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ON MEASURABLE MULTIFUNCTIONS WITH STOCHASTIC DOMAIN

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Abstract

In this paper we prove several random fixed point theorems for multifunctions with a stochastic domain. Then those techniques are used to establish the existence of solutions for random differential inclusions. A useful tool in this process is a stochastic version of the Tietze extension theorems that we prove. Finally we present a stochastic version of the Riesz representation theorem for Hilbert spaces.

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1. Introduction

In this paper we examine single valued and multivalued functions with a stochastic domain. We prove some random fixed point theorems that generalize results existing in the literature. A useful tool in this process is a stochastic version of the Tietze extension theorem. Then, in Section 4, we use the random fixed point theorem to obtain solutions for a random differential inclusion. Finally in Section 5 we examine random linear operators with stochastic domain and we prove a random version of the classical Riesz representation theorem. Our work was motivated from control theory, differential equations and inclusions and mathematical economics, where our results can find useful applications.

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2. Preliminaries

Let (Ω, Σ, μ) be a complete σ-finite measure space and X a Polish space (that is, a complete, separable, metrizable space). We will denote by $d(\cdot,\cdot)$ a metric compatible with the topology of X. Recall (see Castaing-Valadier [12] or Himmelberg [4]) that a multifunction $F\colon 2^X\backslash\{\varnothing\}$ is said to be measurable if either of the following two equivalent statements hold:

(1) for all $U \subseteq X$ open, $\{\omega \in \Omega\colon F(\omega) \cap U \neq \varnothing\} \in \Sigma$,

(2) for all $y \in X$, $\omega \to d(y, F(\omega)) = \inf[d(y,x)\colon x \in F(\omega)]$ is measurable.

If $F(\cdot)$ is closed valued, then (1) and (2) above are equivalent to (3) $\operatorname{Gr} F = \{\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$, $B(X)$ being the Borel σ-field of X (graph measurability).

Following Engl [11] and Schäl [23] we will say that a measurable $F(\cdot)$ is separable if there exists a countable set $D \subseteq X$ such that $F(\omega) = \overline{F(\omega) \cap D}$ for all $\omega \in \Omega$. It is not difficult to see that if $F(.)$ is measurable and $F(\omega) = \overline{\operatorname{int} F(\omega)}$, then it is separable. Also a multifunction $T\colon \operatorname{Gr} F \to 2^X\backslash\{\varnothing\}$ is said to be an "adjective" random map with stochastic domain $F(\cdot)$, if for all $x \in X$ and all $U \subset X$ open, $\{\omega \in \Omega\colon T(\omega,x) \cap U \neq \varnothing,\ x \in F(\omega)\} \in \Sigma$ and for every $\omega \in \Omega, x \to T(\omega, x)$ is "adjective" on $F(\omega)$. Random operators with stochastic domain were first introduced and studied by Engl [8] (single valued case) and [11] (multivalued case).

Recall (see Klein-Thompson [17]), that if Y, Z are Hausdorff topological spaces and $G\colon Y \to 2^Z\backslash\{\varnothing\}$, then we say that $F(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c)) if for all $U \subset Z$ open $\{y \in Y\colon G(y) \subset U\}$ (resp. $\{y \in Y\colon G(y) \cap U \neq \varnothing\}$) is open in Y. If $G(\cdot)$ is both u.s.c. and l.s.c., then we say that it is continuous. Also if Z is a metric space and $G(\cdot)$ is closed valued, we say that $G(\cdot)$ is h-continuous, if it is continuous from Y into the closed subsets of Z with the Hausdorff (generalized) metric $h(\cdot,\cdot)$. Those two concepts of set valued continuity are equivalent when $G(\cdot)$ is compact valued. Finally for Z a metric space, we will say that $G(\cdot)$ is d-continuous if $y \to d(z, G(y))$ is continuous for all $z \in X$. Here $d(\cdot,\cdot)$ is the metric on Z. Note that an h-continuous multifunction is d-continuous and if Z is a reflexive Banach space, then the same is true for closed, convex valued multifunctions that are continuous in the Kuratowski-Mosco convergence (see Tsukada [26]). This last type of continuity is useful in differential inclusions and variational inequalities.

Let X be a Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subset X\colon \text{ nonempty, closed, (convex)}\},$$
$$P_{k(c)}(X) = \{A \subset X\colon \text{ nonempty, compact, (convex)}\}.$$

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Also by $\gamma(\cdot)$ we will denote the Kuratowski measure of noncompactness, that is, if $A \subset X$ is nonempty and bounded then

$$\gamma(A) = \inf\{\varepsilon > 0\colon A \text{ admits a finite covering of sets of diameter } \leq \varepsilon\}.$$

Recall that a map $T\colon X \to 2^X \backslash \{\emptyset\}$ is said to be condensing if for all $A \subset X$ nonempty, bounded with $\gamma(A) > 0$, we have $\gamma(T(A)) < \gamma(A)$.

3. Random fixed point theorems

In this section we prove the existence of random fixed points for multifunctions with stochastic domain. Throughout this section (Ω, Σ, μ) is a complete σ-finite measure space and X a Polish space. Additional hypotheses will be introduced as needed.

We will start with a powerful and general result, that extends to a probabilistic setting a large class of deterministic fixed point theorems. Already such a "general random fixed point theorem" (of the type that a.e. existence of a deterministic fixed point implies existence of a random fixed point) was proved by Engl [9] (Theorem 13) for single valued mappings and X a separable, reflexive Banach space. Then it was extended by Engl [11] (Theorem 13) and [10] (Theorem 6) to multivalued h-continuous mappings defined on a separable Banach space. Here we relax the h-continuity to d-continuity and the space X can be any Polish space. Also our result extends Theorem 4.2 of Cuong [5] and the results of Nowak [19], where a stronger continuity hypothesis was made and Theorem 1 of [21], where the multifunction had a deterministic domain, but the continuity hypotheses were weaker.

THEOREM 3.1. *If $F\colon \Omega \to P_f(X)$ is a separable measurable multifunction, $T\colon \operatorname{Gr} F \to P_f(X)$ is a d-continuous random map with stochastic domain $F(\cdot)$ and for all $\omega \in \Omega$ there exist $x \in F(\omega)$ such that $x \in T(\omega, x)$ then there exists $x\colon \Omega \to X$ measureable, and such that for all $\omega \in \Omega$, $x(\omega) \in F(\omega)$, and $x(\omega) \in T(\omega, x(\omega))$.*

PROOF. Let $G(\omega) = F(\omega) \times X$. Since $F(\cdot)$ is separable, measurable, so is $G(\cdot)$. Let $\varphi\colon \operatorname{Gr} G \to R_+$ be defined by $\varphi(\omega, x, y) = d(y, T(\omega, x))$. Note that for $(x, y) \in X \times X$ and for $\lambda > 0$ we have: $\{\omega \in \Omega\colon \varphi(\omega, x, y) < \lambda, (x, y) \in G(\omega\} = \{\omega \in \Omega\colon T(\omega, x) \cap [z \in X\colon d(z, y) < \lambda] \neq \emptyset, (x, y) \in G(\omega)\} \in \Sigma$. Also note that for fixed $\omega \in \Omega$, if $(x_n, y_n) \to (x, y)$, then

$$\begin{aligned}
&|d(y_n, T(\omega, x_n)) - d(y, T(\omega, x))| \\
&\qquad \leq |d(y_n, T(\omega, x_n)) - d(y, T(\omega, x_n))| + |d(y, T(\omega, x_n)) - d(y, T(\omega, x))| \\
&\qquad \leq d(y_n, y) + |d(y, T(\omega, x_n)) - d(y, T(\omega, x))| \to 0,
\end{aligned}$$

the summand tending to zero because of the d-continuity hypothesis. So $(x,y) \to \varphi(\omega,x,y)$ is continuous. Thus $\varphi(\cdot,\cdot,\cdot)$ is a continuous random operator with stochastic domain the separable, measurable multifunction $G(\cdot)$. Using Corollary 3.2. of Jdanok [15] we can find $\hat{\varphi}\colon \Omega \times X \times X \to R$ a Carathéodory extension of $\varphi(\cdot,\cdot,\cdot)$.

Next let $\hat{T}\colon \Omega \times X \to P_f(X)$ be the multifunction defined by

$$\hat{T}(\omega,x) = \begin{cases} T(\omega,x) & \text{for } \omega,x) \in \operatorname{Gr} F, \\ C & \text{for } (\omega,x) \notin \operatorname{Gr} F, \end{cases}$$

where $C \in P_f(X)$ is arbitrary. Consider the multifunction $R\colon \Omega \to 2^X$ defined by $R(\omega) = \{x \in F(\omega)\colon x \in \hat{T}(\omega,x)\}$. Note that because of our hypothesis, $R(\omega) \neq \varnothing$ for all $\omega \in \Omega$.

Also $\operatorname{Gr} R = \{(\omega,x) \in \operatorname{Gr} F\colon x \in \hat{T}(\omega,x)\} = \{(\omega,x) \in \operatorname{Gr} F\colon \hat{\varphi}(\omega,x,x) = 0\} \in \Sigma \times B(X)$. So we can apply Aumann's selection theorem (see Saint-Beuve [22], Theorem 3) to find $x\colon \Omega \to X$ measurable s.t. $x(\omega) \in R(\omega)$ for all $\omega \in \Omega$. Then $x(\omega) \in F(\omega)$ and $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$.

REMARK. The function $x(\cdot)$ obtained in the above theorem is called a random fixed point for $T(\cdot,\cdot)$.

We will give a small sample of the many useful applications that Theorem 3.1 can have. For the next result assume that X is a reflexive, separable Banach space. By X_w we will denote the space X with the weak topology.

THEOREM 3.2. *If $F\colon \Omega \to P_f(X)$ is a separable measurable multifunction, $T\colon \operatorname{Gr} F \to P_{fc}(X)$ is a continuous, condensing random multifunction, with stochastic domain $F(\cdot)$ and for all $\omega \in \Omega$, $T(\omega, F(\omega)) \subset F(\omega)$ and $T(\omega,F(\omega))$ is bounded, then $T(\cdot,\cdot)$ admits a random fixed point.*

PROOF. Since $T(\omega,\cdot)$ is continuous from $F(\omega)$ into X, it is u.s.c. from $F(\omega)$ into X_w and so for $x_n \xrightarrow{s} x$ in $F(\omega)$, we have $w\text{-}\overline{\lim}_{n\to\infty} T(\omega,x_n) \subset T(\omega,x)$ (see Delahaye-Denel [6]). Also $T(\cdot,\cdot)$ is l.s.c. from $F(\omega)$ into X and so $T(\omega,x) \subset s\text{-}\underline{\lim}\, T(\omega,x_n)$. Then using Theorem 2.2. of Tsukada [26] we get that $T(\omega,\cdot)$ is d-continuous on $F(\omega)$. Finally Theorem 1 of Himmelberg, Porter and Van Vleck [13], together with Theorem 3.1. produce the desired random fixed point.

REMARK. This result extends Theorem 3.1. of Itoh [14], where the multifunction $T(\cdot,\cdot)$ had a deterministic domain and its values were compact, convex sets in X.

For the next result assume that X is a separable Banach space. Recall that a multifunction $G\colon X \to P_f(X)$ is h-contractive, if $h(G(x'),G(x)) < \|x'-x\|$ for all $x', x \in X$. Our result can be viewed as a stochastic version of the fixed point theorem of Assad [1].

THEOREM 3.3. *If $F: \Omega \to P_k(X)$ is a separable, measurable multifunction, $T: \operatorname{Gr} F \to P_f(X)$ is an h-contractive, random operator with stochastic domain $F(\cdot)$ and with bounded values, and for all $(\omega, x) \in \operatorname{Gr}(\operatorname{bd} F(\cdot))$, $T(\omega, x) \subset F(\omega)$, then $T(\cdot,\cdot)$ admits a random fixed point.*

PROOF. This follows from Theorem 1.1 of Assad [1] and Theorem 3.1.

REMARK. The above theorem extends Corollary 15 of Engl [11], where $T(\omega,\cdot)$ was a strict contraction, that is, $k(\omega)$-Lipschitz with $k(\omega) < 1$ for all $\omega \in \Omega$.

So the strategy is clear. Combine a well known deterministic result with Theorem 3.1. This way we can get several other theorems. In the rest of this section we are going to use a particular version of this method, based on the following interesting extension theorem. The same result was also proved by Bocsan, Constantin and Radu [3]. This theorem is an interesting application of an important extension principle for R-valued functions proved by Jdanok [15] under very general hypotheses.

Asume that X, Y are Polish spaces.

THEOREM 3.4. *If $F: \Omega \to P_f(X)$ is a separable measurable multifunction, and $T: \operatorname{Gr} F \to Y$ is a continuous random operator with stochastic domain $F(\cdot)$, then there exists $\hat{T}: \Omega \times X \to Y$ a Carathéodory map such that $\hat{T}|_{\operatorname{Gr} F} = T$.*

PROOF. From Urysohn's theorem (see Dugundji [7], page 195, Corollary 9.2) we know that there exists a homeomorphism $u: Y \to V \subset I^N$ where $I = [0,1]$. So we can view T as a map from $\operatorname{Gr} F$ into V, that is, $T(\omega, x) = (T_1(\omega,x), T_2(\omega,x), \ldots, T_n(\omega,x), \ldots)$.

We claim that for all $n \geq 1$, $T_n(\cdot,\cdot)$ is a continuous random function with stochastic domain $F(\cdot)$. The continuity is clear from the continuity of $T(\omega,\cdot)$. To show measurability, lelt $p_n: I^N \to I$ be the projection on the nth factor. We know that $p_n(\cdot)$ is a continuous, open surjection. Then for every $x \in F(\omega)$ and for every $U \subset I$ open we have $\{\omega \in \Omega: T_n(\omega,x) \in U\} = \{\omega \in \Omega: (p_n \circ T)(\omega, x) \in U, x \in F(\omega)\} \in \Sigma$. For each $n \geq 1$, let $\hat{T}_n: \Omega \times X \to I$ be the Carathéodory extension of T_n, existing by Corollary 3.2 of Jdanok [15]. Let $\hat{T}: \Omega \times X \to Y$ be defined by $\hat{T}(\omega,x) = (\hat{T}_1(\omega,x), \hat{T}_2(\omega,x), \ldots, \hat{T}_n(\omega,x), \ldots)$. Clearly $\hat{T}(\cdot,\cdot)$ is the desired extension of $T(\cdot,\cdot)$.

This result leads us to some other interesting random fixed point theorems for measurable multifunctions with stochastic domain. So let X be a separable Banach space.

THEOREM 3.5. *If $F: \Omega \to P_f(X)$ is a separable multifunction, and $T: \operatorname{Gr} F \to P_{kc}(X)$ is a continuous, bounded, condensing random multifunction with* *stochastic domain* $F(\cdot)$ *such that for all* $\omega \in \Omega$, $T(\omega, \operatorname{bd} F(\omega)) \subset F(\omega)$, *then* $T(\cdot,\cdot)$ *admits a random fixed point.*

PROOF. From Theorem II-14, page 47, of Castaing-Valadier [4], we know that $(P_{kc}(Y), h)$ is a Polish space. Then Theorem III-2 of [4] tells us that $T(\cdot,\cdot)$ is a continuous, random operator with stochastic domain $F(\cdot)$. Using Theorem 3.4. we can find $\hat{T}\colon \Omega \times K \to P_{kc}$, a Carathéodory extension of $T(\cdot,\cdot)$. Let $R(\omega) = \{x \in F(\omega)\colon x \in \hat{T}(\omega, x)\}$. From Corollary 3 of Su-Seghal [25] we know that for all $\omega \in \Omega, R(\omega) \neq \varnothing$. Note that $\operatorname{Gr} R = \{(\omega, x) \in \Omega \times X\colon d(x, \hat{T}(\omega, x)) = 0\} \cap \operatorname{Gr} F$. Since measurablity of $\hat{T}(\cdot, x)$ as a function in the metric space $(P_{kc}(X), h)$ is equivalent to measurablity of $\hat{T}(\cdot, x)$ as a multifunction (see Theorem III-2 in Castaing-Valadier [4]), we see that $(\omega, x) \to d(x, \hat{T}(\omega, x))$ is a Carathéodory function, that is, is measurable in ω and continuous in x. Hence from Lemma III-14 of [4] we get that $(\omega, x) \to d(x, \hat{T}(\omega, x))$ is measurable, and thus $\operatorname{Gr} R \in \Sigma \times B(X)$. So we can apply Aumann's selection theorem and get $x\colon \Omega \to X$, measurable, such that for all $\omega \in \Omega$, $x(\omega) \in R(\omega)$. This is the desired random fixed point.

REMARK. The result remains true if instead we assume "$T\colon \operatorname{Gr} F \to P_{fc}(X)$ is a continuous, bounded, condensing random multifunction with stochastic domain $F(\cdot)$ and with values in a separable subset C of the space of nonempty, bounded, closed, convex subsets of X with the Hausdorff metric".

The proof is the same, if we consider the Polish space (C, h). Similarly, using Theorem 1 of Su-Seghal [25] we can have

THEOREM 3.6. *If* $F(\cdot)$ *is as above, and* $T\colon \operatorname{Gr} F \to P_{kc}(X)$ *is a continuous, bounded, condensing random multifunction such that for all* $(\omega, x) \in \operatorname{Gr} F$, $T(\omega, x) \cap F \neq \varnothing$, *then* $T(\cdot,\cdot)$ *admits a random fixed point*

REMARKS. (1) Again we can assume instead that $T\colon \operatorname{Gr} F \to P_{fc}(X)$ is a continuous, bounded, condensing random multifunction with values in C.

(2) This result extends Theorem 2 of Sehgal and Walters [24], which was stated for single valued functions with domain $\Omega \times X$.

Also Theorem 3.4. leads us to the following implicit function theorem of Filippov type, that can be useful in control theory and mathematical economics. Assume that X, Y are Polish spaces.

THEOREM 3.7. *If* $F\colon \Omega \to P_f(X)$ *is a separable, measurable multifunction,* $T\colon \operatorname{Gr} F \to P_k(Y)$ *is a continuous, random multifunction with stochastic domain* $F(\cdot)$, $G\colon \Omega \to 2^X \backslash \{\varnothing\}$ *is graph measurable and for all* $\omega \in G$, $G(\omega) \subset F(\omega)$ *and if* $f\colon \Omega \to Y$ *is measurable such that for all* $\omega \in \Omega, f(\omega) \in T(\omega, G(\omega))$, *then* *there exists* $g\colon \Omega \to X$ *measurable such that for all* $\omega \in \Omega$, $g(\omega) \in G(\omega)$ *and* $f(\omega) \in T(\omega, g(\omega))$.

PROOF. From Theorem II-8, p. 42 of Castaing and Valadier [4] we know that $(P_k(Y), h)$ is a Polish space. Apply Theorem 3.4. to find $\hat{T}\colon \Omega \times X \to P_k(Y)$, a Carathéodory extension of $T(\cdot,\cdot)$. Let $R\colon \Omega \to 2^X$ be defined by $R(\omega) = \{x \in G(\omega)\colon f(\omega) \in \hat{T}(\omega, x)\}$. Clearly for all $\omega \in \Omega$, $R(\omega) \neq \emptyset$. Also $\operatorname{Gr} R = \{(\omega, x) \in \operatorname{Gr} G\colon d(f(\omega), \hat{T}(\omega, x)) = 0\}$. But recall that $(\omega, x) \to d(f(\omega), \hat{T}(\omega, x))$ is a Carathéodory function, and hence it is jointly measurable. Therefore $\operatorname{Gr} R \in \Sigma \times B(X)$. Once Again Aumann's selection theorem gives us $g\colon \Omega \to X$ measurable such that for all $\omega \in \Omega$, $g(\omega) \in R(\omega)$ implies $g(\omega) \in G(\omega)$ and $f(\omega) \in T(\omega, g(\omega))$.

REMARK. Again we can assume instead that "$T\colon \operatorname{Gr} F \to P_f(X)$ is a continuous, random multifunction with stochastic domain $F(\cdot)$ and values in a separable subset C' of the set of all bounded, closed subsets of X"

Interesting random fixed point theorems for u.s.c. operators can be found in [10] and [11].

4. A random differential inclusion

In this section, using a random fixed point argument, we will establish the existence of solutions for a class of random differential inclusions defined in a separable Banach space.

Our existence theorem extends to infinite dimensions the work of Nowak [20]. Also the orientor field in our case satisfies weaker continuity hypotheses (Carathéodory conditions) and a more general growth assumption. When specialized to single valued functions, our result extends significantly Theorem 4.3 of Itoh [14].

So let (Ω, Σ, μ) be a complete, σ-finite measure space, $T = [0, b]$ a bounded, closed interval in R_+, and X a separable Banach space. Given $F\colon \Omega \times T \times X \to 2^X \backslash \{\emptyset\}$ and $x_0\colon \Omega \to X$ measurable, we consider the following random multivalued Cauchy problem:

$$(*) \qquad \dot{x}(\omega, t) \in F(\omega, t, x(\omega, t)), \qquad x(\omega, 0) = x_0(\omega).$$

By a random solution of $(*)$ we understand process $x\colon \Omega \times T \to X$ which is measurable in ω, absolutely continuous in t and satisfies $(*)$ for all $\omega \in \Omega$ and almost all $t \in T$. We will make the following assumptions concerning $F(\cdot,\cdot,\cdot)$:

(A_1) $F(\cdot,\cdot,\cdot)$ has values in $P_{kc}(Y)$.

(A_2) for all $x \in X$, $(\omega, t) \to F(\omega, t, x)$ is measurable.

(A_3) for all $(\omega, t) \in \Omega \times T$, $x \to F(\omega, t, x)$ is h-continuous.

(A_4) $|F(\omega,t,x)| = \sup\{\|z\|: z \in F(\omega,t,x)\} \leq a(\omega,t) + b(\omega,t)\|x\|$ a.e. for all $\omega \in \Omega$ and all $x \in X$, where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are jointly measurable, and for all $\omega \in \Omega, a(\omega,\cdot), b(\omega,\cdot) \in L^1_+$.

(A_5) For all $B \subset X$ bounded, $\gamma[F(\omega,t,B)] \leq \varphi(\omega,t)\gamma(B)$ a.e. for all $\omega \in \Omega$, where $\varphi(\cdot,\cdot)$ is jointly measurable and for all $\omega \in \Omega$, $\varphi(\omega,\cdot) \in L^1_+$ and $\|\varphi(\omega,\cdot)\|_1 < 1/2$.

THEOREM 4.1. *If* (A_1) *to* (A_5) *hold then* $(*)$ *admits a random solution.*

PROOF. Let $M(\omega) = [\|x_0(\omega)\| + \|a(\omega,\cdot)\|_1]\exp[\|b(\omega,\cdot)\|_1]$. Let $B(\omega) \subseteq C(T,X)$ be defined by $B(\omega) = \{x(\cdot) \in C(T,X): \|x(\cdot)\|_\infty \leq M(\omega)\}$.

Let $R: \Omega \times C(T,X) \to 2^{C(T,X)}$ be defined by $R(\omega,x) = \{y(\cdot) \in C(T,X): y(t) = x_0(\omega) + \int_0^t f(s)\,ds,\ t \in T,\ f(\cdot) \in L^1(T,X),\ f(s) \in F(\omega,s,x(s))$ a.e.$\}$.

Note that because of Aumann's selection theorem and (A_4) it is easy to see that $R(\omega,x) \neq \emptyset$. Next we will show that $\omega \to R(\omega,x)$ is measurable, while $x \to R(\omega,x)$ is h-continuous.

Let $\int_0^t F(\omega,s,x(s))\,ds = \{\int_0^t f(s)\,ds: f(\cdot) \in L^1(T,X), f(s) \in F(\omega,s,x(s))$ a.e.$\}$. As we already said this is nonempty and because of the Rädstrom embedding theorem (see Klein and Thompson [17] it is easy to see that

$$\int_0^t F(\omega,s,x(s))\,ds \in P_{kc}(X) \quad \text{for all } t \in T.$$

So a straightforward application of the Arzela and Ascoli theorem tells us that $R(\cdot,\cdot)$ has values in $P_{kc}(C(T,X))$. Now note that $\operatorname{Gr} R = \{(\omega,x,y) \in \Omega \times C(T,X) \times C(T,X); d(y(t),L(\omega,t,x)) = 0$ for all $t \in T\}$, where $L(\omega,t,x) = x_0(\omega) + \int_0^t F(\omega,s,x(s))\,ds$. Let $u(\omega,t,x,y) = x_0(\omega) + d(e_t(y),L(\omega,t,x))$ where $e_t: C(T,X) \to X$ is the evaluation map at t. From Theorem 2.4, page 260 of Dugundji [7] we know that $e_\cdot(\cdot)$ is continuous. Also from (A_2) $\omega \to L(\omega,t,x)$ is measurable. Furthermore if $(t_n,x_n) \to (t,x)$ in $T \times C(T,X)$, then using (A_3) we have

$$\begin{aligned} h&\left(\int_0^{t_n} F(\omega,s,x_n(s))\,ds, \int_0^t F(\omega,s,x(s))\,ds\right) \\ &\leq h\left(\int_0^{t_n} F(\omega,s,x_n(s))\,ds, \int_0^{t_n} F(\omega,s,x(s))\,ds\right) \\ &\quad + h\left(\int_0^{t_n} F(\omega,s,x(s))\,ds, \int_0^t F(\omega,s,x(s))\,ds\right) \\ &\leq \int_0^{t_n} h(F(\omega,s,x_n(s)),F(\omega,s,x(s)))\,ds \\ &\quad + \int_{t\wedge t_n}^{t\vee t_n} |F(\omega,s,x(s))|\,ds \to 0, \qquad \text{as } n \to \infty. \end{aligned}$$

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Hence $(t,x) \to L(\omega,t,x)$ is h-continuous, so $(\omega,t,x,y) \to u(\omega,t,x,y)$ is measurable in ω and continuous in (t,x,y), whence $u(\cdot,\cdot,\cdot,\cdot)$ is jointly measurable, and so $(\omega,x,y) \to v(\omega,x,y) = \sup_{n\geq 1} u(\omega,t_n,x,y)$ is measurable, where $\{t_n\}_{n\geq 1}$ is dense in T. But $\operatorname{Gr} R = \{(\omega,x,y) \in \Omega \times C(T,X) \times C(T,X) \colon v(\omega,x,y) = 0\} \in \Sigma \times B(C(T,X)) \times B(C(T,X))$ implies that $R(\cdot,\cdot)$ is graph measurable.

Next let $S^1_{F(\omega,\cdot,z(\cdot))} = \{f(\cdot) \in L^1(T,X) \colon f(s) \in F(\omega,s,z(s)) \text{ a.e.}\}$. Also let $x_n(\cdot) \to x(\cdot)$ in $C(T,X)$. Then using Hörmander's formula (see Theorem II-18, page 49, of Castaing and Valadier [4]), we have that (recall $\sigma(x^*,A) = \sup[(x^*,a),\ a \in A]$ for $A \subset X$ and $x^* \in X^*$):

$$\begin{aligned} &h(S^1_{F(\omega,\cdot,x_n(\cdot))}, S^1_{F(\omega,\cdot,x(\cdot))}) \\ &\quad = \sup_{\|v\|_\infty \leq \infty} |\sigma(v, S^1_{F(\omega,\cdot,x_n(\cdot))}) - \sigma(v, S^1_{F(\omega,\cdot,x(\cdot))})| \\ &\quad \leq \sup_{\|v\|_\infty \leq 1} \int_0^t |\sigma(v(s), F(\omega,s,x_n(s))) - \sigma(v(s), F(\omega,s,x(s)))|\, ds \\ &\quad \leq \int_0^t h(F(\omega,s,x_n(s)), F(\omega,s,x(s)))\, ds \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

and so

$$S^1_{F(\omega,\cdot,x_n(\cdot))} \xrightarrow{h} S^1_{F(\omega,\cdot,x(\cdot))} \quad \text{as } n \to \infty. \tag{1}$$

Let $z(\cdot) \in R(\omega,x)$. Then

$$z(t) = x_0(\omega) + \int_0^t f(s)\, ds, \qquad t \in T,$$

with $f(\cdot) \in S^1_{F(\omega,\cdot,x(\cdot))}$. Let $f_n(\cdot) \in S^1_{F(\omega,\cdot,x_n(\cdot))}$ be such that $f_n \xrightarrow{s} f$ in $L^1(X)$. This is possible because of (1). Set

$$z_n(t) = x_0(\omega) + \int_0^t f_n(s)\, ds, \qquad t \in T.$$

Clearly $z_n(\cdot) \in R(\omega,x_n)$ and

$$\|z_n - z\|_\infty \leq \int_0^b \|f_n(s) - f(s)\|\, ds \to 0, \quad \text{as } n \to \infty,$$

so $z \in s - \underline{\lim}_{n\to\infty} R(\omega,x_n)$, whence

$$R(\omega,\cdot) \text{ is l.s.c.} \tag{2}$$

(see Delahaye and Denel [6]).

Next we will show that $R(\omega,\cdot)$ is u.s.c. Because it has compact values it suffices to show that given $\varepsilon > 0$ there exists $\delta(\varepsilon,x) > 0$ such that $R(\omega,y) \subseteq R(\omega,x) + \varepsilon B_1$ for all $y \in B_\delta(x)$, where $B_\delta(x) = \{z(\cdot) \in C(T,X) \colon \|z - x\|_\infty < \delta\}$ and B_1 is the open unit ball. Recall that $x(\cdot) \to S^1_{F(\omega,\cdot,x(\cdot))}$ is h-continuous. So we can find $\delta > 0$ such that for $\|x-y\|_\infty < \delta$ we have $h(S^1_{F(\omega,\cdot,x(\cdot))}S^1_{F(\cdot,y(\cdot),\cdot)}) < \varepsilon$. Our claim is that this $\delta > 0$ will do the job for us. So let $z \in R(\omega, y)$. Then

$$d(z, R(\omega,x)) = \inf_{f\in S^1_{F(\omega,\cdot,x(\cdot))}} \sup_{t\in T} \left\| z(t) - x_0(\omega) - \int_0^t f(s)\,ds \right\|$$
$$= \inf_{f\in S^1_{F(\omega,\cdot,x(\cdot))}} \sup_{t\in T} \left\| x_0(\omega) + \int_0^t g(s)\,ds - x_0(\omega) - \int_0^t f(s)\,ds \right\|$$

where $g(\cdot) \in S^1_{F(\omega,\cdot,y(\cdot))}$ and $z(t) = x_0(\omega) + \int_0^t g(s)\,ds,\ t \in T$. So we have

$$d(z, R(\omega,x)) = \inf_{f\in S^1_{F(\omega,\cdot,x(\cdot))}} \left\| \int_0^t g(s)\,ds - \int_0^t f(s)\,ds \right\|$$
$$\leq \inf_{f\in S^1_{F(\omega,\cdot,x(\cdot))}} \int_0^t \|g(s) - f(s)\|\,ds = d(g, S_{F^1_{(\omega,\cdot,x(\cdot))}}) < \varepsilon,$$

whence $R(\omega, y) \subseteq R(\omega, x) + \varepsilon B_1$, and so

(3) $$R(\omega,\cdot) \text{ is u.s.c.}$$

From (2) and (3) we conclude tht $R(\omega,\cdot)$ is continuous and because it has compact values it is h-continuous.

Next we will show that for each $\omega \in \Omega$ $R(\omega,\cdot)$ has a fixed point in $B(\omega)$. To this end note that because of Gronwall's inequality, for all $x(\cdot) \in B(\omega)$, $R(\omega,x) \subseteq B(\omega)$. So $R(\omega,\cdot)\colon B(\omega) \to B(\omega)$. Next let $A \subseteq B(\omega)$ be nonempty. We have

$$\gamma[R(\omega,A)(t)] \leq \gamma\left[\int_0^t F(\omega,s,A(s))\,ds\right].$$

Let $\{x_n(\cdot)\}_{n\geq 1} \subseteq B(\omega)$ be such that $\mathrm{cl}\,\{x_n(\cdot)\}_{n\geq 1} = \overline{A}$. This set exists because $\overline{A}$ being a closed subset of the separable Banach space $C(T,X)$ is itself separable. We claim that, for all $s \in T$,

$$\mathrm{cl}[F(\omega,s,\overline{A}(s))] = \mathrm{cl}\{F(\omega,s,x_n(s))\}_{n\geq 1}.$$

So let $y \in F(\omega,s,\overline{A}(s))$; then $y \in F(\omega,s,z)$ with $z \in \overline{A}(s)$. Let $\{x_m(\cdot)\}_{m\geq 1} \subseteq \{x_n(\cdot)\}_{n\geq 1}$ be such that $x_m(s) \xrightarrow{s} z$; then $F(\omega,s,x_m(s)) \xrightarrow{h} F(\omega,s,z)$. Let $y_m \in F(\omega,s,x_m(s))$ be such that

$$\|y_m - y\| = d(y, F(\omega,s,x_m(s))).$$

Then $\|y_m - y\| \to 0$ as $m \to \infty$, so $y \in\ \mathrm{cl}\{F(\omega,s,x_n(s))\}_{n\geq 1}$, and so the claim follows. But note that because of (A_2) and (A_3) $s \to F(\omega,s,x_n(s))$ is measurable implies $s \to \overline{\mathrm{conv}} \bigcup_{n\geq 1} F(\omega,s,x_n(s)) = H(\omega,s)$ is measurable. So it admits a Castaing representation $\{h_n(\omega,\cdot)\}_{n\geq 1}$ (see Castaing-Valadier [4], Theorem III-7, page 66). Using it we have that

$$\operatorname{cl} \int_0^t \overline{\operatorname{conv}}\{h_n(\omega, s)\}_{n\geq 1} ds = \int_0^t \overline{\operatorname{conv}}\{h_n(\omega, s)\}_{n\geq 1}\, ds$$
$$= \operatorname{cl} \int_0^t \{h_n(\omega, s)\}_{n\geq 1}\, ds,$$

which implies that

$$\gamma \left[\int_0^t F(\omega, s, A(s))\, ds \right] \leq \gamma \left[\int_0^t h_n(\omega, s)\, ds \colon n \geq 1 \right].$$

Using Lemma 2.2. of Kisielewicz [16] we get that

$$\gamma \left[\int_0^t h_n(\omega, s)\, ds \colon n \geq 1 \right] \leq \int_0^t 2\gamma[h_n(\omega, s) \colon n \geq 1]\, ds$$
$$= \int_0^t 2\gamma(F(\omega, s, \overline{A}(s)))\, ds$$
$$\leq \int_0^t 2\varphi(\omega, s)\gamma(\overline{A}(s))\, ds.$$

Let $\psi(\cdot)$ denote the Kuratowski measure of noncompactness on $C(T, X)$. From a well-known result of Ambrosetti (see for example Lakshmikantham and Leela [18]) we have $\psi(R(\omega, A)) \leq 2\psi(A)\|\varphi(\omega, \cdot)\|_1 < \psi(A)$, whence $R(\omega, \cdot)$ is ψ-condensing.

Apply Theorem 1 of Himmelberg-Porter-Van Vleck [13] to deduce that there exists $x \colon \Omega \to C(T, X)$ measurable, and such that $x(\omega) \in B(\omega)$ and $x(\omega) \in R(\omega, x(\omega))$ for all $\omega \in \Omega$. Because of Proposition 4.2. of Itoh [14] (see also the Lemma in [21]) we have that $x(\omega, t) = x(\omega)(t)$ is a Carathéodory function and clearly is the desired random solution of $(*)$.

5. Random linear functionals

In this final section of the paper we prove a random version of the Riesz representation theorem for Hilbert spaces.

So let (Ω, Σ, μ) be a complete, σ-finite measure space and X a separable Hilbert space.

THEOREM 5.1. *If $L \colon \Omega \to P_f(X)$ is a separable, measurable multifunction with values in the closed subspaces of X and $T \colon \operatorname{Gr} L \to R$ is a linear, continuous, random functional with stochastic domain $L(\cdot)$, then there exists a unique $x^* \colon \Omega \to X$, measurable, and such that for all $\omega \in \Omega$, $x^*(\omega) \in L(\omega)$, $T(\omega, x) = (x^*(\omega), x)$ for all $x \in L(\omega)$, and $\|T(\omega, \cdot)\| = \|x^*(\omega)\|$.*

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PROOF. From Corollary 4.4. of Jdanok [15] we know that there exists $\hat{T}\colon \Omega \times X \to R$, a linear, continuous, measurable functional such that $\hat{T}|_{\operatorname{Gr} L} = T$ and $\|\hat{T}(\omega)\| = \|T(\omega)\|$ for all $\omega \in \Omega$.

Let $R\colon \Omega \to 2^X$ be defined by

$$R(\omega) = \{x^* \in L(\omega)\colon (x^*, x) = \hat{T}(\omega, x) \text{ for all } x \in L(\omega)\}.$$

Since $L(\omega)$ is a Hilbert space itself, from the classical Riesz representation theorem we know that $R(\omega) \neq \varnothing$ for all $\omega \in \Omega$. Let $\{x_n(\cdot)\}_{n\geq 1}$ be a Castaing representation for $L(\cdot)$; that is, for all $n \geq 1$, $x_n\colon \Omega \to X$ is measurable and $L(\omega) = \operatorname{cl}\{x_n(\omega)\}_{n\geq 1}$. Then $\operatorname{Gr} R = \bigcap_{n\geq 1}\{(\omega, x^*) \in \Omega \times X\colon (x^*, x_n(\omega)) - \hat{T}(\omega, x_n(\omega)) = 0\} \cap \operatorname{Gr} L$. Note that for all $n \geq 1$, $u_n(\omega, x^*) = (x^*, x_n(\omega)) - \hat{T}(\omega, x_n(\omega))$ is jointly measurable and so $\operatorname{Gr} R \in \Sigma \times B(X)$. Apply Aumann's selection theorem to find $x^*\colon \Omega \to X$ measurable such that $x^*(\omega) \in R(\omega)$ for all $\omega \in \Omega \Rightarrow x^* \in L(\omega)$ and $(x^*(\omega), x) = T(\omega, x)$ for all $x \in L(\omega)$ and all $\omega \in \Omega$. From this last equality we also get that $\|x^*(\omega)\| = \|T(\omega)\|$ for all $\omega \in \Omega$.

REMARK. If for all $\omega \in \Omega$, $L(\omega) = X$, then we recover the result of Bensoussan [2], page 89, which is useful in filtering theory.

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