orbit of ϕ_t can be encoded by the bi-infinite sequence of sections it crosses. To make this encoding useful, however, sections need to be chosen carefully. Bowen [**Bow73**] and Pollicott [**Pol87**] provide a framework for doing this in the Anosov and metric Anosov flow settings, respectively. Reference [**CLT20a**] further describes this process. The end result is a Markov proper family.

Definition 3.3. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ and $\mathcal{D} = \{D_1, \ldots, D_n\}$ be collections of disjoint sections. Here, $(\mathcal{B}, \mathcal{D})$ form a *Markov proper family at scale* $\alpha > 0$ if they satisfy the following properties:

(1) $\operatorname{diam}(D_i) < \alpha$ and $B_i \subset D_i$ for each $i = 1, \ldots, n$;
(2) $\bigcup_{i=1}^{n} \phi_{(-\alpha,0)} \operatorname{Int}_\phi B_i = Y$;
(3) each B_i is a *rectangle* in D_i (see [**CLT20a**, Definition 3.6] for details);
(4) $\mathcal{B}$ satisfies a *Markov property* (see [**CLT20a**, Definition 3.8] for details).

References [**Bow73**, **CLT20a**, **Pol87**] all discuss conditions under which Markov proper families exist. Condition (3) does not play a role in the present paper, and we will work with a weaker version of condition (4) (see Lemma 3.12 below). For our purposes, the key result is the following.

PROPOSITION 3.4. [**CLT20a**, §4] *Let X be a compact, locally CAT(−1) space. For geodesic flow on GX, at any scale $\alpha > 0$, there exist Markov proper families with Lipschitz return time functions τ.*

3.2. *Markov codings and Markov proper families.* In this section, let $(\mathcal{R}, \mathcal{S})$ be a Markov proper family at scale $\alpha > 0$ for a metric Anosov flow $\{\phi_t\}$ on Y. We discuss how the sections index orbits as sequences in a subshift of finite type.

Definition 3.5. For a Markov proper family $(\mathcal{R}, \mathcal{S})$ for a metric Anosov flow, we define the *canonical coding space* to be

$$\Theta = \Theta(\mathcal{R}) = \left\{ \theta \in \prod_{-\infty}^{\infty} \{1, \ldots, n\} : \text{ for all } l, k \geq 0, \bigcap_{j=-k}^{l} \psi^{-j}(\operatorname{Int}_\phi R_{\theta_j}) \neq \emptyset \right\}.$$

In [**Pol87**, §2.3], the symbolic space $\Theta(\mathcal{R})$ is shown to be a shift of finite type. Let $\sigma : \Theta \to \Theta$ be the associated left shift map.

Definition 3.6. For any $\theta \in \Theta$, we define the local stable and unstable sets of θ to be

$$W^s_{\text{loc}}(\theta) := \{y \in \operatorname{Int}_\phi(R_{\theta_0}) : \text{ for all } k \geq 0, \psi^k(y) \in R_{\theta_k}\},$$
$$W^u_{\text{loc}}(\theta) := \{y \in \operatorname{Int}_\phi(R_{\theta_0}) : \text{ for all } k \geq 0, \psi^{-k}(y) \in R_{\theta_{-k}}\}.$$

There is a canonically defined map $\pi : \Theta(\mathcal{R}) \to \mathcal{R}$ given by

$$\pi(\theta) = W^s_{\text{loc}}(\theta) \cap W^u_{\text{loc}}(\theta) = \bigcap_{j=-\infty}^{\infty} \psi^{-j}(R_{\theta_j}).$$

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Since the flow is expansive (Proposition 2.9), this intersection is a single point on R_{θ_0} and the map π is well defined. However, for any point $y \in \mathcal{R}$, we have a canonical sequence $\theta^y \in \Theta(\mathcal{R})$ given by

$$\theta^y = (\ldots, \theta^y_{-1} \mid \theta^y_0, \theta^y_1, \ldots) \quad \text{where } \psi^k(y) \in R_{\theta^y_k}.$$

Therefore, $\pi(\theta^y) = y$.

3.3. *Sections for flow on GX.* In this section, we utilize attributes of the metric Anosov flow to define the tools and verify the properties needed to construct a 'discretized' subaction. Let (X, d_X) be a locally CAT(-1) space with geodesic flow $\{g_t\}$ which is metric Anosov (Theorem 2.10).

Let Σ be the disjoint union of the rectangle sections $\{\Sigma_i\}_{i\in I}$ in the Markov proper family given by Proposition 3.4. Let $\psi : \Sigma \to \Sigma$ be the Poincaré return map and let $\tau : \Sigma \to (0, \infty)$ be the return time map with respect to the flow $\{g_t\}$. While constructing collections of sections at arbitrarily small scales is straightforward (see Proposition 3.4), the key aspect of the next lemma is the ability to construct collections whose sections have arbitrarily small diameters and such that the associated return time is bounded below by a fixed constant, τ_*.

LEMMA 3.7. (Cf. [**LT05**, Lemma 6]) *There exist constants α^*, τ_*, τ^* and a collection of Poincaré sections $\tilde{\Sigma} = \{\tilde{\Sigma}_i\}_{i\in I}$ with* diam $\tilde{\Sigma}_i \le \alpha^*$ *such that for any $\alpha < \alpha^*$, we can construct a collection of disjoint Poincaré sub-sections $\Sigma = \{\Sigma_i\}_{i\in I}$ satisfying the following properties:*

(1) *for all $i \in I$, $\Sigma_i \subset \tilde{\Sigma}_i$ and* diam $\Sigma_i \le \alpha$*;*
(2) $\bigcup_{i\in I} g_{(0,\tau^*)}\mathrm{Int}_g\,\Sigma_i = GX$*;*
(3) *the return time τ associated to the sub-section Σ is $\ge \tau_*$.*

Proof. We first start with any collection of Poincaré sections $\widehat{\Sigma} = \{\widehat{\Sigma}_i\}_{i\in\hat{I}}$, given by the collection $\mathcal{B}$ in Proposition 3.4, and suppose the associated return time is bounded below and above by t_* and t^*, respectively. Note that $t_* > 0$ since the sections in $\widehat{\Sigma}$ are disjoint and closed. By definition, the $(0, t^*)$-flow boxes $g_{(0,t^*)}\mathrm{Int}_g\,\widehat{\Sigma}_i$ of $\widehat{\Sigma}_i$ cover GX; that is, $GX = \bigcup_{i\in I} g_{(0,t^*)}\mathrm{Int}_g\,\widehat{\Sigma}_i$. Choose α^* small enough that $2\alpha^* < t_*$ and the $(0, t^* - 2\alpha^*)$-flow boxes of $\widehat{\Sigma}'_i$ still cover GX, where

$$\widehat{\Sigma}'_i = \{\gamma \in \widehat{\Sigma}_i : d(\gamma, GX\backslash\widehat{\Sigma}_i) > 2\alpha^*\}.$$

We want to apply Lemma 2.14 to $\widehat{\Sigma}'_i$, but must take some care, as that lemma is proven on $\partial^{(2)}_\infty \tilde{X}$. Consider a lift Σ^* of $\widehat{\Sigma}'_i$ to $G\tilde{X}$. Since $G\tilde{X} \cong \partial^{(2)}_\infty \tilde{X} \times \mathbb{R}$, Σ^* projects to a subset Σ^*_∞ of $\partial^{(2)}_\infty \tilde{X}$ by forgetting the flow coordinate. Apply Lemma 2.14 (with covering constant C) to Σ^*_∞. Then, remembering the flow coordinate associated to Σ^*, we project the results of Lemma 2.14 back to Σ^* and then project down to GX.

The result is as follows. Given any $\alpha < \alpha^*$, we can cover each $\widehat{\Sigma}'_i$ with sets $\{B_{ijk}\}_{j\in J,\,1\le k\le C}$ in such a way that for fixed i, k, the sets $\{B_{ijk}\}_{j\in J}$ are pairwise disjoint, and (using Lemma 2.4) such that for any $t \in [0, \alpha^*]$, diam $g_t B_{ijk} < \alpha$. We note here that in the proof of Lemma 2.14, the sets B_{ijk} are ϵ-balls with respect to the ℓ^∞ product of

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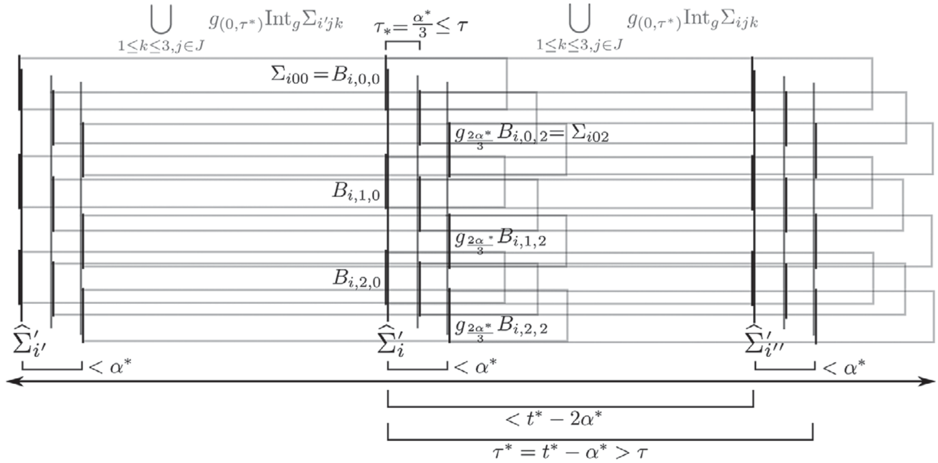

FIGURE 2. A visual aid for the proof of Lemma 3.7. Note that the stack associated to Σ_{ijk} is covered by the flow boxes for $\Sigma_{i'jk}$. Here, $C = 3$.

visual metrics. It is not hard to see that by taking ϵ small enough, we can ensure that the resulting sets have d_{GX}-diameter as small as we need.

We then 'stack' C copies of $\widehat{\Sigma}_i$ along the flow, letting

$$\tilde{\Sigma}_{ijk} = g_{(k/C)\alpha^*}\widehat{\Sigma}_i \quad \text{and} \quad \Sigma_{ijk} = g_{(k/C)\alpha^*} B_{ijk}.$$

Note that by the choice of α^*, $\Sigma_{ijk} \subset \tilde{\Sigma}_{ijk}$. Let τ denote the return time associated to the sections $\Sigma = \{\Sigma_{ijk}\}_{i,j,k}$.

By design, for fixed i, $\{\Sigma_{ijk}\}_{k,j}$ are pairwise disjoint, but additionally, since the maximum height of the stack is $\alpha^* < (t_*/2)$, no two stacks intersect; therefore, $\{\Sigma_{ijk}\}_{i,j,k}$ are all pairwise disjoint. Also, by construction, τ is at least $\alpha^*/C =: \tau_*$. Finally, we note that the longest τ can be is the sum of the maximum return time for $\widehat{\Sigma}'$ and the maximum stack height; that is, $\tau < (t^* - 2\alpha^*) + \alpha^* = t^* - \alpha^* =: \tau^*$, and so the $(0, \tau^*)$-flow boxes for Σ cover GX:

$$\bigcup_{i,j,k} g_{(0,\tau^*)}\mathrm{Int}_g\,\Sigma_{ijk} \supset \bigcup_i g_{(0,t^*-2\alpha^*)}\mathrm{Int}_g\,\widehat{\Sigma}'_i = GX.$$

Importantly, we note that τ_* and τ^* are independent of the size α of the sections Σ (see Figure 2). □

From now on, we fix the sections $\tilde{\Sigma}$ with fixed associated constants α^*, τ_*, τ^* given by Lemma 3.7; in particular, diam $\tilde{\Sigma}_i \leq \alpha^*$ and its associated return time $\tilde{\tau}$ is bounded below and above by the fixed constants τ_* and τ^*, respectively. By Lemma 3.7, we will be able to choose subsections Σ such that diam $\Sigma_i = \alpha$ is arbitrarily small, while its associated return time τ preserves the same bounds.

Definition 3.8. Let $i, j \in I$. We say $i \to j$ is a *simple transition* if there exists $\gamma \in \Sigma_i$ such that $\psi(\gamma) \in \Sigma_j$. Let

$$\mathrm{dom}(\tilde{\psi}_{ij}) = \mathrm{dom}(\tilde{\tau}_{ij}) := \{\gamma \in \tilde{\Sigma}_i : g_t(\gamma) \in \tilde{\Sigma}_j \text{ for some } t \in (0, \tau^*)\}.$$

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We define the extended first return time $\tilde{\tau}_{ij}$ and first return map $\tilde{\psi}_{ij}$ for the simple transition $i \to j$ as follows: for $\gamma \in \mathrm{dom}(\tilde{\psi}_{ij}) = \mathrm{dom}(\tilde{\tau}_{ij})$,

$$\tilde{\tau}_{ij}(\gamma) = \inf\{t \in (0, \tau^*) : g_t\gamma \in \tilde{\Sigma}_j\},$$

$$\tilde{\psi}_{ij}(\gamma) = g_{\tilde{\tau}_{ij}(\gamma)}\gamma.$$

In particular, using Lemma 3.7, we choose α to be small enough that $\Sigma_i \subseteq \mathrm{dom}(\tilde{\psi}_{ij})$ and $\Sigma_j \subseteq \mathrm{range}(\tilde{\psi}_{ij})$ for any simple transition $i \to j$.

Note that for any canonical sequence $\theta \in \Theta(\Sigma)$ (see Definition 3.5), we have $\theta_k \to \theta_{k+1}$ is a simple transition for each $k \in \mathbb{Z}$. Indeed, $\psi(\psi^k(\pi(\theta))) = \psi^{k+1}(\pi(\theta)) \in \Sigma_{\theta_{k+1}}$. We now extend the canonical coding space $\Theta(\Sigma)$ to allow for double-sided sequences of simple transitions.

Definition 3.9. A *pseudo-orbit* is a double-sided sequence of simple transitions. Let Ω be the extended coding space of pseudo-orbits,

$$\Omega = \{\omega = (\ldots, \omega_{-1} \mid \omega_0, \omega_1, \ldots) : \ \omega_k \to \omega_{k+1} \text{ is a simple transition for all } k\}.$$

It is immediate from its definition that Ω is also a sub-shift of finite type, and we denote by $\sigma : \Omega \to \Omega$ the associated left shift map. Note that $\Theta \subset \Omega$. For each $\omega \in \Omega$, define

$$\tilde{\psi}_\omega = \tilde{\psi}_{\omega_0\omega_1} : \Sigma_{\omega_0} \to \tilde{\Sigma}_{\omega_1},$$

$$\tilde{\tau}_\omega = \tilde{\tau}_{\omega_0\omega_1},$$

and, on the domain where the composition is well defined,

$$\tilde{\psi}^k_\omega = \tilde{\psi}_{\sigma^{k-1}(\omega)} \circ \cdots \circ \tilde{\psi}_\omega.$$

As before, we define the local stable and unstable sets of a pseudo-orbit $\omega \in \Omega$. These are where the composition $\tilde{\psi}^k_\omega$ is well defined for all positive (respectively, negative) k.

Definition 3.10. For any $\omega \in \Omega$, we define the local stable and unstable sets of ω to be

$$W^s_{\mathrm{loc}}(\omega) := \{\gamma \in \mathrm{Int}_\phi(\Sigma_{\omega_0}) : \ \text{for all } k \geq 0,\ \tilde{\psi}^k_\omega(\gamma) \in \Sigma_{\omega_k}\},$$
$$W^u_{\mathrm{loc}}(\omega) := \{\gamma \in \mathrm{Int}_\phi(\Sigma_{\omega_0}) : \ \text{for all } k \geq 0,\ \tilde{\psi}^{-k}_\omega(\gamma) \in \Sigma_{\omega_{-k}}\}.$$

For any $\omega, \omega' \in \Omega$ with $\omega_0 = \omega'_0$, $W^s_{\mathrm{loc}}(\omega)$ intersects $W^u_{\mathrm{loc}}(\omega')$ at a unique point. This is due to the fact that α can be made arbitrarily small and Bowen's shadowing lemma: consider the sequence formed by concatenating the past of ω' with the future of ω: $\zeta = (\ldots, \omega'_{-2}, \omega'_{-1} \mid \omega_0, \omega_1, \ldots)$. By definition, for each j, $\zeta_j \to \zeta_{j+1}$, so there exists $\gamma_j \in \Sigma_{\zeta_j}$ such that $\psi_\zeta(\gamma_j) \in \Sigma_{\zeta_{j+1}}$. This $(\gamma_j)_{j\in\mathbb{Z}}$ is a pseudo-orbit and by Bowen's shadowing lemma, there exists a unique element γ whose true orbit $(\psi_\zeta(\gamma))_{j\in\mathbb{Z}}$ shadows $(\gamma_j)_{j\in\mathbb{Z}}$. Therefore, $\{\gamma\} = W^s_{\mathrm{loc}}(\omega) \cap W^u_{\mathrm{loc}}(\omega')$.

Definition 3.11. For any $\omega, \omega' \in \Omega$ with $\omega_0 = \omega'_0$, we denote the unique point in the intersection $W^s_{\mathrm{loc}}(\omega) \cap W^u_{\mathrm{loc}}(\omega')$ by $[\omega, \omega']$. We denote $[\omega, \omega]$ by $\pi(\omega)$.

We now demonstrate that $\tilde{\psi}_\omega$ satisfies a property that is related to and resembles the Markov property (Definition 3.3).

LEMMA 3.12. (Markov property for $\tilde{\psi}_\omega$) *$\tilde{\psi}_\omega$ stabilizes the local stable set:*

$$\tilde{\psi}_\omega(W^s_{\mathrm{loc}}(\omega)) \subset W^s_{\mathrm{loc}}(\sigma(\omega)),$$

and $\tilde{\psi}_\omega^{-1}$ stabilizes the local unstable set:

$$\tilde{\psi}_\omega^{-1}(W^u_{\mathrm{loc}}(\omega)) \subset W^u_{\mathrm{loc}}(\sigma^{-1}(\omega)).$$

Consequently, $\tilde{\psi}_\omega(\pi(\omega)) = \pi(\sigma(\omega))$.

Proof. Let $\gamma \in W^s_{\mathrm{loc}}(\omega)$. Then, for any $k \geq 0$,

$$\tilde{\psi}^k_{\sigma(\omega)}(\tilde{\psi}_\omega(\gamma)) = \tilde{\psi}^{k+1}_\omega(\gamma) \in \Sigma_{\omega_{k+1}} = \Sigma_{(\sigma(\omega))_k},$$

so $\tilde{\psi}_\omega(\gamma) \in W^s_{\mathrm{loc}}(\sigma(\omega))$. Similarly, let $\gamma \in W^u_{\mathrm{loc}}(\omega)$. Then, for any $k \geq 0$,

$$\tilde{\psi}^{-k}_{\sigma^{-1}(\omega)}(\tilde{\psi}^{-1}_\omega(\gamma)) = \tilde{\psi}^{-(k+1)}_\omega(\gamma) \in \Sigma_{\omega_{-(k+1)}} = \Sigma_{(\sigma^{-1}(\omega))_{-k}},$$

so $\tilde{\psi}^{-1}_\omega(\gamma) \in W^u_{\mathrm{loc}}(\sigma^{-1}(\omega))$. □

3.4. *Exponential contraction.* Exponential contraction along $W^s_{\mathrm{loc}}(\theta)$ under the return map ψ for any $\theta \in \Theta$ plays an essential role in later proofs. We will prove our contraction result (an analog for our situation of [**LT05**, Lemma 8(iii)]) for the simple first return map ψ associated to the collection Σ if sections constructed above. We note that the result holds for the 'return map with instructions' $\tilde{\psi}_\omega$ since $\tilde{\psi}_\omega$ is at any point ψ^k for an appropriate k.

LEMMA 3.13. *Let $\gamma, \gamma' \in W^s_{\mathrm{loc}}(\omega)$ for some $\omega \in \Omega$ and let ψ be the first return map for the collection of sections Σ. Then, there exists a constant $C > 0$ (independent of γ, γ', ω) such that for all $k > 0$,*

$$d_{GX}(\psi^k\gamma, \psi^k\gamma') < Ce^{-k\tau_*} d_{GX}(\gamma, \gamma').$$

Similarly, if $\gamma, \gamma' \in W^u_{\mathrm{loc}}(\omega)$, for all $k > 0$,

$$d_{GX}(\psi^{-k}\gamma, \psi^{-k}\gamma') < Ce^{-k\tau_*} d_{GX}(\gamma, \gamma').$$

Remark 3.14. For comparison with [**LT05**] in this and subsequent lemmas, recall that $\lambda = 1$ for geodesic flow on a CAT(-1) space.

For this proof, we will need a sublemma. Consider the following setup (for the stable half of the proof). Suppose that $\gamma, \gamma' \in W^s_{\mathrm{loc}}(\omega) \subset \Sigma_i$ and that $\psi^k\gamma, \psi^k\gamma' \in \Sigma_j$. Since $\gamma, \gamma' \in W^s_{\mathrm{loc}}(\omega)$, γ' is on the weak-stable leaf through γ and hence for some r_1, $g_{r_1}\gamma' \in W^{ss}(\gamma)$. Similarly, as $\psi^k\gamma, \psi^k\gamma' \in W^s_{\mathrm{loc}}(\sigma^k\omega)$, for some r_2, $g_{r_2}\psi^k\gamma' \in W^{ss}(\psi^k\gamma)$. (See Figure 3.)

SUBLEMMA 3.15. *We have $|r_1| \leq 2d_{GX}(\gamma, \gamma')$ and $|r_2| \leq 2d_{GX}(\psi^k\gamma, g_{r_2}\psi^k\gamma')$.*

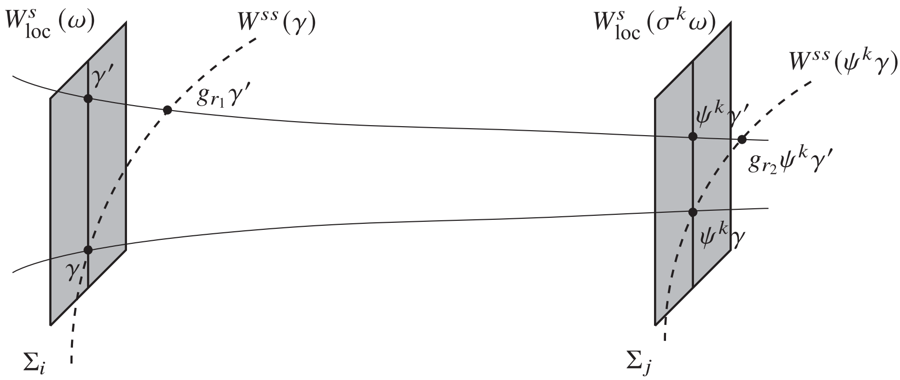

FIGURE 3. The geometric setup for Lemma 3.13.

Proof. We prove the sublemma for r_1; the proof for r_2 is the same.

Note, since the strong stable set $W^{ss}(\gamma)$ is determined by values of $B_{\gamma(0)}(-, \gamma(+\infty))$, $|r_1| = B_{\gamma(0)}(\gamma'(0), \gamma(+\infty))$. Busemann functions are 1-Lipschitz, so $B_{\gamma(0)}(\gamma'(0), \gamma(+\infty)) \leq d_X(\gamma(0), \gamma'(0))$. Therefore (using Lemma 2.3),

$$|r_1| \leq d_X(\gamma(0), \gamma'(0)) \leq 2d_{GX}(\gamma, \gamma'),$$

as desired. □

Proof of Lemma 3.13. We prove the result only for the local stable sets, the proof for the unstable sets being essentially the same. As usual, we lift to the CAT(−1) universal cover to make our geometric arguments.

We continue with the setup and notation illustrated in Figure 3. Let τ, τ' be such that $g_\tau\gamma = \psi^k\gamma$ and $g_{\tau'}\gamma' = \psi^k\gamma'$. Note that by construction of Σ, $\tau, \tau' \geq k\tau_*$. Since g_τ maps $W^{ss}(\gamma)$ to $W^{ss}(\psi^k\gamma)$, we see that $g_{r_2}\psi\gamma' = g_\tau g_{r_1}\gamma'$ and so $\tau' = \tau + (r_1 - r_2)$.

By the triangle inequality,

$$d_{GX}(\gamma, g_{r_1}\gamma') \leq d_{GX}(\gamma, \gamma') + |r_1|,$$

$$d_{GX}(\psi^k\gamma, \psi^k\gamma') \leq d_{GX}(\psi^k\gamma, g_{r_2}\psi^k\gamma') + |r_2|.$$

Since g_t is a metric Anosov flow with $\lambda = 1$, for some uniform $C_1 > 0$,

$$d_{GX}(\psi^k\gamma, g_{r_2}\psi^k\gamma') \leq C_1 e^{-k\tau_*} d_{GX}(\gamma, g_{r_1}\gamma').$$

Combining these inequalities gives

$$\begin{aligned} d_{GX}(\psi^k\gamma, \psi^k\gamma') &\leq d_{GX}(\psi^k\gamma, g_{r_2}\psi^k\gamma') + |r_2| \\ &\leq C_1 e^{-k\tau_*} d_{GX}(\gamma, g_{r_1}\gamma') + |r_2| \\ &\leq C_1 e^{-k\tau_*}[d_{GX}(\gamma, \gamma') + |r_1|] + |r_2|. \end{aligned}$$

Then, using Sublemma 3.15,

$$d_{GX}(\psi^k\gamma, \psi^k\gamma') \leq C_1 e^{-k\tau_*}[d_{GX}(\gamma, \gamma') + 2d_{GX}(\gamma, \gamma')] + 2d_{GX}(\psi^k\gamma, g_{r_2}\psi^k\gamma')$$

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$$
\begin{aligned}
&\le 3C_1 e^{-k\tau_*} d_{GX}(\gamma, \gamma') + 2C_1 e^{-k\lambda\tau_*} d_{GX}(\gamma, g_{r_1}\gamma') \\
&\le 3C_1 e^{-k\tau_*} d_{GX}(\gamma, \gamma') + 2C_1 e^{-k\tau_*} [d_{GX}(\gamma, \gamma') + r_1] \\
&\le 3C_1 e^{-k\tau_*} d_{GX}(\gamma, \gamma') + 2C_1 e^{-k\tau_*} [d_{GX}(\gamma, \gamma') + 2d_{GX}(\gamma, \gamma')] \\
&= 9C_1 e^{-k\tau_*} d_{GX}(\gamma, \gamma'),
\end{aligned}
$$

as desired. □

We also have the following analog of [**LT05**, Lemma 8(iv)] which gives very rough upper and lower bounds on the distance between two points under iterates of our return map $\tilde{\psi}$.

LEMMA 3.16. *There exist constants K^* and $\Lambda_*^s < 0 < \Lambda_*^u$ such that the following holds. Suppose that $x, y \in \Sigma_i$ and that $i = i_0 \to \cdots \to i_n$ is a chain of simple transitions so that $\tilde{\psi}^n := \tilde{\psi}_{i_{n-1}i_n} \circ \cdots \circ \tilde{\psi}_{i_0 i_1}$ exists at x and y. Then,*

$$
(K^*)^{-1} \exp(n\Lambda_*^s) d_{GX}(x, y) \le d_{GX}(\tilde{\psi}^n x, \tilde{\psi}^n y) \le K^* \exp(n\Lambda_*^u) d_{GX}(x, y).
$$

Proof. Say $\tilde{\psi}^n x = g_t x$ and $\tilde{\psi}^n y = g_{t'} y$. Note that $t \le n\tau^*$ and that $|t - t'|$ is the difference in first return times for x and y and under flow from Σ_i to Σ_{i_n}.

By triangle inequality,

$$
d_{GX}(\tilde{\psi}^n x, \tilde{\psi}^n y) \le d_{GX}(g_t x, g_t y) + |t - t'|.
$$

By Lemma 2.4 and the Lipschitz behavior of return times given by Proposition 3.4,

$$
\begin{aligned}
d_{GX}(\tilde{\psi}^n x, \tilde{\psi}^n y) &\le e^{2t} d_{GX}(x, y) + K d_{GX}(x, y) \\
&\le (K + 1) \exp((2\tau^*)n) d_{GX}(x, y).
\end{aligned}
$$

To prove the other bound, we can simply reverse the direction of the flow: $x = g_{-t}\tilde{\psi} x$ and $y = g_{-t'}\tilde{\psi} y$. Applying the same argument, we get

$$
d_{GX}(x, y) \le (K + 1) \exp((2\tau^*)n) d_{GX}(\tilde{\psi}^n x, \tilde{\psi}^n y),
$$

which completes the proof. □

We now prove two lemmas that will be useful in proving that the discretized subaction is Hölder in §4.

LEMMA 3.17. *There exists $\delta > 0$ depending only on Σ such that for any $i \in I$ and any $\gamma, \gamma' \in \Sigma_i$, if $d_{GX}(\gamma, \gamma') < \delta$, then there exists a simple transition $i \to j$ and $m, n \ge 1$ such that*

$$
\psi^m(\gamma) \in \Sigma_j, \ \psi^n(\gamma') \in \Sigma_j.
$$

Furthermore, the first return times from Σ_i to Σ_j satisfy

$$
\sum_{k=0}^{m-1} \tau \circ \psi^k(\gamma) < \tau^*, \quad \sum_{k=0}^{n-1} \tau \circ \psi^k(\gamma') < \tau^*.
$$

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Proof. Let $U_i = g_{(0,\tau^*)}\Sigma_i$. For any simple transition $i \to j$, let

$$\Delta_{ij} = \{\eta \in \Sigma_i : g_t(\eta) \in \Sigma_j \text{ for some } t \in (0, \tau^*)\}$$

and let $t_j : \Delta_{ij} \to (0, \tau^*)$ be the time such that $g_{t_j(\eta)}\eta \in \Sigma_j$. Note that when $\eta \in \Delta_{ij}$, $\phi^n(\eta) \in \Sigma_j$ for some $n \geq 1$ and $t_j(\eta) = \sum_{k=0}^{n-1} \tau \circ \psi^k(\eta)$. Therefore, showing that $\gamma, \gamma' \in \Delta_{ij}$ will prove the lemma.

Since τ^* is the maximum return time, the sets $\{\Delta_{ij}\}_j$ cover Σ_i, which is compact. Therefore, we can find a Lebesgue number $\delta > 0$ for the cover $\{\Delta_{ij}\}_j$. If γ' is contained in the δ-ball around γ, then there exists $j \in I$ such that the ball is contained in Δ_{ij}. In particular, $\gamma, \gamma' \in \Delta_{ij}$. □

In the following, Λ_*^u and K^* are constants given by Lemma 3.16.

LEMMA 3.18. *Suppose $\gamma, \gamma' \in \Sigma_i$. For any $N \geq 1$, if $d_{GX}(\gamma, \gamma') < (\delta/K^*)e^{-N\Lambda_*^u}$, then there exist $\omega, \omega' \in \Omega$ such that $\pi(\omega) = \gamma$, $\pi(\omega') = \gamma'$ and their first $N + 1$ symbols coincide: $\omega_0 = \omega'_0, \ldots, \omega_N = \omega'_N$.*

Proof. Let $\gamma_0 = \gamma$ and $\gamma'_0 = \gamma' \in \Sigma_i$. By Lemma 3.17, since $d_{GX}(\gamma_0, \gamma'_0) < \delta$, there exists m_1, n_1 and $i_1 \in I$ such that

$$\gamma_1 = \psi^{m_1}(\gamma_0), \quad \gamma'_1 = \psi^{n_1}(\gamma'_0) \in \Sigma_{i_1}.$$

By Lemma 3.16,

$$d_{GX}(\gamma_1, \gamma'_1) \leq K^* e^{\Lambda_*^u} d_{GX}(\gamma_0, \gamma'_0) < K^* e^{\Lambda_*^u} \frac{\delta}{K^*} e^{-N\Lambda_*^u} = \delta e^{-(N-1)\Lambda_*^u} < \delta,$$

so we can apply Lemma 3.17 again. We repeat this construction N times to find two sequences $(m_1, m_2, \ldots, m_N)$ and $(n_1, \ldots, n_N)$ such that for each $0 \leq l \leq N$,

$$\gamma_l = \psi^{m_1+\cdots+m_l}(\gamma) \quad \text{and} \quad \gamma'_l = \psi^{n_1+\cdots+n_l}(\gamma')$$

belong to the same section Σ_{i_l} with $d_{GX}(\gamma_l, \gamma'_l) < \delta$.

Let $\theta^\gamma, \theta^{\gamma'} \in \Theta(\Sigma)$ be the canonical sequences associated to γ and γ', respectively, and let

$$m = m_1 + \cdots + m_N, \quad n = n_1 + \cdots + n_N.$$

Define

$$\omega = (\ldots, \theta^\gamma_{-2}, \theta^\gamma_{-1} \mid i_0, i_1, \ldots, i_N, \theta^\gamma_{m+1}, \theta^\gamma_{m+2}, \ldots),$$
$$\omega' = (\ldots, \theta^{\gamma'}_{-2}, \theta^{\gamma'}_{-1} \mid i_0, i_1, \ldots, i_N, \theta^{\gamma'}_{n+1}, \theta^{\gamma'}_{n+2}, \ldots).$$

By construction, $\tilde{\psi}^k_\omega(\gamma) \in \Sigma_{\omega_k}$ and $\tilde{\psi}^k_{\omega'}(\gamma') \in \Sigma_{\omega'_k}$ for all $k \in \mathbb{Z}$, so $\gamma = \pi(\omega)$ and $\gamma' = \pi(\omega')$, and they coincide in the first $N + 1$ times, as desired. □

4. *A sub-action for the discretized system*

Let $A : GX \to \mathbb{R}$ be Hölder and define its *minimal average* by

$$m(A) = \inf\left\{\int A \, d\mu \mid \mu \in \mathcal{M}_1(GX, g_t)\right\},$$

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where $\mathcal{M}_1(GX, g_t)$ denotes the set of all g_t-invariant probability measures on GX. We define the discretized observable $\mathcal{A} : \Sigma \to \mathbb{R}$ by

$$\mathcal{A}(\gamma) = \int_0^{\tau(\gamma)} (A - m(A)) \circ g_t(\gamma)\, dt. \tag{4.1}$$

The goal of this section is to construct a discretized sub-action $\mathcal{V}$ for $\mathcal{A}$.

PROPOSITION 4.1. *There exist sections Σ and a globally Hölder function $\mathcal{V} : \Sigma \to \mathbb{R}$ such that*

$$\mathcal{A}(\gamma) \geq \mathcal{V} \circ \psi(\gamma) - \mathcal{V}(\gamma) \quad \textit{for all } \gamma \in \Sigma.$$

As in [**LT05**], we first extend $\mathcal{A} : \Sigma \to \mathbb{R}$ to the space $\Omega \times \Sigma$. To that end, we introduce the following notation. Define

$$\begin{aligned} D &= \{(\omega,\gamma) \in \Omega \times \Sigma : \gamma \in \Sigma_{\omega_0}\} \\ \tilde{\psi} &: D \to \Omega \times \Sigma \\ (\omega, \gamma) &\mapsto (\sigma(\omega), \tilde{\psi}_\omega(\gamma)) \end{aligned}$$

to be the first return map 'with instructions' and let $\tilde{\tau}$ be the associated first return time $\tilde{\tau}(\omega, \gamma) := \tilde{\tau}_\omega(\gamma)$. We also extend the discretized $\mathcal{A}$ to $\Omega \times \Sigma$ (or, more precisely, to D) by

$$\tilde{\mathcal{A}}(\omega, \gamma) = \int_0^{\tilde{\tau}(\omega,\gamma)} (A - m(A)) \circ g_t(\gamma)\, dt.$$

Recall that for $\gamma \in \Sigma$, its canonical pseudo-orbit $\theta^\gamma \in \Theta(\Sigma)$ given by

$$\theta^\gamma = (\ldots, \theta_{-1} \mid \theta_0, \theta_1, \ldots) \quad \text{where } \psi^k(\gamma) \in \Sigma_{\theta_k}$$

satisfies $\pi(\theta^\gamma) = \gamma$. Define

$$\begin{aligned} \tilde{\theta} &: \Sigma \to \Omega \times \Sigma \\ \gamma &\mapsto (\theta^\gamma, \gamma). \end{aligned}$$

Then, we note that

$$\tilde{\theta} \circ \psi = \tilde{\psi} \circ \tilde{\theta}.$$

Since $\tilde{\tau}(\theta^\gamma, \gamma) = \tau(\gamma)$, we note that $\tilde{\mathcal{A}}$ is an extension of $\mathcal{A}$:

$$\tilde{\mathcal{A}} \circ \tilde{\theta} = \mathcal{A}.$$

Definition 4.2. For $\omega \in \Omega$ and $\eta \in W^s_{\mathrm{loc}}(\omega)$, define

$$\begin{aligned} b^s(\omega, \eta) &= B_{\pi(\omega)}(\eta(0)), \\ w^s(\omega, \eta) &= g_{b^s(\omega,\eta)}\eta, \\ \Delta^s(\omega, \eta) &= \sum_{n \geq 0} (\tilde{\mathcal{A}} \circ \tilde{\psi}^n(\omega, \eta) - \tilde{\mathcal{A}} \circ \tilde{\psi}^n(\omega, \pi(\omega))). \end{aligned}$$

Here, $\Delta^s(\omega, \eta)$ is often called the stable cocycle. Note that $b^s(\omega, \eta)$ is 1-Lipschitz in the distance between the basepoints of η and $\pi(\omega)$ and so goes to zero as this distance goes to zero.

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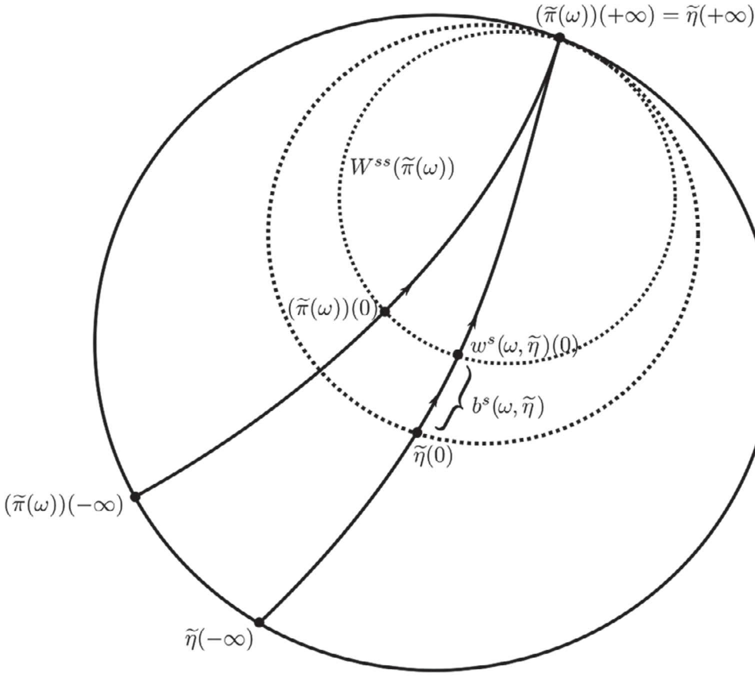

FIGURE 4. An illustration of some of the definitions from Definition 4.2. Note that $\eta \in W^s_{\text{loc}}(\omega)$ so $\tilde{\eta} \in W^{cs}(\tilde{\pi}(\omega))$.

Remark 4.3. In the definition above, we are abusing notation slightly: it should be understood that $B_{\pi(\omega)}(\eta(0))$ is in fact $B_{\tilde{\pi}(\omega)}(\tilde{\eta}(0))$, where $\tilde{\pi}(\omega)$ is a fixed lift of $\pi(\omega)$ to the CAT(-1) universal cover and $\tilde{\eta} \in W^{cs}(\tilde{\pi}(\omega))$ is the lift of η that has the same endpoint at infinity $\tilde{\pi}(\omega)(+\infty)$ (see Figure 4).

LEMMA 4.4. *Let $\omega \in \Omega$.*

(i) *The map $W^s_{\text{loc}}(\omega) \to W^{ss}(\pi(\omega))$ given by $\eta \mapsto w^s(\omega, \eta)$ is a parameterization of $W^{ss}(\pi(\omega))$.*

(ii) *The stable cocycle $\Delta^s(\omega, \eta)$ admits the equivalent form*

$$\Delta^s(\omega, \eta) = \int_0^\infty (A \circ g_t w^s(\omega, \eta) - A \circ g_t \pi(\omega))\, dt - \int_0^{b^s(\omega,\eta)} (A - m(A)) \circ g_t \eta\, dt.$$

In particular, $b^s(\omega, \eta)$, $W^s_{\text{loc}}(\omega)$, and $\Delta^s(\omega, \eta)$ depend on $\omega \in \Omega$ only via $\pi(\omega)$.

Proof. For part (i), note that $\eta \in W^{cs}(\pi(\omega))$, so using Lemma 2.8(2),

$$\begin{aligned} B_{\pi(\omega)}(w^s(\omega, \eta)(0)) &= B_{\pi(\omega)}(\eta(b^s(\omega, \eta))) \\ &= B_{\pi(\omega)}(\eta(0)) + (0 - b^s(\omega, \eta)) \\ &= 0. \end{aligned}$$

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Therefore, $w^s(\omega, \eta) \in W^{ss}(\pi(\omega))$. Conversely, any point on $W^{ss}(\pi(\omega))$ is of the form $g_s(\eta)$ for some $\eta \in \Sigma_{\omega_0}$ and small s. Since they are on the same stable leaf, ω is a valid set of forward instructions that η can take. Therefore, $\eta \in W^s_{\mathrm{loc}}(\omega)$.

To prove part (ii), we first establish the following sublemma.

SUBLEMMA 4.5. *For $\eta \in W^s_{\mathrm{loc}}(\omega)$, the cocycle equation*

$$b^s(\omega, \eta) + \tau_n(\pi(\omega)) = b^s(\sigma^n\omega, \tilde{\psi}^n_\omega(\eta)) + \tau_n(\eta) \tag{4.2}$$

holds, where $\tau_n(-) = \sum_{k=0}^{n-1} \tau \circ \tilde{\psi}^k_\omega(-)$.

Proof of sublemma. Using Lemma 2.8(1) and (2),

$$\begin{aligned} b^s(\omega, \eta) - b^s(\sigma^n\omega, \tilde{\psi}^n_\omega(\eta)) &= B_{\pi(\omega)}(\eta(0)) - B_{\pi(\sigma^n(\omega))}((\tilde{\psi}^n_\omega\eta)(0)) \\ &= B_{\pi(\omega)}(\eta(0)) - B_{g_{\tau_n(\pi(\omega))}\pi(\omega)}((\tilde{\psi}^n_\omega\eta)(0)) \\ &= B_{\pi(\omega)}(\eta(0)) - [B_{\pi(\omega)}((\tilde{\psi}^n_\omega\eta)(0)) + \tau_n(\pi(\omega))] \\ &= \tau_n(\eta) - \tau_n(\pi(\omega)). \end{aligned}$$

□

The proof of part (ii) then follows that of [**LT05**, Proposition 12(iii)] verbatim. To summarize, one considers the difference of partial sums approximating $\Delta^s(\omega, \eta)$:

$$\sum_{n=0}^{N} \tilde{\mathcal{A}} \circ \tilde{\psi}^n(\omega, \eta) - \sum_{n=0}^{N} \tilde{\mathcal{A}} \circ \tilde{\psi}^n(\omega, \pi(\omega)).$$

These are the integrals of $A - m(A)$ over orbit segments, specifically the time $[0, \tau_N(\eta)]$ segment of η and the $[0, \tau_N(\pi(\omega))]$ segment of $\pi(\omega)$, respectively. Large portions of these segments lie on common stable leaves, specifically the time $[0, \tau_N(\pi(\omega))]$ segments of $\pi(\omega)$ and $w^s(\omega, \eta)$. Grouping these contributions in a single integral, and recording the contributions of the non-overlapping segments at the beginning and end with the help of equation (4.2), the difference of the partial sums is

$$\begin{aligned} &\int_0^{b^s(\omega,\eta)} (A - m(A)) \circ g_t\eta \, dt + \int_0^{\tau_N(\pi(\omega))} (A \circ g_t w^s(\omega, \eta) - A \circ g_t\pi(\omega)) \, dt \\ &\quad - \int_{\tau_N(\pi(\omega)) - b^s(\sigma^N\omega, \tilde{\psi}^N_\omega(\eta))}^{\tau_N(\pi(\omega))} (A - m(A)) \circ g_t w^s(\omega, \eta) \, dt. \end{aligned}$$

As $N \to \infty$, the distance between the basepoints of $\sigma^N\omega$ and $\tilde{\psi}^N_\omega(\eta)$ goes to zero, so $b^s(\sigma^N\omega, \tilde{\psi}^N_\omega(\eta)) \to 0$, giving the result.

Finally, if $\pi(\omega) = \pi(\omega')$, then by definition, for any $\eta \in W^s_{\mathrm{loc}}(\omega) \cap W^s_{\mathrm{loc}}(\omega')$, $b^s(\omega, \eta) = b^s(\omega', \eta)$. Then, part (i) gives us that $W^s_{\mathrm{loc}}(\omega) = W^s_{\mathrm{loc}}(\omega')$, and part (ii) gives us that $\Delta^s(\omega, \eta) = \Delta^s(\omega', \eta)$. □

We now define a discretized sub-action $\tilde{\mathcal{V}}$ on the space Ω. Let $S_n\tilde{\mathcal{A}} = \sum_{k=0}^{n-1} \tilde{\mathcal{A}} \circ \tilde{\psi}^k$ be the Birkhoff sum of $\tilde{\mathcal{A}}$.

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Definition 4.6. For any $\omega \in \Omega$, define

$$\tilde{\mathcal{V}}(\omega) = \inf\{S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega, \zeta]) + \Delta^s(\omega, [\omega, \zeta]) \mid n \geq 0, \zeta \in \Omega, \zeta_0 = \omega_0\}.$$

First, we prove the following lemma.

LEMMA 4.7. *For any $\omega \in \Omega$, $\tilde{\mathcal{V}}(\omega)$ is finite.*

Proof. Write $\tilde{\tau}_n = \tilde{\tau}_n(\tilde{\psi}^{-n}(\zeta, [\omega, \zeta]))$ and $\eta = g_{-\tilde{\tau}_n}[\omega, \zeta]$. By definition,

$$S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega, \zeta]) = \int_0^{\tilde{\tau}_n} (A - m(A)) \circ g_t\eta \, dt.$$

By [**CLT20b**, Lemma 4.5], geodesic flows on locally CAT(−1) spaces satisfy the *weak periodic orbit closing property* (see [**CLT20b**, Definition 4.4]). Applying this to the orbit segment $g_{[0,\tilde{\tau}_n]}\eta$, there exists a closed geodesic γ with period T satisfying $|T - \tilde{\tau}_n| \leq M_1$ such that

$$\max\{d_{GX}(g_s\gamma, g_s\eta) : s \in [0, \tilde{\tau}_n]\} < \epsilon.$$

Here, M_1 depends only on ϵ. (In [**CLT20b**], only an upper bound on T was of interest, but the lower bound follows from the same proof.) By [**CLT20b**, Proposition 4.3], after choosing ϵ sufficiently small and using the fact that A is Hölder, there is some uniform M_2 such that

$$\left| \int_0^{\tilde{\tau}_n} A \circ g_t\eta \, dt - \int_0^{\tilde{\tau}_n} A \circ g_t\gamma \, dt \right| \leq M_2.$$

Therefore,

$$\left| \int_0^{\tilde{\tau}_n} A \circ g_t\eta \, dt - \int_0^{T} A \circ g_t\gamma \, dt \right| \leq M_2 + M_1\|A\|_\infty.$$

By the definition of $m(A)$, $\int_0^T A \circ g_t\gamma \, dt \geq Tm(A)$. Therefore (with $\pm$ depending on the sign of $m(A)$),

$$\begin{aligned} S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega, \zeta]) &\geq \int_0^T A \circ g_t\gamma \, dt - (M_2 + M_1\|A\|_\infty) - m(A)(T \pm M_1) \\ &\geq \mathrm{Tm}(A) - (M_2 + M_1\|A\|_\infty) - \mathrm{Tm}(A) \pm M_1 m(A) \\ &= -(M_2 + M_1\|A\|_\infty) \pm M_1 m(A), \end{aligned}$$

giving a uniform lower bound.

By part (ii) of Lemma 4.4,

$$\begin{aligned} \Delta^s(\omega, [\omega, \zeta]) = &\int_0^\infty (A \circ g_t w^s(\omega, [\omega, \eta]) - A \circ g_t\pi(\omega)) \, dt \\ &- \int_0^{b^s(\omega,[\omega,\eta])} (A - m(A)) \circ g_t[\omega, \eta] \, dt. \end{aligned}$$

Since they both belong to Σ_{ω_0}, $d_{GX}(\pi(\omega), [\omega, \eta]) < \operatorname{diam}\Sigma_{\omega_0} < \alpha^*$. As Busemann functions are 1-Lipschitz, $|b^s(\omega, [\omega, \eta])| < \alpha^*$, so the second integral is bounded by $(\|A\|_\infty - m(A))\alpha^*$. Furthermore, since $w^s(\omega, [\omega, \eta]) = g_{b^s(\omega,[\omega,\eta])}[\omega, \eta]$, the distance

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between the stably related $g_t\pi(\omega)$ and $g_t w^s(\omega, [\omega, \eta])$ is bounded by $2\alpha^*$ at $t = 0$ and goes to zero exponentially fast as $t \to \infty$. Applying [**CLT20b**] again (this time in the limiting case where the shadowing interval is half-infinite), we can uniformly bound the first integral, proving that $\tilde{\mathcal{V}}(\omega)$ is finite. □

The next proposition will allow us to complete the goal of this section.

PROPOSITION 4.8. *For any $\omega \in \Omega$, define $\tilde{\mathcal{V}}(\omega, \pi(\omega)) := \tilde{\mathcal{V}}(\omega)$, extending $\tilde{\mathcal{V}}$ to a function on* graph$(\pi) \subset D \subset \Omega \times \Sigma$.

- (i) *With the notational convention above, $\tilde{\mathcal{A}} \geq \tilde{\mathcal{V}} \circ \tilde{\psi} - \tilde{\mathcal{V}}$ on* graph(π).
- (ii) *There exists $\mathcal{V} : \Sigma \to \mathbb{R}$ such that $\tilde{\mathcal{V}}(\omega) = \mathcal{V} \circ \pi(\omega)$ for all $\omega \in \Omega$.*
- (iii) *$\mathcal{V} : \Sigma \to \mathbb{R}$ is globally Hölder.*

Before proving Proposition 4.8, we see that Proposition 4.1 then follows easily from it.

Proof of Proposition 4.1. We get the Hölder map $\mathcal{V}$ from Proposition 4.8, which satisfies $\tilde{\mathcal{V}} \circ \tilde{\theta}(\gamma) = \tilde{\mathcal{V}}(\theta^\gamma) = \mathcal{V} \circ \pi(\theta^\gamma) = \mathcal{V}(\gamma)$, so that $\tilde{\mathcal{V}} \circ \tilde{\theta} = \mathcal{V}$. Since we also have $\tilde{\mathcal{A}} \circ \tilde{\theta} = \mathcal{A}$ and $\tilde{\theta} \circ \psi = \tilde{\psi} \circ \tilde{\theta}$, we obtain

$$\begin{aligned}\mathcal{A}(\gamma) = \tilde{\mathcal{A}} \circ \tilde{\theta}(\gamma) &\geq \tilde{\mathcal{V}} \circ \tilde{\psi}(\tilde{\theta}(\gamma)) - \tilde{\mathcal{V}}(\tilde{\theta}(\gamma)) \\ &= \tilde{\mathcal{V}} \circ \tilde{\theta} \circ \psi(\gamma) - \mathcal{V}(\gamma) \\ &= \mathcal{V}(\psi(\gamma)) - \mathcal{V}(\gamma).\end{aligned}$$ □

Proof of Proposition 4.8. Fix $\omega \in \Omega$ and let $\zeta \in \Omega$ such that $\zeta_0 = \omega_0$. Let

$$\xi = (\ldots, \zeta_{-2}, \zeta_{-1} \mid \omega_0, \omega_1, \omega_2, \ldots).$$

Then, $\pi(\xi) = [\omega, \zeta]$ and since $\tilde{\mathcal{A}}(-, \pi(\zeta))$ depends only on the forward encoding of $-$,

$$\tilde{\mathcal{A}}(\xi, \pi(\xi)) = \tilde{\mathcal{A}}(\omega, \pi(\xi)).$$

Then,

$$\begin{aligned}&S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\xi, [\omega, \zeta]) + \Delta^s(\omega, [\omega, \zeta]) + \tilde{\mathcal{A}}(\omega, \pi(\omega)) \\ &\quad = S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\xi, \pi(\xi)) + [\tilde{\mathcal{A}}(\omega, [\omega, \zeta]) - \tilde{\mathcal{A}}(\omega, \pi(\omega)) \\ &\qquad + \Delta^s(\sigma(\omega), \pi \circ \sigma(\xi))] + \tilde{\mathcal{A}}(\omega, \pi(\omega)) \\ &\quad = S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\xi, \pi(\xi)) + \tilde{\mathcal{A}}(\xi, \pi(\xi)) + \Delta^s(\sigma(\omega), \pi \circ \sigma(\xi)) \\ &\quad = S_{n+1}\tilde{\mathcal{A}} \circ \tilde{\psi}^{-(n+1)}(\sigma(\xi), \pi \circ \sigma(\xi)) + \Delta^s(\sigma(\omega), \pi \circ \sigma(\xi)).\end{aligned}$$

By the Markov property, $\pi \circ \sigma(\xi) \in W^s_{\text{loc}}(\sigma(\omega))$, so we obtain

$$\tilde{\mathcal{V}}(\omega, \pi(\omega)) + \tilde{\mathcal{A}}(\omega, \pi(\omega)) \geq \tilde{\mathcal{V}} \circ \tilde{\psi}(\omega, \pi(\omega))$$

or, equivalently,

$$\tilde{\mathcal{V}}(\omega) + \tilde{\mathcal{A}}(\omega, \pi(\omega)) \geq \tilde{\mathcal{V}} \circ \sigma(\omega).$$

For part (ii), let $\omega, \omega' \in \Omega$ with $\omega_0 = \omega'_0$ and $\pi(\omega) = \pi(\omega')$. For any ζ with $\omega_0 = \zeta_0 = \omega'_0$, we have $[\omega, \zeta] = [\omega', \zeta]$ so $S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega, \zeta]) = S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega', \zeta])$.

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Also, by Lemma 4.4, $\Delta^s(\omega, [\omega, \zeta]) = \Delta^s(\omega', [\omega', \zeta])$. By definition of $\tilde{\mathcal{V}}$, we then have $\tilde{\mathcal{V}}(\omega) = \tilde{\mathcal{V}}(\omega')$. Therefore, $\tilde{\mathcal{V}}(\omega)$ depends on ω actually only via its unique point $\pi(\omega)$; so we can define a map $\mathcal{V} : \Sigma \to \mathbb{R}$ by

$$\mathcal{V}(\pi(\omega)) := \tilde{\mathcal{V}}(\omega).$$

For part (iii), let $\gamma, \gamma' \in \Sigma$. In the following, the constants K^* and $\Lambda_*^s = -2\tau^* < 0 < 2\tau^* = \Lambda_*^u$ are from Lemma 3.16, the constants $C, \lambda_*^s := -\tau_* < 0 < \tau_* =: \lambda_*^u$ are from Lemma 3.13, and the constant δ is from Lemma 3.18.

Since the problems for being Hölder show up at small scales, we assume $d_{GX}(\gamma, \gamma') < (\delta/K^*)$. Let $N = N(\gamma, \gamma')$ be the unique positive integer satisfying

$$\frac{\delta}{K^*} e^{-(N+1)(\Lambda_*^u - \lambda_*^s)} \le d_{GX}(\gamma, \gamma') \le \frac{\delta}{K^*} e^{-N(\Lambda_*^u - \lambda_*^s)}.$$

By Lemma 3.18, there exist pseudo-orbits ω, ω' such that $\pi(\omega) = \gamma$, $\pi(\omega') = \gamma'$, and their first $N + 1$ symbols coincide: $\omega_0 = \omega_0', \ldots, \omega_N = \omega_N'$.

Let $n \ge 0$, $\zeta \in \Omega$ with $\zeta_0 = \omega_0 = \omega_0'$, and let $\eta = [\omega, \zeta]$, $\eta' = [\omega', \zeta]$. Since η and η' are in the stable sets of ω and ω', respectively, we may assert that ζ is encoded so that η and η' hit exactly the same $N + 1$ sections: $\zeta_l = \omega_l (= \omega_l')$ for all $l = 0, \ldots, N$. Since taking an infimum of

$$[S_n\tilde{A} \circ \tilde{\psi}^{-n}(\zeta, \eta) + \Delta^s(\omega, \eta)] - [S_n\tilde{A} \circ \tilde{\psi}^{-n}(\zeta, \eta') + \Delta^s(\omega, \eta')] \tag{4.3}$$

over all $n \ge 0$ and all such ζ gives us $\tilde{\mathcal{V}}(\omega) - \tilde{\mathcal{V}}(\omega') = \mathcal{V}(\gamma) - \mathcal{V}(\gamma')$, we want to bound the magnitude of the expression in equation (4.3) above in terms of $d_{GX}(\gamma, \gamma')$.

We use the following notation:

$$\gamma_n = \tilde{\psi}_\omega^n(\gamma); \quad \gamma_n' = \tilde{\psi}_\omega^n(\gamma'); \quad \eta_n = \tilde{\psi}_\omega^n(\eta), \quad \eta_n' = \tilde{\psi}_\omega^n(\eta').$$

The definition of N gives us

$$N + 1 \ge \frac{1}{\Lambda_*^u - \lambda_*^s} \ln\left(\frac{\delta}{K^* d_{GX}(\gamma, \gamma')}\right),$$

which gives us

$$e^{N\lambda_*^s} \le K_0(\delta) d_{GX}(\gamma, \gamma')^\beta, \quad e^{-N\lambda_*^u} \le L_0(\delta) d_{GX}(\gamma, \gamma')^{-\beta} \tag{4.4}$$

for some constants $K_0(\delta) := e^{-\lambda_*^s}(K^*/\delta)^\beta$, $L_0(\delta) := e^{\lambda_*^u}(K^*/\delta)^{-\beta}$, and $\beta := -(\lambda_*^s/\Lambda_*^u - \lambda_*^s) \in (0, 1)$. Since $\gamma, \eta \in W^s_{loc(\omega)}$ and $\gamma', \eta' \in W^s_{loc(\omega')}$, assuming $\operatorname{diam}(\Sigma) \le 1$,

$$d_{GX}(\eta_N, \gamma_N) \le Ce^{N\lambda_*^s}, \quad d_{GX}(\eta_N', \gamma_N') \le Ce^{N\lambda_*^s}. \tag{4.5}$$

By Lemma 3.16 and equation (4.4),

$$d_{GX}(\gamma_N, \gamma_N') \le \delta e^{N\lambda_*^s}. \tag{4.6}$$

We also obtain from equations (4.5), (4.6) and (4.4) (and assuming $\delta \le C$),

$$d_{GX}(\gamma_N, \gamma_N'),\ d_{GX}(\eta_N, \eta_N'),\ d_{GX}(\gamma_N, \eta_N),\ d_{GX}(\gamma_N', \eta_N') \le M_0 d_{GX}(\gamma, \gamma')^\beta, \tag{4.7}$$

where $M_0 = 3CK_0$.

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Even though $\mathcal{A}$ is highly discontinuous, for any given $\omega \in \Omega$, the map $\tilde{\mathcal{A}}(\omega, \gamma)$ is Hölder continuous in γ, since $\tilde{\mathcal{A}}$ only considers pairs (ω, γ) such that γ can follow the orbits for which ω gives instructions. Therefore, we have a Hölder constant $\mathrm{Hold}_\alpha(\tilde{\mathcal{A}}) = \sup_{\gamma,\gamma' \in \Sigma, \omega \in \Omega}\{|\tilde{\mathcal{A}}(\omega, \gamma) - \tilde{\mathcal{A}}(\omega, \gamma')|/d(\gamma, \gamma')^\alpha\}$, where α is the exponent constant from the Hölder conditions of A and $\tilde{\tau}$.

We now split the expression in equation (4.3) into a sum of five parts:

$$(4.3)_1 = S_n\tilde{\mathcal{A}}^{-n}(\zeta, \eta) - S_n\tilde{\mathcal{A}}^{-n}(\zeta, \eta'),$$

$$(4.3)_2 = \sum_{k=0}^{N-1}[\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, \eta) - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', \eta')],$$

$$(4.3)_3 = \sum_{k=0}^{N-1}[\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', \gamma') - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, \gamma)],$$

$$(4.3)_4 = \sum_{k\geq N}[\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, \eta) - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, \gamma)],$$

$$(4.3)_5 = \sum_{k\geq N}[\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', \eta') - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', \gamma')].$$

The sum of the first two terms can be rewritten over the first $(n + N)$ backward iterates of η_N and η'_N:

$$\begin{aligned}|(4.3)_1 + (4.3)_2| &= |S_{n+N}\tilde{\mathcal{A}} \circ \tilde{\psi}^{-(n+N)}(\sigma^N(\zeta), \eta_N) - S_{n+N}\tilde{\mathcal{A}} \circ \tilde{\psi}^{-(n+N)}(\sigma^N(\zeta), \eta'_N)| \\ &= \bigg|\sum_{k=1}^{n+N}(\tilde{\mathcal{A}} \circ \tilde{\psi}^{k-n-N}(\sigma^N(\zeta), \eta_N) - \tilde{\mathcal{A}} \circ \tilde{\psi}^{k-n-N}(\sigma^N(\zeta), \eta'_N))\bigg|.\end{aligned}$$

It can then be estimated as follows:

$$\begin{aligned}|(4.3)_1 + (4.3)_2| &\leq \sum_{k=1}^{n+N}\mathrm{Hold}_\alpha(\tilde{\mathcal{A}})\, d_{GX}(\tilde{\psi}^{k-n-N}(\sigma^N(\zeta), \eta_N), \tilde{\psi}^{k-n-N}(\sigma^N(\zeta), \eta'_N)^\alpha \\ &\leq \mathrm{Hold}_\alpha(\tilde{\mathcal{A}})\sum_{k=1}^{n+N}|Ce^{-k\lambda_*^u}d_{GX}(\eta_N, \eta'_N)|^\alpha \quad \text{(by Lemma 3.13)} \\ &\leq \mathrm{Hold}_\alpha(\tilde{\mathcal{A}})\sum_{k=1}^{\infty}|Ce^{-k\lambda_*^u}|^\alpha M_0^{\alpha\beta}d_{GX}(\gamma, \gamma')^{\alpha\beta} \\ &= K_2(\delta)\, d_{GX}(\gamma, \gamma')^{\alpha\beta}.\end{aligned}$$

For the third term, we use $\omega_k = \omega'_k$ for $0 \leq k \leq N$ so that by Lemma 3.16 and equation (4.4),

$$\begin{aligned}d_{GX}(\gamma_k, \gamma'_k) &\leq K^*e^{k\Lambda_*^u}d_{GX}(\gamma, \gamma') \\ &\leq K^*e^{k\Lambda_*^u}\frac{\delta}{K^*}e^{-N(\Lambda_*^u - \lambda_*^s)} \\ &= \delta e^{e^{-(N-k)\Lambda_*^u}}e^{N\lambda_*^s}e^{N\lambda_*^s} \\ &\leq \delta e^{e^{-(N-k)\Lambda_*^u}}e^{N\lambda_*^s}K_0(\delta)d_{GX}(\gamma, \gamma')^\beta.\end{aligned}$$

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Then, we have an estimate for the third term as follows:

$$
\begin{aligned}
|(4.3)_3| &\le \sum_{k=0}^{N-1} \mathrm{Hold}_\alpha(\tilde{\mathcal{A}}) d_{GX}(\gamma_k, \gamma_k')^\alpha) \\
&\le \sum_{k=0}^{N-1} \mathrm{Hold}_\alpha(\tilde{\mathcal{A}})(\delta e^{e^{-(N-k)\Lambda_*^u}} e^{N\lambda_*^s} K_0(\delta) d_{GX}(\gamma, \gamma')^\beta)^\alpha \\
&\le \mathrm{Hold}_\alpha(\tilde{\mathcal{A}})(\delta K_0(\delta))^\alpha \sum_{l=1}^{\infty} e^{-l\Lambda_*^u} d_{GX}(\gamma, \gamma')^{\alpha\beta} \\
&= K_3(\delta) d_{GX}(\gamma, \gamma')^{\alpha\beta}.
\end{aligned}
$$

Now, since $\gamma, \eta \in W^s_{\mathrm{loc}}(\omega)$ and $\gamma', \eta' \in W^s_{\mathrm{loc}}(\omega')$, using Lemma 3.16 and equation (4.7), the fourth term can be estimated as follows:

$$
\begin{aligned}
|(4.3)_4| &\le \mathrm{Hold}_\alpha(\tilde{\mathcal{A}}) \sum_{k\ge N} d_{GX}(\eta_k, \gamma_k)^\alpha \\
&= \mathrm{Hold}_\alpha(\tilde{\mathcal{A}}) \sum_{k=0}^{\infty} d_{GX}(\eta_{N+k}, \gamma_{N+k})^\alpha \\
&\le \mathrm{Hold}_\alpha(\tilde{\mathcal{A}}) \sum_{k=0}^{\infty} (C e^{k\lambda_*^s} d_{GX}(\eta_N, \gamma_N))^\alpha \\
&\le \mathrm{Hold}_\alpha(\tilde{\mathcal{A}}) C^\alpha \sum_{k=0}^{\infty} e^{k\lambda_*^s\alpha} (M_0(\delta) d_{GX}(\gamma, \gamma')^\beta)^\alpha \\
&= K_4(\delta) d_{GX}(\gamma, \gamma')^{\alpha\beta}.
\end{aligned}
$$

Similarly, using γ', η' and ω' in place of γ, η and ω above, we obtain an estimate for the fifth term:

$$
|(4.3)_5| \le K_4(\delta) d_{GX}(\gamma, \gamma')^{\alpha\beta}.
$$

Therefore,

$$
|\mathcal{V}(\gamma) - \mathcal{V}(\gamma')| \le (K_2 + K_3 + K_4 + K_4) d_{GX}(\gamma, \gamma')^{\alpha\beta},
$$

as desired. □

5. *Proof of the main theorem*

With Proposition 4.1, we have solved a discretized version of the sub-action problem. Recall that we have a collection of sections Σ, and a globally Hölder function $\mathcal{V} : \Sigma \to \mathbb{R}$ such that

$$
\mathcal{A}(\gamma) \ge \mathcal{V} \circ \psi(\gamma) - \mathcal{V}(\gamma) \quad \text{for all } \gamma \in \Sigma,
$$

where ψ is the first-return map for Σ. To prove Theorem 1.1, we will prove the following proposition.

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PROPOSITION 5.1. *There is a collection of Poincaré subsections $\Sigma' \subset \Sigma$ (with first-return map ψ' and first-return time τ') and a function $H' : GX \to \mathbb{R}_{\geq 0}$ which is globally Hölder and smooth in the flow direction satisfying the following integrability condition:*

$$\int_0^{\tau'(\gamma)} (A - m(A)) \circ g_t(\gamma)\, dt - (\mathcal{V} \circ \psi'(\gamma) - \mathcal{V}(\gamma)) = \int_0^{\tau'(\gamma)} H' \circ g_t(\gamma)\, dt$$

for all $\gamma \in \Sigma'$.

Proposition 5.1 proves Theorem 1.1 as follows.

Proof of Theorem 1.1. Given Σ' and $\gamma \in GX$, let $T'(\gamma) = \inf\{t \geq 0 : g_{-t}\gamma \in \Sigma'\}$. Note that t is small, bounded by the scale of the Markov proper family from which Σ' is constructed (see Definition 3.3). Then, given H', define

$$V(\gamma) = \mathcal{V} \circ g_{-T'(\gamma)}(\gamma) + \int_{-T'(\gamma)}^{0} (A - m(A) - H') \circ g_t\gamma\, dt.$$

First, we note that if $\gamma_1 \in \Sigma'$ and $\gamma_1 = \psi'(\gamma_2)$, then $T'(\gamma_1) = 0$ and so, using the definition of V and the integrability condition given in Proposition 5.1, we get

$$V(\gamma_1) = \mathcal{V}(\gamma_1) = \mathcal{V}(\gamma_2) + \int_0^{\tau'(\gamma_2)} (A - m(A) - H') \circ g_t\gamma_2\, dt.$$

With this, we have the following well-defined expression for V:

$$V(\gamma) = \mathcal{V} \circ g_{-T}\gamma + \int_{-T}^{0} (A - m(A) - H') \circ g_t\gamma\, dt \tag{5.1}$$

for all γ and any T such that $g_{-T}(\gamma) \in \Sigma'$ or, indeed, in Σ. Fix any such T for a given γ. Then,

$$\begin{aligned} V(g_t\gamma) &= \mathcal{V} \circ g_{-(T+t)}g_t(\gamma) + \int_{-(T+t)}^{0} (A - m(A) - H') \circ g_s g_t(\gamma)\, ds \\ &= \mathcal{V} \circ g_{-T}(\gamma) + \int_0^{T+t} (A - m(A) - H') \circ g_s g_{-T}(\gamma)\, ds. \end{aligned}$$

Taking $d/dt|_{t=0}$ of this expression, we get

$$\frac{d}{dt}|_{t=0} V(g_t\gamma) = (A - m(A) - H')(\gamma).$$

Re-arranging terms, this gives the second statement of Theorem 1.1 with H' as the H required in that theorem. Integrating this equation over any geodesic segment yields the first statement of Theorem 1.1.

For the regularity statements, we work with equation (5.1). Here, $\mathcal{V}$ is Hölder on Σ. On an open set around any $\gamma \in \Sigma'$, we can choose T as a function of γ so that $g_{-T}\gamma$ belongs to a single section in Σ. Using the fact that for the sections we are using, constructed via the methods of [**CLT20a**], the return map under the flow to a section is Hölder [**CLT20a**, Proposition 4.9], $g_{-T}\gamma$ is Hölder in γ. Finally, A is Hölder by assumption and H' is Hölder by Proposition 5.1, so V is globally Hölder. Along orbits of the flow, V is differentiable, with the expression for the derivative given above. □

5.1. *Constructing subsections.* To prove Proposition 5.1, following the ideas of [**LT05**], we will construct a nested sequence of Poincaré sections. The following lemma is a basic tool necessary in these arguments.

LEMMA 5.2. (Cf. [**LT05**, Lemma 15]) *Given a collection of sections Σ as in Lemma 3.7, there exists a collection of subsections Σ' such that:*

(i) $\overline{\Sigma}'_i \subset \Sigma_i$;

(ii) $\{U'_i := g_{(0,\tau^*)}\Sigma'_i\}$ *cover* GX;

(iii) *for all* i, j, *if* $U_i \cap \Sigma_j \neq \emptyset$, *then* $U'_i \cap \Sigma'_j \neq \emptyset$.

Proof. Since it is a Poincaré section, $\{U_i := g_{(0,\tau^*)\Sigma_i}\}$ is a finite open cover of GX. Let ϵ be a Lebesgue covering number for $\{U_i\}$.

Using the fact that g_t is continuous, given ϵ, there exists $\delta(\epsilon) > 0$ such that whenever $d_{GX}(\gamma_1, \gamma_2) < \delta$ and $t \in [0, \tau^*]$, $d_{GX}(g_t(\gamma_1), g_t(\gamma_2)) < \epsilon$.

For each i, let $\Sigma'_i := \{\gamma \in \Sigma_i : d_{GX}(\gamma_1, (\bigcup_{t\in(-\alpha,\alpha)} g_t\Sigma_i)^c) > \delta(\epsilon)/2\}$. That is, we 'shrink' Σ_i by $\delta(\epsilon)/2$. Clearly part (i) is satisfied. For part (ii), let $\gamma_3 \in GX$. For some i, $B_\epsilon(\gamma_3) \subset U_i$ since ϵ is a Lebesgue number for the covering $\{U_i\}$. Write $\gamma_3 = g_t\gamma_1$ for $(\gamma_1, t) \in \Sigma_i \times (0, \tau^*)$. Suppose that $\gamma_1 \notin \Sigma'_i$. Then, there is some $\gamma_2 \notin \bigcup_{t\in(-\alpha,\alpha)} g_t\Sigma_i$ such that $d_{GX}(\gamma_1, \gamma_2) < \delta/2$. By the choice of δ, $d_{GX}(\gamma_3, g_t\gamma_2) < \epsilon$, implying that $B_\epsilon(\gamma_3) \nsubseteq U_i$, which is a contradiction. This proves part (ii).

For part (iii), if $U_i \cap \Sigma_j \neq \emptyset$, there exists an $\epsilon_{ij} > 0$ and $\gamma_3 \in \Sigma_j$ such that $B_{\epsilon_{ij}}(\gamma_3) \subset U_i \cap \bigcup_{t\in(-\alpha,\alpha)} g_t\Sigma_j$. By the same argument used for part (ii), if we form Σ'_i and Σ'_j by shrinking Σ_i and Σ_j by less than ϵ_{ij} and less than $\delta(\epsilon_{ij}/2)$, then $\gamma_3 \in U'_i \cap \Sigma'_j$. Therefore, if we shrink all our Σ_i by $\min\{\delta(\epsilon)/2, \delta(\epsilon_{ij})/2, \epsilon_{ij}\}$, part (iii) is satisfied along with part (ii). □

Below, let Σ' be a subsection of Σ as in Lemma 5.2. Let τ' and ψ' be the corresponding first-return time and first-return map. For $\gamma \in \Sigma'$, define

$$\mathcal{H}'(\gamma) := \int_0^{\tau'(\gamma)} (A - m(A)) \circ g_t\gamma \, dt - (\mathcal{V} \circ \psi'(\gamma) - \mathcal{V}(\gamma)).$$

By Proposition 4.1, $\mathcal{H}' \geq 0$. To prove Proposition 5.1, we need to extend $\mathcal{H}'$ to a function H' which is defined on all of GX.

5.2. *A smoothing function.* A key element in the proof of Proposition 5.1 is the 'smoothing function' h provided by Lemma 5.5, which is stated later in this section. This function will allow us to take the values of $\mathcal{H}$, currently concentrated on the section Σ and smooth them out over orbits of the geodesic flow.

We first prove the key lemma necessary for the proof of Lemma 5.5.

LEMMA 5.3. *As before, let* $U_i = g_{(-\alpha,0)}\mathrm{Int}_g(\Sigma_i) \cap (\bigcup_{j=1}^n g_{(-\delta,\delta)}(\Sigma_j))^c$. *Then, for every* U_i, *there exists a non-negative function* $h_i : GX \to \mathbb{R}_{\geq 0}$ *such that:*

(1) $U_i \subseteq \mathrm{supp}(h_i) \subseteq GX \setminus (\bigcup_{j=1}^n g_{(-\delta,\delta)}\Sigma_j)$;

(2) h_i *is Lipschitz continuous;*

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(3) *h_i is smooth in the flow direction;*
(4) *for all γ such that $B_r(\gamma) \subset U_i = W_i$, $\int_0^{\tau(\gamma)} (h_i \circ g_t)(\gamma)\, dt \geq C(r)$ for some constant $C(r)$ depending only on r.*

We now specify our input functions.

Let $\mathcal{U}$ be a cover for a metric space X. Recall that a cover $\mathcal{U}$ has *multiplicity* at most $k \geq 0$ if any $x \in X$ belongs to at most k members of $\mathcal{U}$. With these definitions in mind, we consider the following proposition.

PROPOSITION 5.4. [**DG07**, Proposition 4.1] *Let $\mathcal{U}$ be a cover of a metric space X with multiplicity at most $k+1$ (where $k \geq 0$) and Lebesgue number $L > 0$. For $U \in \mathcal{U}$, define*

$$\varphi_U(x) = \frac{d(x, X \setminus U)}{\sum\limits_{V \in \mathcal{U}} d(x, X \setminus V)}.$$

Then, $\{\varphi_U\}_{U \in \mathcal{U}}$ is a partition of unity on X subordinated to the cover $\mathcal{U}$. Moreover, each φ_U satisfies, for all $x, y \in X$,

$$|\varphi_U(x) - \varphi_U(y)| \leq \frac{2k+3}{L}\, d(x, y).$$

Furthermore, the family $(\varphi_U)_{U \in \mathcal{U}}$ satisfies, for all $x, y \in X$,

$$\sum_{U \in \mathcal{U}} |\varphi_U(x) - \varphi_U(y)| \leq \frac{(2k+2)(2k+3)}{L}\, d(x, y).$$

Proof of Lemma 5.3. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a bump function supported on the interval $(-\epsilon, \epsilon)$ with the property that $\int_{-\epsilon}^{\epsilon} \psi(\gamma)\, dx = 1$, where $\epsilon << \delta$. Let $W_i = g_{(-\alpha,0)}\mathrm{Int}B_i \cap (\bigcup_{j=1}^n g_{(-\delta-\epsilon,\delta+\epsilon)} B_j)^c$. Let us denote φ_{W_i} from Proposition 5.4 as φ_i. Given $\gamma \in GX$, we define the function,

$$h_i(\gamma) := ((\varphi_i \circ g_t)(\gamma) * \psi)(0) = \int_{\mathbb{R}} (\varphi_i \circ g_{-t})(\gamma)\psi(t)\, dt = \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma)\psi(-t)\, dt.$$

Note that the two integrals are equal by symmetry of convolution.

(1) *Support of* $\mathbf{h_i}$. First, observe that since $\mathrm{supp}(\psi) = (-\epsilon, \epsilon)$, it follows that

$$h_i(\gamma) = \int_{\mathbb{R}} (\varphi_i \circ g_{-t})(\gamma)\psi(t)\, dt = \int_{-\epsilon}^{\epsilon} (\varphi_i \circ g_t)(\gamma)\psi(-t)\, dt.$$

Let $\gamma \in U_i$. Then, there exists some union of non-empty open intervals $\mathcal{U}$ that includes an open interval around 0 such that $(\varphi_i \circ g_{-t})(\gamma) \neq 0$ for all $t \in \mathcal{U}$. As a result,

$$h_i(\gamma) = \int_{\mathcal{U} \cap (-\epsilon,\epsilon)} (\varphi_i \circ g_{-t})(\gamma)\psi(t)\, dt > 0.$$

This shows that $\gamma \in \mathrm{supp}(h_i)$, so $U_i \subseteq \mathrm{supp}(h_i)$.

If $\gamma \in \bigcup_{j=1}^n g_{(-\delta,\delta)} B_j$, then for $t \in (-\epsilon, \epsilon)$, $g_t(\gamma) \in \bigcup_{j=1}^n g_{(-\delta-\epsilon,\delta+\epsilon)} B_j$, which is not in the support of φ_i. Thus,

$$h_i(\gamma) = \int_{-\epsilon}^{\epsilon} (\varphi_i \circ g_{-t})(\gamma)\psi(t)\, dt = 0.$$

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(2) *Lipschitz continuity.* Let $\gamma_1, \gamma_2 \in GX$. Assuming that γ_1 and γ_2 are chosen so that $\operatorname{supp}(\psi) \cap \operatorname{supp}((\varphi_i \circ g_{-t})(\gamma_1) - (\varphi_i \circ g_{-t})(\gamma_2))$ is non-empty (otherwise, the inequality is trivial), then for some $T > \epsilon$,

$$\begin{aligned} |h_i(\gamma_1) - h_i(\gamma_2)| &= \left| \int_{\mathbb{R}} (\varphi_i \circ g_{-t})(\gamma_1)\psi(t) - (\varphi_i \circ g_{-t})(\gamma_2)\psi(t)\, dt \right| \\ &= \left| \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma_1)\psi(-t) - (\varphi_i \circ g_t)(\gamma_2)\psi(-t)\, dt \right| \\ &\le \int_{\mathbb{R}} |((\varphi_i \circ g_t)(\gamma_1) - (\varphi_i \circ g_t)(\gamma_2))\psi(-t)|\, dt \\ &\le \int_{\mathbb{R}} \left(\frac{2K+3}{L}\right) d_{GX}(g_t(\gamma_1), g_t(\gamma_2))|\psi(-t)|\, dt \qquad \text{(Proposition 5.4)} \\ &\le \left(\frac{2K+3}{L}\right) e^{2T} d_{GX}(\gamma_1, \gamma_2) \int_{\mathbb{R}} |\psi(-t)|\, dt \qquad \text{(Lemma 2.4)} \\ &= \left(\frac{2K+3}{L}\right) e^{2T} d_{GX}(\gamma_1, \gamma_2). \end{aligned}$$

We now explain our choice of $T > 0$. Notice that since we are integrating over a subset of $\operatorname{supp}(\psi) = (-\epsilon, \epsilon)$, it follows that any $T > \epsilon$ will suffice.

(3) *Smoothness along the flow direction.* First, we show that h_i is smooth along the flow direction. To do so, we show the infinite differentiability of the function (for *any* $s \in \mathbb{R}$, not just $s = 0$):

$$((\varphi_i \circ g_t)(\gamma) * \psi)(s) = \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma)\psi(s-t)\, dt.$$

The proof proceeds similarly to the classical case. Recall that

$$\begin{aligned} \frac{\partial}{\partial s}((\varphi_i \circ g_t)(\gamma) * \psi)(s) &= \lim_{h\to 0} \frac{((\varphi_i \circ g_t) * \psi)(\gamma, s+h) - ((\varphi_i \circ g_t) * \psi)(\gamma, s)}{h} \\ &= \lim_{h\to 0} \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma)\left(\frac{\psi(s+h-t) - \psi(s-t)}{h}\right) dt. \end{aligned}$$

The next step is to apply the dominated convergence theorem. Since ψ is smooth with compact support, all the derivatives of ψ are bounded, so we can set $M > 0$ to be some number such that $d/dt|\psi(t)| \le M$. We claim that $M(\varphi_i \circ g_t)(\gamma)$ is an appropriate dominating function. Note that it suffices to show $(\varphi_i \circ g_t)(\gamma)$ is integrable. Suppose $\gamma \notin W_i$. Then,

$$\begin{aligned} \int_{-\infty}^{\infty} |(\varphi_i \circ g_t)(\gamma)|\, dt &\le \int_{-\alpha}^{\alpha} |(\varphi_i \circ g_t)(\gamma) - \underbrace{\varphi_i(\gamma)}_{0}|\, dt \\ &\le \int_{-\alpha}^{\alpha} \frac{2K+3}{L} d_{G\widetilde{X}}(g_t(\gamma), \gamma)\, dt \qquad \text{(Proposition 5.4)} \\ &= \int_{-\alpha}^{\alpha} \frac{(2K+3)t}{L}\, dt < \infty. \end{aligned}$$

Otherwise, if $\gamma \in W_i$, then simply replace $\varphi_i(\gamma)$ with $(\varphi_i \circ g_{-2\alpha})(\gamma) = 0$, for example, so t in the last line will be replaced with $t + 2\alpha$, in which case the integral is still finite.

This allows us to finish showing the derivative exists:

$$\frac{\partial}{\partial s}((\varphi_i \circ g_t)(\gamma) * \psi)(s) = \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma)\psi'(s-t)\,dt.$$

By induction, we have that

$$\frac{\partial^n((\varphi_i \circ g_t)(\gamma) * \psi)}{\partial s}(s) = \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma)\frac{\partial^n}{\partial s}\psi(s-t)\,dt.$$

Thus, for any $s \in \mathbb{R}$, $((\varphi_i \circ g_t) * \psi)(\gamma, s)$ is smooth.

(4) *Boundedness of the integral.* It remains to bound $\int_0^{\tau(\gamma)} (h_i \circ g_t)(\gamma)\,dt$ below under the condition that $B_r(\gamma) \subset U_i = W_i$.

Recall that $\mathcal{U} = \{g_{(-\alpha,0)}B_i \setminus (\bigcup_{j=1}^n g_{(-\delta-\epsilon,\delta+\epsilon)}B_j)\}_{i=1}^n$. Note that $D_0 := \max_{\gamma\in GX} \sum_{U\in\mathcal{U}} d(\gamma, GX \setminus U)$ exists as the sum is finite and the diameter of GX is finite.

Then, we have

$$\begin{aligned}
&\int_0^{\tau(\gamma)} h_i(\gamma)\,ds \\
&= \int_0^{\tau(\gamma)} \int_{\mathbb{R}} (\varphi_i \circ g_{-t} \circ g_s)(\gamma)\psi(t)\,dt\,ds \\
&= \int_0^{\tau(\gamma)} \int_{\mathbb{R}} (\varphi_i \circ g_{s-t})(\gamma)\psi(t)\,dt\,ds = \int_0^{\tau(\gamma)} \int_{\mathbb{R}} (\varphi_i \circ g_t)(\gamma)\psi(s-t)\,dt\,ds \\
&\geq \int_0^{\tau_*} (\varphi_i \circ g_t)(\gamma) \int_{t-\epsilon}^{t+\epsilon} \psi(s-t)\,dt\,ds \geq \int_0^{\epsilon} (\varphi_i \circ g_t)(\gamma) \int_{t-\epsilon}^{t+\epsilon} \psi(s-t)\,dt\,ds \\
&\geq \int_0^{\epsilon} (\varphi_i \circ g_t)(\gamma) \int_{-\epsilon}^{\epsilon} \psi(u)\,du\,dt = \int_0^{\epsilon} (\varphi_i \circ g_t)(\gamma)\,dt \\
&\geq \int_0^{\epsilon} \frac{d_{G\gamma}(g_t(\gamma), GX \setminus W_i)}{D_0}\,dt.
\end{aligned}$$

To complete the argument, we simply need to bound $\int_0^{\epsilon} d_{GX}(g_t(\gamma), GX \setminus W_i)$ below. Since $B_r(\gamma) \in W_i$, and the geodesic flow is unit speed for d_{GX},

$$d(g_t\gamma, GX \setminus W_i) \geq r - |t| \text{ for } t \in [-r, r].$$

Consider two cases.

Case I: $r > \epsilon$.

$$\int_0^{\epsilon} d_{GX}(g_t\gamma, GX \setminus W_i)\,dt \geq \int_0^{\epsilon} r - |t|\,dt = r\epsilon - \frac{\epsilon^2}{2} > \frac{\epsilon^2}{2}.$$

Case II: $r \leq \epsilon$.

$$\int_0^{\epsilon} d_{GX}(g_t\gamma, GX \setminus W_i)\,dt \geq \int_0^{r} r - |t|\,dt = \frac{r^2}{2}.$$

Letting $C(r) = \min\{\epsilon^2/2, r^2/2\} > 0$, we have the desired result. □

We are now ready to prove Lemma 5.5.

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LEMMA 5.5. *There exists a globally Lipschitz continuous, smooth along orbits, non-negative function $h : GX \to \mathbb{R}_{\geq 0}$ that is null in a neighborhood of $\bigcup_{j=1}^{n} \Sigma_j$ such that for all $\gamma \in GX$,*

$$\int_0^{\tau(\gamma)} h \circ g_t(\gamma)\, dt \geq C$$

for some constant $C > 0$.

Proof. We prove the lemma assuming Lemma 5.3. Let $N = GX \setminus (\bigcup_{j=1}^{n} g_{(-\delta,\delta)} \Sigma_j)$, where $\delta \ll \tau_*$. From Definition 3.3, we know

$$N = \bigg(\bigcup_{i=1}^{n} g_{(-\alpha,0)} \mathrm{Int}_g(\Sigma_i) \bigg) \cap \bigg(\bigcup_{j=1}^{n} g_{(-\delta,\delta)} \Sigma_j \bigg)^c$$
$$= \bigcup_{i=1}^{n} \bigg(g_{(-\alpha,0)} \mathrm{Int}_g(\Sigma_i) \cap \bigg(\bigcup_{j=1}^{n} g_{(-\delta,\delta)} (\Sigma_j) \bigg)^c \bigg).$$

Set $U_i =: g_{(-\alpha,0)} \mathrm{Int}_g(\Sigma_i) \cap \big(\bigcup_{j=1}^{n} g_{(-\delta,\delta)} (\Sigma_j) \big)^c$. By Lemma 5.3, there exists some $h_i : GX \to \mathbb{R}_{\geq 0}$ whose support contains U_i and is contained in N that is smooth in the flow direction and Lipschitz continuous. Let $h = \sum_{i=1}^{n} h_i$. It is smooth in the flow direction, Lipschitz continuous, and null on a neighborhood of $\bigcup_{j=1}^{n} \Sigma_j$.

Note that $\mathcal{U} = \{U_i\}$ forms an open cover of the compact space GX. Let $\rho > 0$ be a Lebesgue number for this cover. Then, for every $\gamma \in GX$, there is some i such that $B_\rho(\gamma) \subset U_i$. Applying part (4) of Lemma 5.3, we find that

$$\int_0^{\tau(\gamma)} h \circ g_t(\gamma)\, dt \geq \int_0^{\tau(\gamma)} h_i \circ g_t(\gamma)\, dt \geq C(\rho) > 0.$$

Here, $C(\rho)$ depends only on ρ, and hence only on the geometry of the sections, not on γ, so it can serve as the constant C. □

5.3. *Inductive extension.* To extend the discretized sub-action, we will need to augment the concept of simple transitions.

Definition 5.6. Let $i, j \in I$. We say $i \Rightarrow j$ is a *multiple transition* if there exist $\gamma \in \Sigma_i$ and $n \geq 1$ such that $\psi^n(\gamma) \in \Sigma_j$ and $\tau_{ij}(\gamma) = \sum_{k=0}^{n-1} \tau \circ \psi^k(\gamma) < \tau^*$. The *rank* of the multiple transition $i \Rightarrow j$ is the largest $n \geq 1$ such that there exists a chain $i = i_0 \to i_1 \to \cdots \to i_n = j$ of simple transitions of length n starting at i and ending at j.

Lopes and Thieullen demonstrated (see [**LT05**, Lemma 17]) that if we can make the diameter α of the subsections $\Sigma \subset \tilde{\Sigma}$ sufficiently small—as permitted by Lemma 3.7—then the rank of any multiple transition is bounded above by $2\tau^*/\tau_*$. This allows us to sensibly define N as the maximum rank of any multiple transition for $\{\Sigma_i\}$.

To extend $\mathcal{H}$ to the function H' specified in Proposition 5.1, we follow the inductive scheme of [**LT05**]. The reason for this inductive argument is ensuring the regularity of H'. Extending $\mathcal{H}$ to a flow box based on an individual Σ_i is straightforward (see Lemma 5.7

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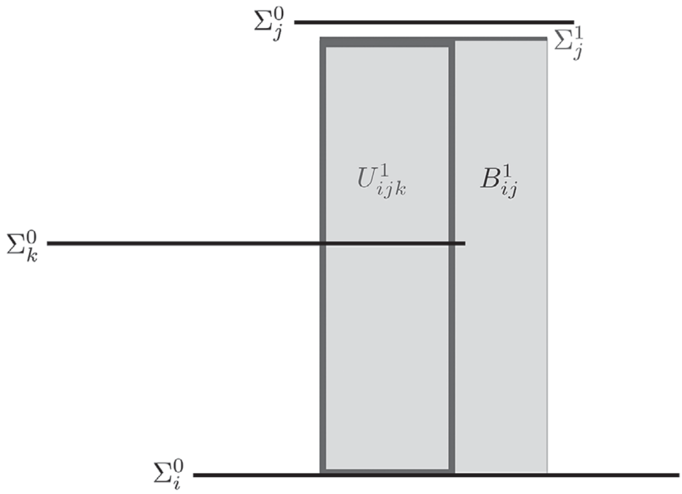

FIGURE 5. An illustration of the flow boxes U^1_{ijk} and B^1_{ij}.

below). However, when two of these flow boxes overlap, ensuring regularity of the resulting extension requires care.

We begin by defining sets used in the construction. Let $i \Rightarrow j$ be a rank n transition. Using Lemma 5.2, let $\{\Sigma_i^k\}_i$ for $k = 0, \ldots, N$ be a sequence of Poincaré sections such that for all i, $\Sigma_i^0 = \Sigma_i$ and $\Sigma_i^{k+1} \subset \bar{\Sigma}_i^{k+1} \subset \Sigma_i^k$ for $k = 0, \ldots N-1$. Let $\Sigma_{ij}^k = \{\gamma \in \Sigma_i^k : \psi_{ij}(\gamma) \in \Sigma_j^k\}$. The *flow box of rank n and size k* for $i \Rightarrow j$ is

$$B_{ij}^k = \{g_t\gamma : \gamma \in \Sigma_{ij}^k, 0 \le t \le \tau_{ij}(\gamma)\}.$$

Let

$$\Sigma_{ijk}^n = \{\gamma \in \Sigma_{ij}^n : g_t\gamma \in \Sigma_k^n \quad \text{for some } 0 \le t \le \tau_{ij}(\gamma)\}$$

and

$$U_{ijk}^n = \{g_t\gamma : \gamma \in \Sigma_{ijk}^n, 0 \le t \le \tau_{ij}(n)\}.$$

These are the points in Σ_{ij}^k (and corresponding partial flow box) which hit Σ_k^n between Σ_i and Σ_j (see Figure 5). Note that

$$U_{ijk}^n = \{g_t\gamma : \gamma \in \Sigma_{ijk}^n, 0 \le t \le \tau_{ik}(\gamma)\} \\ \cup \{g_t\gamma : \gamma \in \Sigma_{ijk}^n, \tau_{ik}(\gamma) \le t \le \tau_{ij}(\gamma)\}$$

is the union of two partial flow boxes each of rank less than n.

Let

$$\mathcal{U}^n = \bigcup_{\text{rank}(i \Rightarrow j) \le n} B_{ij}^n$$

be the union of all flow boxes of size n and rank $\le n$. Note that $\mathcal{U}^N = GX$.

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We now inductively build H^n on $\mathcal{U}^n$; H^N will be the H' asked for in Proposition 5.1. For $\gamma \in \Sigma^0_{ij}$, let

$$\mathcal{H}_{ij}(\gamma) := \int_0^{\tau^0_{ij}(\gamma)} (A - m(A)) \circ g_t\gamma \, dt - (\mathcal{V} \circ \psi^0_{ij}(\gamma) - \mathcal{V}(\gamma)).$$

Since $\mathcal{H}_{ij}$ is a sum of $\mathcal{H} \circ \psi^k$, it is non-negative. Let $i \Rightarrow j$ be a multiple transition of any rank. On the flow box B^0_{ij}, define H^0_{ij} by

$$H^0_{ij} \circ g_t(\gamma) = \mathcal{H}_{ij}(\gamma) \frac{h \circ g_t(\gamma)}{\int_0^{\tau_{ij}(\gamma)} h \circ g_t(\gamma) \, dt}$$

for all $\gamma \in \Sigma^0_{ij}$, where h is the function provided by Lemma 5.5.

LEMMA 5.7. *H^0_{ij} is Lipschitz, smooth in the flow direction, null in a neighborhood of $\bigcup_{j=1}^n \Sigma_j$, and satisfies*

$$\mathcal{H}_{ij}(\gamma) = \int_0^{\tau_{ij}(\gamma)} H^0_{ij} \circ g_t(\gamma) \, dt$$

for all $\gamma \in \Sigma^0_{ij}$.

Proof. The proof follows from the properties of h provided by Lemma 5.5, the fact that τ_{ij} is Lipschitz, and straightforward direct computation. □

The core idea of the construction of H' is in the definition of H^0_{ij}. However, $\{H^0_{ij}\}$ do not jointly define a well-defined function, as there is no reason for them to agree on the overlaps of the $\{B^0_{ij}\}$. Even if this issue is fixed, at the transitions between flow boxes, there is no reason for a function patched together from $\{H^0_{ij}\}$ to satisfy the necessary regularity conditions. The inductive construction of [**LT05**] is designed to fix these issues.

Definition 5.8. Let $H^1 : \bigcup_{\mathrm{rank}(i \to j)=1} B^1_{ij} \to \mathbb{R}$ by $H^1|_{B^1_{ij}} = H^0_{ij}$.

The only overlaps between the sets $\{B^1_{ij} : \mathrm{rank}(i \to j) = 1\}$ occur on $\bigcup_i \Sigma_i$. Since h is zero in a neighborhood of this set, H^1 is well defined. It satisfies the integrability condition and regularity conditions thanks to Lemma 5.7.

Now suppose that $H^n : \mathcal{U}^n \to \mathbb{R}$, satisfying the integrability and regularity conditions have been defined. Write H^n_{ij} for the restriction of H^n to B^n_{ij}.

To define H^{n+1}, suppose $\mathrm{rank}(i \Rightarrow j) = n + 1$. For all k such that $\Sigma^n_{ijk} \neq \emptyset$, as noted above, U^n_{ijk} is a union of two partial flow boxes, each of rank $\leq n$. Hence, H^n is already defined on these partial flow boxes. Therefore, on the partial flow box $\{g_t\gamma : \gamma \in \bigcup_k \Sigma^n_{ijk}, 0 \leq t \leq \tau_{ij}(\gamma)\}$, H^n provides a well-defined function H^n_{ij} satisfying the integrability and regularity conditions.

On $\{g_t\gamma : \gamma \in \Sigma^n_{ij} \setminus \bigcup_k \Sigma^n_{ijk}, 0 \leq t \leq \tau_{ij}(\gamma)\}$, we have only H^0_{ij}. We glue these two functions together with a partition of unity. Let $p, q : \Sigma^n_i \to [0, 1]$ be Lipschitz functions such that $p + q = 1$ on $\overline{\Sigma}^{n+1}_{ij}$ and so that

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$$\mathrm{supp}(p) \subset \bigcup_k \Sigma_{ijk}^{n+1}$$
$$\mathrm{supp}(q) \subset \Sigma_{ij}^{n} \setminus \bigcup_k \Sigma_{ijk}^{n+1}.$$

Define H_{ij}^{n+1} on B_{ij}^{n+1} by

$$H_{ij}^{n+1}(g_t\gamma) = p(\gamma)H_{ij}^{n}(g_t\gamma) + q(\gamma)H_{ij}^{0}(g_t\gamma).$$

Then, $H^{n+1} : \mathcal{U}^{n+1} \to \mathbb{R}$ is well defined by setting $H^{n+1}|_{B_{ij}^{n+1}} = H_{ij}^{n+1}$. It is straightforward to check that it satisfies the integrability condition given that H_{ij}^{n}, H_{ij}^{0} do and that $p + q = 1$ on Σ_{ij}^{n+1}. The regularity follows from the regularity of H_{ij}^{n} and H_{ij}^{0}, and the fact that p and q are Lipschitz. This finishes the proof of Proposition 5.1.

6. *Volume rigidity*

In the following section, we will show a more general version of the Volume Rigidity Corollary stated in the introduction.

6.1. *Surface amalgams.*

The following definition is adapted from [**Laf07**, Definition 2.3], which instead uses the terminology *two-dimensional P-manifolds*.

Definition 6.1. (Negatively curved surface amalgams) A compact metric space X is a *negatively curved surface amalgam* if there exists a closed subset $Y \subset X$ (the *gluing curves* of X) that satisfies the following:

(1) each connected component of Y is homeomorphic to S^1;
(2) the closure of each connected component of $X - Y$ is homeomorphic to a compact surface with boundary endowed with a negatively curved (Riemannian) metric, and the homeomorphism takes the component of $X - Y$ to the interior of a surface with boundary. We will call each $\overline{X - Y}$ a *chamber* in X;
(3) there exists a negatively curved metric on each chamber which coincides with the original metric.

If Y forms a totally geodesic subspace of X consisting of disjoint simple closed curves, we say that X is *simple*. If each connected component of Y (gluing curve) is attached to at least three distinct boundary components of chambers, then we say X is *thick*. Like Lafont in [**Laf07**], we will only be considering simple, thick, negatively curved surface amalgams, as doing so ensures the surface amalgam is locally CAT(-1) (see Figure 6).

We will equip a simple, thick negatively curved surface amalgam X with a metric in a class we denote as $\mathcal{M}_{\leq}$, following the notation from [**CL19**]. Roughly speaking, metrics in $\mathcal{M}_{\leq}$ are piecewise Riemannian metrics with an additional condition that limits pathological behavior around the gluing curves of X. More precisely, we say $g \in \mathcal{M}_{\leq}$ if g satisfies the following properties:

(1) each chamber of $C \subset X$ is equipped with a negatively curved Riemannian metric with sectional curvature bounded above by -1 so that C has geodesic boundary components;

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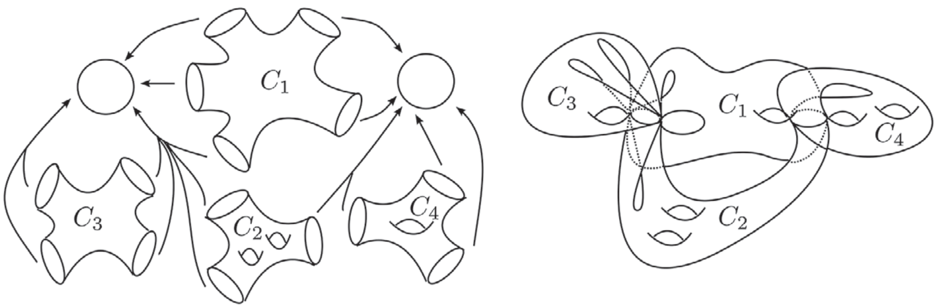

FIGURE 6. An example of a simple, thick surface amalgam with four chambers.

(2) the restrictions of g to the chambers of X are 'compatible' in the sense that if two boundary components b_1 and b_2 of two (possibly the same) chambers C_1 and C_2 are both attached to a gluing curve $\gamma \subset X$, then the gluing maps $b_1 \hookrightarrow \gamma$ and $b_2 \hookrightarrow \gamma$ are isometries (in particular, we do not allow circle maps of degree two;

(3) for any two boundary components $b_1 \in C_1$ and $b_2 \in C_2$, the restriction of g to $N_{b_1} \bigcup_{b_1 \sim b_2} N_{b_2}$ is a negatively curved smooth Riemannian metric with sectional curvature bounded above by -1, where N_{b_1} and N_{b_2} are ϵ-neighborhoods around b_1 and b_2, respectively, for some $\epsilon > 0$.

We impose the third condition to ensure that we can exploit previous marked length spectrum rigidity results for surfaces which, in particular, require Riemannian negatively curved metrics with at most a finite number of cone singularities. We now discuss some properties of $\mathcal{M}_{\leq}$ that will be useful in the proof of Theorem 6.6.

Remark 6.2. If (X, g) is a negatively curved surface amalgam where $g \in \mathcal{M}_{\leq}$, then (X, g) is locally CAT(-1).

Indeed, suppose X is equipped with a metric $g \in \mathcal{M}_{\leq}$ and $C \subset X$ is a chamber in X. Recall a generalization of the Cartan–Hadamard theorem which states that a smooth Riemannian manifold M has sectional curvature $\leq \kappa$ if and only if M is locally CAT(κ) (see [**BH99**, Theorem 1A.6]). As a result, the restriction of g to C is locally CAT(-1) since C is endowed with a negatively curved metric with sectional curvature bounded above by -1. If $\kappa \in \mathbb{R}$, and X_1 and X_2 are locally CAT(κ) spaces glued isometrically along a convex, complete metric subspace $A \subset X_1 \cap X_2$, then $X_1 \sqcup_A X_2$ is locally CAT(κ) (see [**BH99**, Theorem 2.11.1]). As a result, a negatively curved surface amalgam (X, g) with locally CAT(-1) chambers will also be locally CAT(-1), as claimed.

6.2. *Volume rigidity.* Let $\mathcal{G}\widetilde{X}$ be the set of unparameterized, unoriented geodesics in $\widetilde{X}$. A *geodesic current* on (X, g) is a π_1-invariant Radon measure on $\mathcal{G}\widetilde{X}$.

There are two especially important examples of geodesic currents. We will assume, for simplicity, that (X, g) is locally CAT(-1). Furthermore, we assume there is a well-defined notion of *transversality* in $\mathcal{G}\widetilde{X}$. If (X, g) is a negatively curved surface, there is a natural such notion since $\partial^\infty \widetilde{X}$ is a circle. For other settings, transversality must be defined with more care; for examples, see [**CL19**, §6] and [**Wu23**, §2.4.1].

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First, given a homotopy class $[\alpha] \in \pi_1(X)$ with geodesic representative α, there is a *counting current* associated with α that assigns to each Borel set $E \subset \mathscr{G}\widetilde{X}$ the measure

$$\mu_\alpha(E) = |E \cap \{\widetilde{\alpha}\}|,$$

where $\{\widetilde{\alpha}\}$ denotes the set of lifts of α to $\widetilde{X}$. We follow convention and, with abuse of notation, write $\alpha := \mu_\alpha$, which makes clear the fact that the collection of closed geodesics of (X, g) embeds naturally into the space of geodesic currents.

Second, there is a *Liouville current* which, roughly speaking, captures lengths of closed geodesics. There is no general definition of a Liouville current; rather, the Liouville current should capture important properties, which we state in Assumption 6.3. Before stating the assumption, we need to introduce the notion of *intersection numbers*.

In the case of negatively curved Riemannian surfaces, the geometric intersection number of two closed geodesics (viewed as geodesic currents) extends to a symmetric, bilinear form (see [**Bon86**, Proposition 4.5]), the *intersection number* of two geodesic currents. Explicitly, given two geodesic currents μ, ν, one can define the intersection number of μ and ν as

$$i(\mu, \nu) = \int_{DGX} d\mu \times d\nu,$$

where DGX is the set of all pairs of transversally intersecting geodesics in GX.

A straightforward computation shows that for two counting currents $[\alpha]$ and $[\beta]$, $i(\alpha, \beta)$ is exactly their geometric intersection number. Metric-independent generalizations of intersection numbers can be found for compact quotients of certain Fuchsian buildings (see [**CL19**]) and surface amalgams (see [**Wu23**]).

We now state a few requirements for the Liouville current necessary for the proof of Theorem 6.6.

Assumption 6.3. Let (X, g) be a locally CAT(-1) metric space with a well-defined notion of transversality, a well-defined intersection number function, as well as a *Liouville current* Λ_g, e.g. a geodesic current that satisfies the following two properties:
(1) $i(\alpha, \Lambda_g) = C_\alpha \ell_g(\alpha) = C_\alpha \mathscr{L}_g([\alpha])$;
(2) $i(\Lambda_g, \Lambda_g) = K\pi \operatorname{Vol}_g(X)$,
where C_α and K are two positive real constants.

Assumption 6.3 holds in the case of negatively curved Riemannian surfaces (see [**Ota90**]), and for negatively curved surfaces with large angle cone points as in [**HP97**]. In this case (and with Otal's choice of scale factor), $C_\alpha = 1$ for all α and $K = \frac{1}{2}$. A Liouville measure, a flow-invariant Radon measure on *oriented* unit-speed geodesics, is defined for non-positively curved orbihedra in [**BB95**], which is used in [**CL19**, **Wu23**] for quotients of Fuchsian buildings and surface amalgams. Both sets of authors choose to scale the Liouville current, viewed as a measure on *unoriented* unit-speed geodesics, to match the scale factor of the Liouville measure defined in [**BB95**]. However, if one scales the Liouville measure from [**BB95**] with a factor of $\frac{1}{2}$ to match the scale factor in [**Ota90**], one can see that for the cases of surface amalgams and quotients of Fuchsian buildings, $C_\alpha = 2$ and $K = 1$, given that α is *not* a branching geodesic or gluing curve. (We remark that the original factor $K = 4$ from [CL19] should be $K = 2$ and has been calculated without the scale factor of $\frac{1}{2}$). So Assumption 6.3 is satisfied.

We also require the intersection pairing to satisfy a specific form of continuity.

Assumption 6.4. (Weak continuity of the intersection number function) Given g and g', two metrics on X, and a sequence of currents $(\mu_k)_{k\in\mathbb{N}}$ on X such that $\mu_k \to \Lambda_g$ with respect to the weak-$*$ topology, $i(\mu_k, \Lambda_{g'}) \to i(\Lambda_g, \Lambda_{g'})$.

In the case of closed non-positively curved surfaces (with or without cone points), the intersection number given in [Bon86] is continuous everywhere and thus automatically satisfies Assumption 6.4. In the cases of compact quotients of certain Fuchsian buildings and surface amalgams, the intersection number functions are *not* continuous everywhere (see [CL19, Lemma 10.2]). However, by [CL19, Proposition 10.1], they satisfy Assumption 6.4; the proof follows verbatim in the case of surface amalgams.

Finally, to prove Theorem 6.6, we need one final assumption about the Liouville current.

Assumption 6.5. (X, g) is equipped with a Liouville current that is realized as the weak-$*$ limit of scalar multiples of counting currents $a_n\alpha_n$ with $C_{\alpha_n} = 2K$.

In the cases of negatively curved closed surfaces (with or without cone points), quotients of Fuchsian buildings from [CL19], and surface amalgams from [Wu23], the following property is satisfied: scalar multiples of counting currents are dense in the space of all geodesic currents equipped with the weak-$*$ topology. One can see this in a number of different ways.

Bonahon proves this statement in the setting of Gromov hyperbolic spaces in [Bon91, Theorem 7]. While he works with Cayley graphs of Gromov hyperbolic groups, his argument mainly relies on the existence of a free, cocompact, properly discontinuous, isometric action on the space as well as properties of quasigeodesics in CAT(−1) and Gromov hyperbolic spaces.

Alternately, as noted in [Bon88, Proposition 2], this statement follows from [Sig74, Theorem 1] or [Sig72, Theorem 1], which only use the weak specification property of the geodesic flow. Weak specification holds for geodesic flow on any compact, locally CAT(−1) space [CLT20b, Theorem A]. Sigmund's density proof uses weak-specification to approximate the Liouville measure with measures supported on closed geodesics. In Fuchsian buildings and surface amalgams, the only α for which $C_\alpha \neq 2K$ are the branching geodesics or gluing curves. These are proper, closed subsets in GX and it is easy to see from the definition of the Liouville measure developed by [BB95, CL19, Wu23] that such geodesics have zero Liouville measure. Therefore, any such geodesics can be omitted from the approximating sequence $(a_n\alpha_n) \to \Lambda_g$ obtained via Sigmund's argument. Therefore, the examples we are interested in satisfy Assumption 6.5.

Recall that a class of CAT(−1) metrics $\mathcal{M}$ is *marked length spectrum rigid* if whenever (X, g_1) and (X, g_2) (where $g_1, g_2 \in \mathcal{M}$) have the same marked length spectrum, (X, g_1) and (X, g_2) are isometric. There are some well-documented classes of marked length spectrum rigid metrics. Negatively curved Riemannian metrics on surfaces are known to be marked length spectrum rigid due to [Cro90, Ota90]. Due to [CL19], certain classes of

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piecewise negatively curved Riemannian metrics, including piecewise hyperbolic metrics, on compact quotients of Fuchsian buildings are marked length spectrum rigid. Piecewise negatively curved Riemannian metrics satisfying certain smoothness conditions on simple, thick surface amalgams are also marked length spectrum rigid due to [**Wu23**]. Marked length spectrum rigidity is a key tool in proving volume rigidity; thus, we will state it as an additional assumption.

Recall also from §1 that given a metric g on a metric space (X, g), $\mathscr{L}_g$ denotes the marked length length spectrum of (X, g). We are now ready to state and prove our volume rigidity result, which is a generalization of the Volume Rigidity Corollary stated in the introduction.

THEOREM 6.6. (Volume rigidity) *Let (X, g_0) and (X, g_1) be two locally CAT(-1) spaces satisfying Assumptions 6.3, 6.4, and 6.5. Furthermore, suppose g_0 and g_1 belong to a class of marked length spectrum rigid metrics. Let $\mathscr{L}_{g_0} \leq \mathscr{L}_{g_1}$. Then, $\mathrm{Vol}_{g_0}(X) \leq \mathrm{Vol}_{g_1}(X)$. Furthermore, if $\mathrm{Vol}_{g_0} = \mathrm{Vol}_{g_1}$, then (X, g_0) and (X, g_1) are isometric.*

Proof. There exists a $\pi_1(X)$-equivariant homeomorphism $\Phi : (G\tilde{X}, \tilde{g}_0) \to (G\tilde{X}, \tilde{g}_1)$. Let $\Phi^*\Lambda_{g_1}$ be the pullback of Λ_{g_1} under Φ so that $\Phi^*\Lambda_{g_1}$ is a geodesic current of (X, g_0). (Technically, we are using the homeomorphism induced by Φ on unoriented, unparameterized geodesics.) By Assumption 6.3(1) and the hypothesis, for any $[\alpha] \in \pi_1(X)$,

$$\begin{aligned} i(\alpha, \Lambda_{g_0}) = C_\alpha \mathscr{L}_{g_0}([\alpha]) &\leq C_\alpha \mathscr{L}_{g_1}([\alpha]) \\ &= i(\alpha, \Lambda_{g_1}) = i(\Phi^*\alpha, \Phi^*\Lambda_{g_1}) = i(\alpha, \Phi^*\Lambda_{g_1}). \end{aligned}$$

Using the fact that the Liouville current is realized as a limit of counting currents (Assumption 6.5) and the continuity of $i(-,-)$ at the Liouville current (Assumption 6.4),

$$i(\Lambda_{g_0}, \Lambda_{g_0}) \leq i(\Lambda_{g_0}, \Phi^*\Lambda_{g_1}).$$

Finally, using the symmetry of the intersection form, another application of Assumption 6.5, and Assumption 6.3(2),

$$K\pi \mathrm{Vol}_{g_0} = i(\Lambda_{g_0}, \Lambda_{g_0}) \leq i(\Lambda_{g_0}, \Phi^*\Lambda_{g_1}) \leq i(\Phi^*\Lambda_{g_1}, \Phi^*\Lambda_{g_1}) = K\pi \mathrm{Vol}_{g_1}$$

and we get the desired volume inequality. Note that the condition on C and K in Assumption 6.5 is not necessary for this part of the proof.

Now, suppose there is a volume equality: $\mathrm{Vol}_{g_0} = \mathrm{Vol}_{g_1}$. Then, in particular, we have the following equality:

$$i(\Lambda_{g_0}, \Phi^*\Lambda_{g_1}) = K\pi \mathrm{Vol}_{g_0}. \tag{6.1}$$

Let g_t^0 and g_t^1 be the geodesic flows on (GX, g_0) and (GX, g_1), respectively. Let $F : (GX, g_0) \to (GX, g_1)$ be an orbit equivalence of these flows. Then, there exists a reparameterization map $T : (GX, g_0) \times \mathbb{R} \to \mathbb{R}$ such that we have

$$F(g_t^0(\gamma)) = g^1_{T(\gamma,t)}(F(\gamma)).$$

By the general stability theory (see [**KH95**, Ch. 19]), the orbit equivalence F (and consequently T) can be chosen to be Hölder continuous. The key requirements for this

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are the existence of stable and unstable foliations (Theorem 2.10) for the flow, and the Lipschitz continuity of the geodesic flows (Lemma 2.4).

Since $T(\gamma, s+1) = T(\gamma, s) + T(g_s^0(\gamma), 1)$ for any $[\alpha] \in \pi_1(S)$ if we denote $L := \mathscr{L}_{g_0}([\alpha])$, then

$$\begin{aligned}\int_0^L T(g_s^0(\alpha), 1)\, ds &= \int_0^L T(\alpha, s+1) - T(\alpha, s)\, ds \\ &= \int_0^1 T(\alpha, L+s) - T(\alpha, s)\, ds \\ &= T(g_L^0(\alpha), L) \\ &= T(\alpha, L) = \mathscr{L}_{g_1}([\alpha]) \\ &= \frac{1}{C_\alpha} i(\alpha, \Phi^* \Lambda_{g_1}).\end{aligned}$$

Therefore, for any $[\alpha] \in \pi_1(X)$,

$$i(\alpha, \Phi^* \Lambda_{g_1}) = C_\alpha \int_0^{\mathscr{L}_{g_0}([\alpha])} T(g_s^0(\alpha), 1)\, ds. \tag{6.2}$$

We now want to interpret the right-hand side of the above equation as an integral over GX with respect to a geodesic-flow invariant measure. This requires some care.

As before, let $\partial_\infty^{(2)} \tilde{X} = (\partial^\infty \tilde{X} \times \partial^\infty \tilde{X}) \setminus \Delta$ be the set of distinct, ordered pairs of points in $\partial^\infty \tilde{X}$. This set specifies the *oriented*, unparameterized geodesics in $\tilde{X}$ and there is a natural 2-to-1 map $\Psi : \partial_\infty^{(2)} \tilde{X} \to \mathscr{G}\tilde{X}$ sending $(\xi, \eta) \mapsto \{\xi, \eta\}$ mapping to the *unoriented*, unparameterized geodesics $\mathscr{G}\tilde{X}$. For any measure μ on $\partial_\infty^{(2)} \tilde{X}$, the push-forward $\Psi_* \mu$ is a measure on $\mathscr{G}\tilde{X}$ defined by $\Psi_*\mu(A) = \mu(\Psi^{-1} A)$. If μ is Γ-invariant, so is $\Psi_*\mu$, and hence it is a current.

Let $\tilde{\Lambda}_g$ be the measure on $\partial_\infty^{(2)} \tilde{X}$ defined in local coordinates along a geodesic segment exactly as the Liouville current is. For instance, on a surface, its expression would be $\frac{1}{2}|\sin\theta|\, d\theta\, dx$, where $\theta \in [0, 2\pi]$, with a corresponding expressions for a Fuchsian building or surface amalgam as described in [**Wu23**] or [**CL19**] (up to the choice of scale factor $\frac{1}{2}$). Note that since this measure is defined on the space of *oriented* geodesics, the angular coordinate must run over $[0, 2\pi]$ instead of its domain for $\mathscr{G}\tilde{X}$, $[0, \pi]$.

Since Ψ is 2-to-1,

$$\Psi_* \tilde{\Lambda}_g = \sin\theta\, d\theta\, dx = 2\Lambda_g,$$

(where now $\theta \in [0, \pi]$). If $\alpha \in \pi_1(X)$, then (again abusing notation a bit) let α be the measure on $\partial_\infty^{(2)} \tilde{X}$ given by the counting measure on *oriented* lifts of the geodesic representative of α, with the orientation induced by α—namely oriented from the repelling to the attracting endpoint of (the correct conjugate of) α for its action on $\partial^\infty \tilde{X}$. Then, $\Psi_*\alpha = \alpha$, where, on the right-hand side of this expression, α is the counting measure on *unoriented* lifts of α. This equation holds because each unoriented lift of α has exactly one pre-image under Ψ in the support of α as a measure on $\partial_\infty^{(2)} \tilde{X}$, namely the same lift oriented according to the action of (the correct conjugate of) α.

Under Assumption 6.5, we can take a sequence $a_n\alpha_n \to \tilde{\Lambda}_{g_0}$ in the weak-$*$ topology for measures on $\partial^{(2)}_\infty \tilde{X}$ for $a_n \in \mathbb{R}^+$, $\alpha_n \in \pi_1(X)$, and $C_{\alpha_n} = 2K$. Using the notes of the previous paragraph,

$$\Psi_*(a_n\alpha_n) = a_n\alpha_n \to \Psi_*\tilde{\Lambda}_{g_0} = 2\Lambda_{g_0}$$

in the weak-$*$ topology for geodesic currents.

Consider the measure $\tilde{\Lambda}_{g_0} \times dt$ on $\partial^{(2)}_\infty \tilde{X} \times \mathbb{R} \cong G\tilde{X}$. In [Bon91, (A3)], Bonahon examines this measure in its local coordinates and shows that it is $\frac{1}{2}\lambda_g$, where λ_g is the Liouville measure. Returning to equation (6.2), using continuity of the intersection pairing at a Liouville current, the fact that when $a_n\alpha_n \to \tilde{\Lambda}_{g_0}$, $a_n\alpha_n \to 2\Lambda_{g_0}$, and the continuity of T, we get

$$\begin{aligned} 2i(\Lambda_{g_0}, \Phi^*\Lambda_{g_1}) &= \lim_{n\to\infty} C_{\alpha_n} a_n \int_0^{\mathscr{L}_{g_0}([\alpha_n])} T(g_s^0(\alpha_n), 1)\, ds \\ &= 2K \lim_{n\to\infty} \int_{GX \cong (\partial^{(2)}\tilde{X}\times\mathbb{R})/\Gamma} T(g_t^0(\gamma), 1) d(a_n\alpha_n)\, dt \\ &= 2K \int_{GX} T(\gamma, 1)\, d\tilde{\Lambda}_{g_0}\, dt \\ &= \frac{2K}{2} \int_{GX} T(\gamma, 1)\, d\lambda_{g_0}. \end{aligned}$$

Therefore,

$$i(\Lambda_{g_0}, \Phi^*\Lambda_{g_1}) = \frac{K}{2} \int_{GX} T(\gamma, 1)\, d\lambda_{g_0}. \tag{6.3}$$

Using equations (6.3) and (6.1),

$$\begin{aligned} \int_{GX} (T(\gamma, 1) - 1)\, d\lambda_{g_0} &= \frac{2}{K} i(\Lambda_{g_0}, \Phi^*\Lambda_{g_1}) - 2\pi \mathrm{Vol}_{g_0} \\ &= \frac{2}{K} K\pi \mathrm{Vol}_{g_0} - 2\pi \mathrm{Vol}_{g_0} = 0. \end{aligned} \tag{6.4}$$

We now use the Main Theorem with $A : GX \to \mathbb{R}$ defined as $A(\gamma) := T(\gamma, 1) - 1$. We first claim that the integral of A along any closed geodesic α is zero. By the Main Theorem, we have a sub-action $V : GX \to \mathbb{R}$ that is smooth in the flow direction such that

$$A(\gamma) = m + \left(\frac{d}{dt}\right)\bigg|_{t=0} V(g_t^0(\gamma)) + H(\gamma).$$

Note that H is a non-negative function. Moreover, by the marked length spectrum inequality assumption, for any closed geodesic α, $\int_0^{\mathscr{L}_{g_0}([\alpha])} T(g_s^0(\alpha), 1) - 1\, ds \geq 0$, so using Sigmund's theorem [Sig72, Theorem 1], we get that the minimal average m of A is also non-negative; indeed by equation (6.4), it is 0. Therefore, using the Lie derivative notation $L_X V(\gamma) := (d/dt)|_{t=0} V(g_t^0(\gamma))$, we have $A \geq L_X V$. Together with equation (6.4), this gives

$$0 \leq \int_{GX} \big(A(\gamma) - L_X V(\gamma)\big)\, d\lambda_{g_0} = \int_{GX} A(\gamma)\, d\lambda_{g_0} = 0,$$

and hence that $A(\gamma) = L_X V(\gamma)$. For any closed geodesic α,

$$\int_0^{\ell_{g_0}(\alpha)} T(g_t^0(\alpha), 1) - 1\, dt = \int_0^{\ell_{g_0}(\alpha)} A(g_t^0(\alpha))\, dt = \int_0^{\ell_{g_0}(\alpha)} L_X V(g_t^0(\alpha))\, dt = 0,$$

as claimed. Now, for any free homotopy class $[\alpha] \in \pi_1(X)$,

$$\mathscr{L}_{g_0}([\alpha]) = \int_0^{\ell_{g_0}(\alpha)} 1\, dt = \int_0^{\ell_{g_0}(\alpha)} T(g_t^0(\alpha), 1)\, dt = \mathscr{L}_{g_1}([\alpha]).$$

Finally, by our assumption, marked length spectrum rigidity for the class of metrics containing (X, g_i), g_0 and g_1 are isometric. □

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