

Boundary-value problems for weakly nonlinear ordinary differential equations

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In this paper, we announce results on the Dirichlet and periodic boundary-value problems for the equation $-x''(t) = g(x(t)) - f(t)$ on $[0, \pi]$. We consider degenerate cases not covered by the author's earlier work.

Suppose that $g : R \rightarrow R$ is continuous and $f \in L^1[0, \pi]$. In this paper, we announce results on the existence of solutions of

$$(1) \quad -x''(t) = g(x(t)) - f(t)$$

on $[0, \pi]$ which satisfy *either*

$$x(0) = x(\pi) = 0 \quad (\text{Dirichlet boundary conditions})$$

or

$$x(0) = x(\pi), \quad x'(0) = x'(\pi) \quad (\text{periodic boundary conditions}).$$

We denote these two boundary-value problems by (1_D) and (1_P) . *We only consider cases which the results in [1] and [2] do not cover.* Our results, when combined with those in [1] and [2], enable us to decide whether, for nearly all well-behaved functions g , (1_D) (or (1_P)) is solvable for every f in $L^1[0, \pi]$. (Our results suggest that some of the degenerate cases studied here are more complicated than those in [2].)

The study of the degenerate cases considered in this paper is of

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additional interest because it offers hope of progress for the corresponding problems for elliptic partial differential equations. These problems are much more difficult and most of the methods in this paper do not extend to these cases.

Throughout the paper, we assume that the limits $\mu = \lim_{y \rightarrow \infty} y^{-1}g(y)$ and $\nu = \lim_{y \rightarrow -\infty} y^{-1}g(y)$ exist (possibly $\pm\infty$). Define

$$X_D = \{u \in C^1[0, \pi] : u(0) = u(\pi) = 0, u'' \in L^1[0, \pi]\}$$

and define $H_D : X_D \rightarrow L^1[0, \pi]$ by $(H_D x)(t) = -x''(t) + g(x(t))$. Note that X_D is a Banach space under the norm $\|x\| + \|x''\|$, where $\|x\|$ denotes the L^1 norm of x . Let R_D denote the range of H_D , that is, $R_D = \{f \in L^1[0, \pi] : (1_D) \text{ is solvable}\}$. X_P, H_P , and R_P are defined analogously (for periodic boundary conditions).

1. The case where μ and ν are finite

We assume that the limits $I^\pm = \lim_{y \rightarrow \pm\infty} (g(y) - \mu y^+ + \nu y^-)$ exist, where $y^+ = \max\{y, 0\}$, $y^- = \max\{-y, 0\}$, and the limits are allowed to be infinite.

Let ϕ_β denote the solution of $-x''(t) = \mu(x(t))^+ - \nu(x(t))^-$ for which $\phi_\beta(0) = 0$ and $\phi'_\beta(0) = \beta$.

We first consider the Dirichlet problem. The results in [2] cover all cases except where $\mu \neq \nu$ and there exist positive integers k and l such that $|k-l| \leq 1$ and $k\mu^{-\frac{1}{2}} + l\nu^{-\frac{1}{2}} = 1$.

THEOREM 1. *Suppose that $k = l$.*

(i) *If $I^+ \neq I^-$ and at least one of I^+ and I^- is infinite, H_D is proper and onto.*

(ii) *If I^+ and I^- are both finite, let*

$$T = \left\{ f \in L^1[0, \pi] : \int_0^\pi f_{+1}(t)dt \int_0^\pi f_{-1}(t)dt > 0 \right\},$$

where $f_{\pm 1}(t) = f(t)\phi_{\pm 1}(t) - I^+(\phi_{\pm 1}(t))^+ + I^-(\phi_{\pm 1}(t))^-$. Then T is non-empty and $T \subseteq R_D$. Moreover, R_D contains a relatively closed unbounded proper subset of $L^1[0, \pi] \setminus \bar{T}$.

THEOREM 2. Suppose that $k - l = 1$ and $\mu > \nu$.

(i) If $I^+ \neq I^-$ and either $I^+ = -\infty$ or $I^- = \infty$, then H_D is proper and onto.

(ii) If $I^+ \neq I^-$ and either $I^+ = \infty$ or $I^- = -\infty$, then H_D is proper, R_D is closed, and $R_D \neq L^1[0, \pi]$.

(iii) If I^- and I^+ are both finite, let

$$\bar{T} = \left\{ f \in L^1[0, \pi] : \int_0^\pi f_{+1}(t)dt > 0 \right\}.$$

Then T is non-empty and $T \subseteq R_D$. Moreover, R_D contains a relatively closed unbounded proper subset of $L^1[0, \pi] \setminus \bar{T}$.

Natural analogues of Theorem 2 hold for the other cases in which $|k-l| = 1$.

The main method used in the proof of these two theorems is a shooting argument. The following lemma is basic to this argument. For simplicity, we only state the lemma in the case where I^+ is finite.

LEMMA 1. Suppose that $0 \leq t_0 \leq t_1 \leq \pi$, u is a non-negative solution of (1) on $[t_0, t_1]$, and $u(t_0) = u(t_1) = 0$. There exists a $K > 0$ and $\varepsilon : (0, \infty) \rightarrow (0, \infty)$ with $\varepsilon(r) \rightarrow 0$ as $r \rightarrow \infty$ such that, if $u(t_0) = \alpha > K$, then

$$\left| t_1 - t_0 - \frac{\pi}{\sqrt{\mu}} + \alpha^{-1} \int_{t_0}^{t_1} (I^+ - f(s))\phi_{+1}(s-t_0)ds \right| \leq \alpha^{-1}\varepsilon(\alpha).$$

Moreover, for fixed g , ε and K depend only on $\|f\|$.

To prove that $L^1[0, \pi] \setminus \bar{T} \not\subseteq R_D$ in Theorem 1 (ii), we first use a similar argument to that in the proof of Proposition 1 (i) in [2] to prove the result when g is positive homogeneous. We then use a similar argument to that in the proof of Theorem 2 in [2] to deduce the general case. In the positive homogeneous case, we prove that $R_D \setminus \bar{T}$ is non-empty by using the inequalities obtained by taking the scalar product of (1) with ϕ_{+1} (and ϕ_{-1}) to show that $H_D(u) \notin \bar{T}$ if u is a suitable function with small support. As before, it can then be deduced that $R_D \setminus \bar{T}$ is non-empty in the general case. Similar arguments are used in the proof of Theorem 2 (iii).

We now consider the periodic boundary-value problem. The only case not covered by the results in [2] is where $\mu \neq \nu$ and there is a positive integer k such that $k\mu^{-\frac{1}{2}} + k\nu^{-\frac{1}{2}} = 1$.

THEOREM 3. (i) *If at least one of I^+ and I^- is infinite and $I^- \neq I^+$ then H_p is proper and onto.*

(ii) *If both I^- and I^+ are finite, let*

$$F(\theta) = \int_0^\pi f(t)\phi_{+1}(t+\theta)dt - 2k(\mu^{-1}I^+ - \nu^{-1}I^-),$$

$$T = \{f \in L^1[0, \pi] : F(\theta) \neq 0 \text{ for every } \theta \text{ in } [0, \pi]\},$$

and

$$S = \{f \in L^1[0, \pi] : (F(\theta))^2 + (F'(\theta))^2 > 0 \text{ for every } \theta \text{ in } [0, \pi]\}.$$

Then T is non-empty, $T \subseteq R_p$, and $(R_p \cap S) \setminus T$ is a relatively closed non-empty proper subset of $S \setminus T$.

The proof of this uses similar ideas to those in the proof of Theorem 1. However, to prove $f \in R_p$, we firstly use the results in [2] to solve the equation when g is replaced by g_n (where $g_n(y) = g(y) \pm n^{-1}y^-$). We next use a variant of Lemma 1 to obtain a uniform bound for these solutions and then pass to the limit. (It seems possible that this type of argument can be used in more general situations.)

The methods of this section can be used to give an alternative proof of Theorem 8 in [2] for equation (1_D) and its natural analogue for (1_p) . Thus we have an alternative proof of the main result in Lazer and Leach [5].

2. The case where exactly one of μ and ν is finite

In this section we consider the case where $\mu = \infty$ and $\nu = i^2$ where i is a positive integer for (1_D) and a non-negative integer for (1_p) . The case where $\mu = i^2$ and $\nu = \infty$ is similar. The remaining cases are covered by the results of [2]. We assume that $I^- = \lim_{y \rightarrow -\infty} g_i(y)$ exists

(where $g_i(y) = g(y) - i^2 y$) and that there exists an $M > 0$ such that $g(y) + M(y-x) + M \geq g(x)$ if $y \geq x \geq M$. If $I^- = -\infty$, we also assume that for every $\varepsilon > 0$ there exist $N_1, N_2 > 0$ such that $g_i(y) \leq (1-\varepsilon)g_i(x)$ if $y \leq x \leq -N_1$ and $g_i(y) \geq (1+\varepsilon)g_i(x)$ if $N_2 x \leq y \leq x \leq -N_1$.

We first consider the Dirichlet problem. If $I^- = -\infty$ and $i > 0$, define

$$P^l = \liminf_{\alpha \rightarrow \infty} T(\alpha)(S(\alpha))^{-1} \quad \text{and} \quad P^u = \limsup_{\alpha \rightarrow \infty} T(\alpha)(S(\alpha))^{-1},$$

where

$$S(\alpha) = -(i\alpha)^{-1} \int_0^{i^{-1}\pi} g_i(-\alpha i^{-1} \sin it) \sin it dt,$$

and $T(\alpha)$ is the first positive zero of the solution of $-x''(t) = g(x(t))$, $x(0) = 0$, $x'(0) = \alpha$.

THEOREM 4. (i) If $i = 1$ and I^- is finite, then

$$\left\{ f \in L^1[0, \pi] : \int_0^\pi f(t) \sin t dt > 2I^- \right\} \subseteq R_D.$$

Moreover, if $g(y) - y > I^-$ for all y , equality holds.

(ii) Suppose that

(a) $i > 1$ and $I^- > -\infty$, or

(b) $i = 1$ and $I^- = \infty$, or

(c) $i = 1$, $I^- = -\infty$ and $P^\mu < \frac{1}{2}$, or

(d) $i > 1$, $I^- = -\infty$ and either $P^l > i(i-1)^{-1}$ or $P^\mu < i(i+1)^{-1}$.

Then H_D is proper, R_D is closed, and $R_D \neq L^1[0, \pi]$.

(iii) Suppose that

(a) $i = 1$, $I^- = -\infty$ and either $P^l > 1$ or $\frac{1}{2} < P^l \leq P^\mu < 1$, or

(b) $i > 1$, $I^- = -\infty$ and either $i(i+1)^{-1} < P^l \leq P^\mu < 1$ or $1 < P^l \leq P^\mu < i(i-1)^{-1}$.

Then H_D is proper and onto.

Much of the proof of this theorem is similar to the proof of Theorem 1. We use Lemma 1 and the following lemma.

LEMMA 2. Suppose that $y(w_0) = y(w_1) = 0$ and y is a non-negative solution of (1) on $[w_0, w_1]$. There exist $K_1 > 0$ and $\varepsilon : (0, \infty) \rightarrow (0, \infty)$ with $\varepsilon(r) \rightarrow 0$ as $r \rightarrow \infty$ such that, if $y(w_0) = \alpha > K_1$, then $|w_1 - w_0 - T(\alpha)| \leq \varepsilon(\alpha)T(\alpha)$. Moreover, for given g , K_1 and ε depend only on $\|f\|$.

The other idea in the proof of Theorem 4 (ii) is to use Lemmas 1 and 2 and a limit argument to show that, if H_D were onto, then (1_D) would have a generalized solution when $f = c\delta_a$. Here δ_a is the Dirac delta function with support at $a \equiv \pi - (4i)^{-1}\pi$. We can use comparison arguments to show that, if $|c|$ is large and c has suitable sign, then (1_D) is not solvable for this f . Thus H_D is not onto.

Now consider the periodic boundary-value problem.

THEOREM 5. (i) If $i = 0$, (1_p) is solvable if $\int_0^\pi f(t)dt > \pi I^-$.

This condition is also necessary if I^- is finite and $g(y) > I^+$ for all y .

(ii) Suppose that $i \geq 1$ and

(a) $I^- > -\infty$ or

(b) $I^- = -\infty$ and either $P^\mu < 1$ or $P^l > 1$.

Then H_p is proper and onto.

The proof of this result is similar to the proof of Theorem 3.

It is possible to construct examples to show that, if $i \geq 1$ and the conditions on P^l and P^μ in Theorem 4 (Theorem 5) do not hold, then $(H_D)^{-1}(0)$ $\left[(H_p)^{-1}(0) \right]$ may be unbounded.

The cases where μ and ν are both infinite are covered by the results in [2], Ehrmann [3], and Fučík and Lovicar [4].

As the full proofs are long and complicated, the author does not at present plan to publish them, but will reconsider this if he has sufficient evidence of interest in them. They are, however, freely available to those who request them from the author.

References

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