LATTICE-ORDERED GROUPS HAVING AT MOST TWO DISJOINT ELEMENTS†

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1. Introduction. Let $L = L(+, \vee, \wedge)$ be a lattice-ordered group, or *l-group* (Birkhoff [**1**, p. 214]). Two elements a and b of L will be called *disjoint* if $a > 0$, $b > 0$, and $a \wedge b = 0$. It is easily seen that if L does not contain two disjoint elements, then it is linearly ordered (and, of course, conversely). What can we say about l-groups containing two but not more than two mutually disjoint elements?

Let A and B be linearly ordered groups (*o-groups*), and let $A \oplus B$ be the cardinal sum of A and B. That is, $A \oplus B$ is the direct sum of A and B, and (a, b) is positive in $A+B$ if and only if a is positive in A and b is positive in B. An l-group L containing $A \oplus B$ as a convex normal subgroup (or l-ideal) is called a *lexico-extension* of $A \oplus B$ if every positive element of L not in $A \oplus B$ exceeds every element of $A \oplus B$. It then follows (subsection 2.9 below) that $L/(A \oplus B)$ is an o-group. Such an l-group L is easily seen to satisfy the following condition :

(D) *There exists a pair of disjoint elements in* L, *but no triple of pairwise disjoint elements exists in* L.

The following theorem shows that condition (D) characterizes L.

THEOREM‡. *An l-group satisfying* (D) *is a lexico-extension of the cardinal sum* $A \oplus B$ *of two linearly ordered subgroups* A *and* B *of* L *by an o-group* C.

The following steps in the proof of this theorem would follow from results of Jaffard [**2**] *if* L *were abelian* : that G_p and G_q are linearly ordered from his Theorem 1, p. 235 ; that the subgroup of L generated by L_p and L_q is the cardinal sum $G_p + G_q$ from his Proposition 3, p. 241. These occur in subsections 2.7 and 2.8 respectively.

2. Proof of the theorem. Let $L^* = \{x \in L : x > 0\}$. Select p and q in L^* such that $p \wedge q = 0$. Let $L_p = \{x \in L : x \wedge q = 0\}$ and let $L_q = \{x \in L : x \wedge p = 0\}$.

2.1. *L_p and L_q are linearly ordered convex subsemigroups of L.*

If $0 \leqslant x \leqslant a \in L_p$, then $0 \leqslant x \wedge q \leqslant a \wedge q = 0$. Thus $x \in L_p$, and hence L_p is convex. L_p is a semigroup [**1**, p. 219]. Let x and y be two non-zero elements in L_p. Then $x \wedge y > 0$, for otherwise $0 = x \wedge y = x \wedge q = y \wedge q$, contrary to (D). Now $x = x' + (x \wedge y)$ and $y = y' + (x \wedge y)$ with $x' \wedge y' = 0$. Since L_p is convex, x' and y' belong to L_p. Thus either $x' = 0$ and $x \leqslant y$ or $y' = 0$ and $y \leqslant x$.

2.2. *The subsemigroup of L generated by $L_p \cup L_q$ is the direct sum $L_p \oplus L_q$, and it is convex.*

If $x \in L_p \cap L_q$, then $x \wedge p = x \wedge q = p \wedge q = 0$, and hence $x = 0$ by (D). If $x \in L_p$ and $y \in L_q$, then $x \wedge y \in L_p \cap L_q$ because L_p and L_q are convex. Thus $x \wedge y = 0$, and so $x + y = x \vee y$

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‡ Added in proof. This result has subsequently been extended to l-groups with n disjoint elements but not $n+1$ such elements, and, in fact, to l-groups in which each element is greater than at most a finite number of disjoint elements.

$= y \vee x = y + x$. If $0 \leqslant x \leqslant a + b$, where $a \in L_p$ and $b \in L_q$, then $x \wedge a \in L_p$ and $x \wedge b \in L_q$. Thus

$$x = x \wedge (a+b) = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) = (x \wedge a) + (x \wedge b) \in L_p \oplus L_q,$$

and hence $L_p \oplus L_q$ is convex.

2.3. *If* $0 < p' \in L_p$ *and* $0 < q' \in L_q$, *then* $L_p = L_{p'}$ *and* $L_q = L_{q'}$, *where*

$$L_{p'} = \{x \in L : x \wedge q' = 0\} \quad \text{and} \quad L_{q'} = \{x \in L : x \wedge p' = 0\}.$$

If $x \in L_p$, then, since $q' \in L_q$, $x \wedge q' = 0$. Thus $x \in L_{p'}$ and hence $L_p \subseteq L_{p'}$. Similarly $L_q \subseteq L_{q'}$. In particular, $0 < p \in L_{p'}$ and $0 < q \in L_{q'}$. By reversing this argument, we have $L_{p'} \subseteq L_p$ and $L_{q'} \subseteq L_q$.

2.4. *If* $a \in L^*$ *and* $a \notin L_p \oplus L_q$, *then* $a > L_p \oplus L_q$.

We first show $a > p$. Let $d = a \wedge p$. Then $d \in L_p$, $a = d + \bar{a}$, $p = d + \bar{p}$, and $\bar{a} \wedge \bar{p} = 0$. If $\bar{p} = 0$, then $p = d < a$. Now $p \wedge \bar{a} > 0$, for otherwise $\bar{a} \in L_q$ and hence $a = d + \bar{a} \in L_p \oplus L_q$. If $\bar{p} > 0$, it follows that $\bar{p}$ and $p \wedge \bar{a}$ are strictly positive elements in the linearly ordered semigroup L_p, and hence $0 < p \wedge \bar{a} \wedge \bar{p} = p \wedge 0 = 0$. This contradiction shows that $a > p$, and similarly $a > q$. Therefore $a > p \vee q = p + q$. It follows from 2.3 that $a > p' + q'$ for every p' in L_p and every q' in L_q.

2.5. *If* $a, b \in L^*$ *and* $a \wedge b = 0$, *then* $a, b \in L_p \oplus L_q$.

If neither a nor b belongs to $L_p \oplus L_q$ then, by 2.4, $a \wedge b > p > 0$. If, say, a belongs to $L_p \oplus L_q$ but b does not, then $b > a$ by 2.4, and $a \wedge b = a > 0$. Hence they must both belong to $L_p \oplus L_q$.

2.6. *The semigroup* $L_p \oplus L_q$ *is invariant under o-automorphisms of* L (*in particular under inner automorphisms of* L).

If π is an o-automorphism of L, then $p\pi \wedge q\pi = (p \wedge q)\pi = 0\pi = 0$. By 2.5, $p\pi$ and $q\pi$ both belong to $L_p \oplus L_q$. But, by 2.6, we can replace p by any non-zero element in L_p, and q by any non-zero element in L_q. Thus $L_p\pi$ and $L_q\pi$ are contained in $L_p \oplus L_q$, and hence so is $(L_p \oplus L_q)\pi$.

2.7. *The set* $G_p = \{x \in L : x \wedge q = 0 \text{ or } x \vee (-q) = 0\}$ *is a convex, linearly ordered subgroup of* L.

Clearly $G_p = L_p \cup N_p$, where $N_p = \{x \in L : x \vee (-q) = 0\} = \{-x : x \in L_p\}$, and N_p is a convex, linearly ordered subsemigroup of L. Evidently G_p is linearly ordered. To show that G_p is convex, suppose that $x < y < z$, where $x, z \in G_p$ and $y \in L$. If $x < y \leqslant 0$ or $0 \leqslant y < z$, then $y \in N_p$ or $y \in L_p$, respectively, since these sets are convex. Suppose (by way of contradiction) that y is not comparable with 0. Then $x < 0 < z$, and hence z, $-x$, and $z - x$ all belong to L_p. From $0 < y - x < z - x$ and the convexity of L_p, we conclude that $y - x \in L_p$. Since L_p is linearly ordered, $y - x \leqslant -x$ or $y - x \geqslant -x$; hence $y \leqslant 0$ or $y \geqslant 0$. Hence G_p is convex. Clearly G_p is closed with respect to taking inverses. Thus to prove that G_p is a group, it suffices (by symmetry) to show that if $a \in N_p$ and $b \in L_p$, then $a + b \in G_p$. But $a \leqslant a + b \leqslant b$, and therefore $a + b \in G_p$ because G_p is convex.

Similarly, the set $G_q = \{x \in L : x \wedge p = 0 \text{ or } x \vee (-p) = 0\}$ is a convex, linearly ordered subgroup of L.

2.8. *The subgroup of* L *generated by* G_p *and* G_q *is their cardinal sum* $G_p \oplus G_q$, *and is a convex normal subgroup of* L.

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It is clear from the corresponding properties of L_p and L_q shown in 2.2 above that $G_p \cap G_q = 0$, and that G_p and G_q commute elementwise with each other. Hence the group generated by $G_p \cup G_q$ is their direct sum $G_p \oplus G_q$. It is now clear that $G_p \oplus G_q$ is the difference group of $L_p \oplus L_q$, and the difference group of any normal convex subsemigroup of L^* is a normal convex subgroup of L. But $L_p \oplus L_q$ is normal and convex by 2.2 and 2.6, and hence the same holds for $G_p \oplus G_q$. Finally, to show that $G_p \oplus G_q$ is cardinally ordered, we must show that if $x+y \geqslant 0$, with x in G_p and y in G_q, then $x \geqslant 0$ and $y \geqslant 0$. Since x and y cannot both be strictly negative, we may assume (by symmetry) that $x \geqslant 0$. We must now show that $y \leqslant 0$ implies that $y = 0$. But $y \leqslant 0$ implies that $0 \leqslant x + y \leqslant x \in G_p$, and so $x+y \in G_p$ by convexity. But this and $x \in G_p$ imply that $y \in G_p$, and hence that $y \in G_p \cap G_q = 0$.

2.9. Setting $A = G_p$ and $B = G_q$, we have now established that the subgroup of L generated by A and B is their cardinal sum $A \oplus B$, and is a normal convex subgroup of L. By 2.4, L is a lexico-extension of $A \oplus B$; for if an element of L exceeds every element of $L_p \oplus L_q$, it evidently exceeds every element of $G_p \oplus G_q$. We now show that $C = L/(A \oplus B)$ *is linearly ordered.* Otherwise C would contain two disjoint elements $X = x+(A \oplus B)$ and $Y = y+(A \oplus B)$. Denote by $\bar{0}$ the identity element $A \oplus B$ of C. Since $X > \bar{0}$ and $Y > \bar{0}$, we can assume that x and y are positive elements of L not in $A \oplus B$, and hence exceeding every element of $A \oplus B$. But then $x \wedge y$ exceeds every element of $A \oplus B$. But $X \wedge Y = \bar{0}$ would require $x \wedge y \in A \oplus B$, which is plainly impossible.

3. An example. Let $L = I \times I \times I$, where I is the additive group of integers. For (a, b, c) and (a', b', c') in L we define

$$(a, b, c)+(a', b', c') = \begin{cases}(a+a', b+b', c+c') & \text{if } c' \text{ is even,}\\ (b+a', a+b', c+c') & \text{is } c' \text{ is odd.}\end{cases}$$

We define (a, b, c) to be positive if $c > 0$ or else $c = 0$ and both a and b are $\geqslant 0$. This is the one and only non-abelian splitting lexico-extension of the cardinal sum $I \oplus I$ by I.

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