THE LINEAR j-DIFFERENTIAL EQUATION

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Introduction

The basic reciprocity of j-differential and LM-integral

$$jdy = f(x) \,.\, dg(x) \longleftrightarrow y(x) - y(a) = \int_a^x f(x) \,.\, dg(x)$$

for bounded functions $f(x)$ with simple discontinuities but continuous on the left at each point and for $g(x)$ in the somewhat restricted class B'' of functions of bounded variation and also left-continuous, was established in (**2**) and (**3**); the dot here indicates the lower product of $(\bar{f}, f(x+))$ and $(jg, g'^{+}(x+)dx)$, with $\bar{f} = \frac{1}{2}(f(x)+f(x+))$, and the integral indicated is the $RJDS\sigma$-integral, equivalent to $(LM)\int_a^x fdg$.

We consider now the linear j-differential problem

$$dy = dH(x)y + dE(x)f(x)$$

and show that a solution exists when $H(x)$ and $E(x)$ are in B'' and left-continuous, $f(x)$ continuous (a perhaps unnecessary restriction), and when there is a determinantal condition on jH. The integral equivalent of this problem in the homogeneous case, $y(x)-y(a) = (LM)\int_a^x dHy$, is seen to be not equivalent to the problem $y(x)-y(a) = (Y)\int_a^x dHy$ solved by Hildebrandt in (**1**) because of identity $\int_a^x dH \,.\, y = (Y)\int_a^x dHy + \frac{1}{2}\sum_a^x jHjy$. Nevertheless, our method of solution takes numerous cues from his. Where he would link up semi-open intervals of continuity, in introducing the discontinuities one at a time, by means of the equation $y(x+) = (I+jH)y(x)$, we link them similarly by means of the equation $y(x+) = Jy(x)$ (and supply the ΠJ-theorem to cope with this complication). An equality due to him is a necessary element of our proof and the advantages of his definition of norm and total variation of sets are also here exploited.

1. Definitions

A lower-case letter here represents an individual function but more often a one-index set of functions; an upper-case letter, a two-index set. Following

Hildebrandt, we employ the *norms*:

$$\| y \| = \max_i | y_i |, \quad \| M \| = \max_i \Sigma_j | m_{ij} |,$$

and note that the norm of the sum is less than or equal to the sum of the norms, and similarly for the product. We employ also his definition of *total variation* over $[ax]$, for any lower-case letter:

$$\text{var } y \,|_a^x = \text{l.u.b.}_{\sigma/[ax]} \Sigma_{k/\sigma} \| y(x_{k+1}) - y(x_k) \|,$$

where σ is a finite set of points x_i subdividing $[ax]$ and including the end-points of this interval; for y we may substitute M.

The mean-value inequality. When, for each value of the index h,

$$\lim_\sigma \Sigma_{k/\sigma} \Sigma_i \Sigma_j \bar{w}_{hi}(x_k) \Delta_k m_{ij}(x_k) \bar{v}_j(x_k) = \int_a^x WdMv$$

exists, then

$$\left\| \int_a^x WdMv \right\| \leqq \text{l.u.b.}_{\xi/[ax]} \| W(\xi) \| \,.\, \text{var } M \,|_a^x \,.\, \text{l.u.b.}_{\alpha/[ax]} \| v(\alpha) \|.$$

In particular, if W is in D_1, $M(x)$ is in B and $v(x)$ is continuous, then the integral exists and is an LM-integral; if $W = I$, $M(x)$ is in B and $v(x)$ is in D_1, then

$$\left\| (LM) \int_a^x dMv \right\| \leqq \text{var } M \,|_a^x \,.\, \text{l.u.b.}_{\alpha/[ax]} \| v(\alpha) \|.$$

I call $\boldsymbol{y}$, $\boldsymbol{M}$, $\boldsymbol{dy}$, $\boldsymbol{dM}$ vectorisations of y, M, dy, dM, respectively. The class of such vectorised sets forms a ring, with respect to the norm as above defined, in the case where the product of two vectors is the middle product but not in the cases of the lower and upper products and for this reason all products of such vectors are middle products here unless otherwise indicated, as in the Introduction where the lower product is indicated by a dot.

2. Definitions of J and the ΠJ theorem

The homogeneous vector equation $\boldsymbol{dy} = \boldsymbol{dH}(x)\boldsymbol{y}$ and its adjoint $\boldsymbol{dz} = -\boldsymbol{z}\boldsymbol{dH}(x)$ imply, respectively, the equations

$$y(x+) = Jy(x), \quad z(x+) = z(x)K,$$

in which $J = (I - \frac{1}{2}jH)^{-1}(I + \frac{1}{2}jH)$, $K = (I - \frac{1}{2}jH)(I + \frac{1}{2}jH)^{-1}$. To secure one-to-one correspondence both between $y(x)$ and $y(x+)$ and between $z(x)$ and $z(x+)$ for all values of x, we impose the *discontinuity condition*

$$\det(I \pm \tfrac{1}{2}jH) \neq 0.$$

If this condition holds, both J and K exist on (ab) and $JK = I$.

The following theorem on products of the $J(x_i)$'s and on products of the $K(x_i)$'s, where x_i is a point of any finite set σ which, by the definition of the σ-limit, tends to include all the points of discontinuity in $H(x)$, is required at a later point.

ΠJ theorem. *Each of the limits* $\lim_\sigma \Pi_\sigma J_i$, $\lim_\sigma \Pi_\sigma K_i$ *exists and is not null. When the discontinuity condition is satisfied, each of the limits* $\lim_\sigma \Pi_\sigma \| J_i \|$, $\lim_\sigma \Pi_\sigma \| K_i \|$ *exists and is not zero.*

Proof. Let s be a set of points on $[ab]$ over which the total variation t_{ij} of jh_{ij} is not more than $1/(4m)$ for $i, j = 1, 2, \ldots, m$. For all sums $\sum_{x \in s} \| jH \| \leqq \frac{1}{4}$. Let $(I - \frac{1}{2} jH)^{-1} = I + \frac{1}{2}E$. Then, for all n,

$$\tfrac{1}{2}E(I - (\tfrac{1}{2}jH)^n) = \sum_{k=1}^{n} (\tfrac{1}{2}jH)^k.$$

For x any point of s, $E(x)$ has a convergent series representation and

$$J(x) = I + E(x).$$

Moreover, if $x \in s$, then for any finite sum,

$$\Sigma_x \tfrac{1}{2}E(x) = \Sigma_x \tfrac{1}{2}jH + \Sigma_x(\tfrac{1}{2}jH)^2 + \Sigma_x(\tfrac{1}{2}jH)^3 + \ldots$$

and, for all n, $\Sigma_x(\frac{1}{2} | jH |)^n \leqq (\Sigma_x \frac{1}{2} | jH |)^n \leqq 2^{-n} T_s^n$, where $| jH |$ stands for a matrix each element of which is the absolute value of the corresponding element of jH and where T_s is a matrix each element of which is the total variation of the corresponding element of $jH(x)$ over the set s. If we use the symbol $((a))$ to represent a matrix each element of which is a, then $((1/m))^n = ((1/m))$ for any $(m \times m)$-matrix and natural number n; we have, accordingly,

$$T_s \leqq \tfrac{1}{4}((1/m)), \quad T_s^n \leqq 4^{-n}((1/m)), \quad \sum_{n=1}^{\infty} T_s^n \leqq \tfrac{1}{3}((1/m)).$$

Because of the inequality $\frac{1}{2} | E | \leqq \sum_{n=1}^{\infty} | \frac{1}{2} jH |^n$, we have, for every sum,

$$| \Sigma_{x \in s} E(x) | \leqq \Sigma_{x \in s} | E(x) | \leqq \sum_{n=1}^{\infty} 2T_s^n/2^n \leqq \sum_{n=1}^{\infty} T_s^n \leqq \tfrac{1}{3}((1/m)).$$

We note also that $\| E \| \leqq 2 \| jH \|$ and, because E is a null matrix except at a denumerable set of points, $\| \Sigma_x | E | \| \leqq \Sigma_x \| E \| \leqq \frac{1}{2}$.

The absolute convergence of the series $\Sigma_{x \in s} E(x)$ implies the absolute convergence of the infinite products $\Pi_{x \in s} J(x)$. To see this, suppose we pick points x_i at random among the points of discontinuity of $H(x)$ in s and consider the corresponding products:

$$J(x_1), \quad J(x_1)J(x_2), \quad J(x_1)J(x_2)J(x_3), \quad \ldots$$

i.e. $I + E(x_1)$, $(I + E(x_1))(I + E(x_2))$, $(I + E(x_1))(I + E(x_2))(I + E(x_3))$, ...

Each member of the latter sequence is among the terms of the series

$$I + \sum_{1}^{\infty} E(x_i) + \sum_{j=1+i}^{\infty} \sum_{i=1}^{\infty} E(x_i)E(x_j) + \sum_{k=1+j}^{\infty} \sum_{j=1+i}^{\infty} \sum_{i=1}^{\infty} E(x_i)E(x_j)E(x_k) + \ldots$$

which is dominated by the series

$$I + \sum_{1}^{\infty} | E_i | + \left(\sum_{i=1}^{\infty} | E_i | \right)^2 + \left(\sum_{i=1}^{\infty} | E_i | \right)^3 + \ldots$$

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which itself is dominated by the series $I+\sum_1^\infty (\frac{1}{3})^n((1/m))$. Thus the class of products considered, i.e.

$$\left\{\prod_1^n J(x),\ n=1,2,\ldots,\ x\in s,\ \text{all orders of multiplication}\right\},$$

is seen to be uniformly bounded. Because the elements of $H(x)$ are in B, s differs from the set $[ab]$ at no more than a number of points at each of which $J(x)$ is defined (by the discontinuity condition), and it follows that the infinite product $\prod_1^\infty J(x)$ is uniformly bounded on $[ab]$ with respect to all orders of multiplication.

The boundedness of $|E_1|+|E_2|+\ldots$ implies that of $\Pi(I+|E_i|)$;

$$\therefore \lim_\sigma \Pi_\sigma(I+|E(x)|) \text{ exists}; \quad \therefore \lim_\sigma \Pi_\sigma J(x_i) \text{ exists}.$$

Similarly $\lim_\sigma \Pi_\sigma K(x_i)$ exists. Since $JK=I$ at all points, it follows that $\lim_\sigma \Pi_\sigma J(x_i)\neq 0$, $\lim_\sigma \Pi_\sigma K(x_i)\neq 0$.

We see next that $\lim_\sigma \Pi_\sigma \|J(x)\|$ exists and is not zero. For any number of factors,

$$\Pi_{x\in s}\|J\|\leqq \Pi_x\{1+\|E\|\}\leqq 1+\sum_{n=1}^\infty (\Sigma_x\|E\|)^n\leqq 2; \quad \therefore \lim_\sigma \Pi_\sigma\|J\|<k_1.$$

Similarly, $\lim_\sigma \Pi_\sigma\|K\|<k_2$. Since $1\leqq \Pi\|J\|\,.\,\Pi\|K\|<k_1k_2$, with any number of factors, it is obvious that $k_1\neq 0\neq k_2$.

3. Preliminary existence theory

In the Picard-Liouville succession of LM-integral determinations:

$$v_1(x)=y_0(a)+\int_a^x dH(x)y_0(x)+\int_a^x dEf-y_0(x),$$

$$v_k(x)=\int_a^x dHv_{k-1}, \quad k>1,$$

one finds that $y_0(x)+\sum_1^\infty v_k(x)=y_\infty(x)$ is a solution of the equation

$$y(x)-y(a)=\int_a^x dHy+\int_a^x dEf$$

whenever the series is uniformly convergent. When $H(x)=H_c(x)$ is continuous, the series is both uniformly and absolutely convergent in the case of every arbitrary function $y_0(x)$ which is bounded and with simple discontinuities. For, on letting η_0 and c be constants such that $\|v_1\|<\eta_0$ and $\|H'+(x+)\|<c$, we have that $\Sigma\|v_k\|<\eta_0 e^{c(x-a)}$. The solution is unique with respect to the initial value $y(a)$. In the homogeneous case, with $y_0(x)=y_0=$ const., the series for $y_\infty(x)$ may be written $y_\infty(x)=y(x)=Y(a,x)y_0$; the matric factor Y here is the so-called Peano series and we say that y_0 is a *generator* of this solution. Since y_0 may be given the components of any column of the matrix I, we have that $Y(a, x)I$ is a solution of the equation $Y(a, x) = I + \int_a^x dH_c(\xi)Y(a, \xi)$.

Similarly, the Picard-Liouville process for the problem $dz = -zdH_c(x)$, with the rows of I as generators, gives a solution $Z(a, x)$ of the equation

$$Z = I - \int_a^x ZdH_c.$$

If we multiply the equation $dY = dH(x)Y$ on the left by $Z(x)$, the equation $dZ = -ZdH(x)$ on the right by $Y(x)$, and add, we find that

$$ZdY + dZ\,.\,Y = d(ZY) = 0, \quad \therefore\ Z(x)Y(x) = C.$$

Similar operations on $dy = dH(x)y + dEf$, $dZ = -ZdH(x)$, give

$$Zdy + dZy = d(Zy) = ZdEf, \quad \therefore\ y(x) = Y(x)\int_a^x ZdEf.$$

This particular solution, in which $Y(x)$ and $Z(x)$ are both continuous, has the points of discontinuity of $E(x)$. We now connect Z and Y by the equation $Z(x)Y(x) = I$, C being arbitrary and note that det $Y(x) \neq 0$ on $[ab]$.

4. The case with $H(x) = H_\sigma(x)$ discontinuous at n points of σ.

We define the approximation $H_\sigma(x)$ to any matrix $H(x)$ in B and continuous on the left by the equation

$$H_\sigma(x) = H_c(x) + \Sigma_{\xi \in \sigma/(ax)}[H(\xi+) - H(\xi)]$$

where $H_c(x) = H(x) - \lim_{\sigma/(ax)}\Sigma_{\xi \in \sigma} jH(\xi)$ is obviously continuous; σ is assumed to contain points of discontinuity x_i of $H(x)$ in the open interval (ab).

We shall now find an expression for the solution in the case $H_\sigma(x)$ and show that its σ-limit is the solution in the case $H(x)$. We do this in six steps, i.e. we establish that

(1) the y_σ's are bounded on $[ab]$ uniformly in σ,
(2) they have a limit $y(x)$,
(3) $\lim_\sigma \int_a^x dHy_\sigma$ exists,
(4) $y(x)$ is in D_1 and so $\int_a^x dHy$ exists,
(5) the two integrals have the same value,
(6) $y(x)$ satisfies the equation $dy = dH(x)y$.

Let the points of discontinuity be x_1, $i = 1, 2, \ldots, n$, ordered from left to right and with $a < x_1$, $x_n < b$. Then the P-L process, with $y_0 = \text{const.}$, gives

$$y(x) = Y_c(a, x)y_0, \quad a \leqq x \leqq x_1,$$

a solution for the first interval, generated by y_0, and where the subscript "c" indicates that the variable is in an interval of continuity. In the case of a discontinuity at $x = a$, one replaces $Y_c(a, x)y_0$ by $Y_c(a+, x)J_a y_0$. The generator

$J_1 Y_c(a, x_1)y$ gives the solution for the second interval:

$$y(x) = Y_c(x_1+, x)J_1 Y_c(a, x_1)y_0, \quad x_1 < x \leqq x_2,$$

etc., so that in general, for the $i+1$st interval

$$y(x) = Y_c(x_i+, x)J_i Y_c(x_{i-1}+, x_i)\dots J_2 Y_c(x_1+, x_2)J_1 Y_c(a, x_1)y_0, \ x_i < x \leqq x_{i+1}.$$

This formula defines an m-parameter family of solutions $y_\sigma(y_0;\ x)$ of the equation $\boldsymbol{dy} = \boldsymbol{dH_\sigma}(x)\boldsymbol{y}$.

Let I generate $Y_\sigma(x)$ and, similarly, also generate $Z_\sigma(x)$, i.e.,

$$Z_\sigma(x) = IZ_c(a, x_1)K_1 Z_c(x_1+, x_2)K_2 \dots K_i Z_c(x_i+, x), \quad x_i < x \leqq x_{i+1}.$$

Since $I = Z_c(a, x_1)Y_c(a, x_1) = Z_c(x_1+, x_2)Y_c(x_1+, x_2) = \dots$ etc., and $K_i J_i = I$, for $i = 1, 2, \dots, n$, it follows that, for all x and σ,

$$Z_\sigma(x)Y_\sigma(x) = I,$$

and $Y_\sigma(x)$ is seen to have a reciprocal at all points; thus det $Y_\sigma(x) \neq 0$ on $[ab]$ and the columns of $Y_\sigma(x)$ form a fundamental set of solutions.

In order to establish the existence of a solution in the limiting case, we we invoke the ΠJ-theorem, note that $\lim_{\sigma/(ab)}\Pi_{x_i \in \sigma} \| J(x_i) \| < k$ for some constant k, and show that $\lim_\sigma y_\sigma(x)$ exists. We find first that there is the inequality

$$\| y_\sigma(x) \| \leqq \lim_{\sigma/(ab)}\Pi_{x \in \sigma} \| J(x) \| . \exp \operatorname{var} H_c \Big|_a^b . \| y_0 \|,$$

which follows from applications of the inequality

$$\| Y_c(x_{i-1}+, x_i) \| . \| Y_c(x_i+, x_{i+1}) \| \leqq \exp \operatorname{var} H_c \Big|_{x_{i+1}}^{x_i} . \exp \operatorname{var} H_c \Big|_{x_i}^{x_{i+1}}$$

$$= \exp \operatorname{var} H_c \Big|_{x_{i-1}}^{x_{i+1}},$$

which is the composition of an inequality due to Hildebrandt (**1**, p. 357) and which shows that the y_σ's are bounded on $[ab]$ uniformly in σ.

The limit $y(x) = \lim_\sigma y_\sigma(x)$ may be approached by taking into account one point of discontinuity at a time in an order of non-increasing magnitude of norm of J. In this approach, to say that a point x' is counted as a point of discontinuity in the formula for $y_{\sigma_{n+1}}(x)$ but as a point of continuity in the formula for $y_{\sigma_n}(x)$ is to say that where $y_{\sigma_{n+1}}(x)$ has the factor

$$Y_c(x'+, x'')J(x'')Y_c(x''+, x''\,'),$$

$y_{\sigma_n}(x)$ has the factor $Y_c(x'+, x'')IY_c(x''+, x''\,') = Y_c(x'+, x''\,')$; in the first case $x' = x_i$, $x'' = x_{i+1}$, $x''\,' = x_{i+2}$ with x_i, x_{i+1}, x_{i+2} elements of σ_{n+1} but, in the second, $x' = x_i$, $x''\,' = x_{i+1}$ with x_i, x_{i+1}, but not x'', elements of σ_n. Making use of an inequality due to Hildebrandt (**1**, p. 362), it is seen that there is the inequality

$$\| Y_{\sigma_{n+p}}(x) - Y_{\sigma_n}(x) \| \leqq k' \cdot \sum_{i=n+1}^{n+p} \| J_i - I \|$$

in which k' can be taken to be $\| Y_c(a, x) \| \prod_{i=1}^{\infty} \| J_i \|$. Let k'' be the maximum of $\| [I - \frac{1}{2} jH(x_i)]^{-1} \|$, $x_i \in \sigma/[ab]$. Then, since $J_i - I = (I - \frac{1}{2} j_i H)^{-1} j_i H$ and since $\sum_{i=n+1}^{\infty} \| j_i H \| \leqq \varepsilon$ for all $n > n_0(\varepsilon)$, it follows that, for all natural numbers p,

$$\| Y_{\sigma_{n+p}}(x) - Y_{\sigma_n}(x) \| < k'k''\varepsilon, \quad n > n_0(\varepsilon).$$

Accordingly, the Y_σ's have a limit $Y(x)$ and $y(x) = Y(x)y_0$ exists; similarly, $\lim_\sigma Z_\sigma(x) = Z(x)$ exists. One notes that $Y(x)Z(x) \equiv I$. The limit $y(x)$ satisfies the homogeneous equation because $y_\sigma(x)$ satisfies it approximately. For in the equation

$$y_\sigma(x) - y_0 - \int_a^x dHy_\sigma = \int_a^x d(H_\sigma - H)y_\sigma$$

the right member vanishes in the limit because, by the definition of H_σ,

$$H(x) - H_\sigma(x) = \Sigma_{\xi \in C\sigma/[ax]} jH(\xi)$$

and therefore there is the inequality

$$\| H(x) - H_\sigma(x) \| \leqq \max_i \Sigma_j \Sigma_{\xi \in C\sigma/[ax]} \left| jh_{ij}(\xi) \right| = \beta_\sigma(x)$$

with $\beta_\sigma(x)$ a positive non-decreasing function of x; $\therefore \operatorname{var} (H - H_\sigma)|_a^b \leqq \beta_\sigma(b)$. It is obvious that $\lim_\sigma \beta_\sigma(b) = 0$; $\therefore \lim \operatorname{var} (H - H_\sigma)|_a^b = 0$. But

$$\left\| \int_a^x d(H_\sigma - H)y_\sigma \right\| \leqq \operatorname{var} (H_\sigma - H)|_a^b . \text{l.u.b.}_{\xi/[ab]} \| y_\sigma(\xi) \|$$

and, as has been seen, the y_σ's are bounded on $[ab]$ uniformly in σ; $\therefore \lim_\sigma \int_a^x dHy_\sigma$ exists.

By hypothesis, $H(x)$ is continuous on the left and therefore, from their definition, so are the H_σ's; $\therefore Y_\sigma(x-)$ exists in (ab). Since $H_\sigma(x)$ is discontinuous at isolated points only, $Y_\sigma(x+)$ exists in (ab) and, in fact

$$\| Y_{\sigma'}(x+) - Y_{\sigma''}(x+) \| < \varepsilon$$

for all $\sigma' \supset \sigma_0(\varepsilon)$, $\sigma'' \supset \sigma_0(\varepsilon)$ and therefore $Y(x+)$ exists; $\therefore Y(x)$ and $y(x)$ are in D_1. (One notes also that $Y_\sigma'^{+}(x+)$ exists and is equal to $Y_c'^{+}(a, x+)$ for all σ; $\therefore \lim_\sigma Y_\sigma'^{+}(x+) = Y_c'^{+}(a, x+)$.)

Since $y(x)$ is in D_1, $\int_a^x dHy$ exists. Moreover, $\lim_\sigma \int_a^x dHy_\sigma = \int_a^x dHy$ because of the inequality

$$\left\| \lim_\sigma \int_a^x dHy_\sigma - \int_a^x dHy \right\| \leqq \left\| \lim_\sigma \int_a^x dHy_\sigma - \int_a^x dHy_\sigma \right\| + \left\| \int_a^x dHy_\sigma - \int_a^x dHy \right\|$$

in which the first term on the right is less than ε for $\sigma \supset \sigma_0'(\varepsilon)$ and in which the second term is less, for all x, than

$$\operatorname{var} H|_a^b . \text{l.u.b.}_{\xi/[ab]} \| y_\sigma(\xi) - y(\xi) \|,$$

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a quantity itself less than var $H\,|_a^b\,\varepsilon$ for all $\sigma \supset \sigma_0''(\varepsilon)$.

It is now seen that $y(x)-y_0-\int_a^x dHy = 0$, y_0 an arbitrary set of m numbers, and that there is an m-parameter family of linearly independent solutions of the equation $\boldsymbol{dy} = \boldsymbol{dH}(x)\boldsymbol{y}$.

5. The general case

Theorem. *When $H(x)$ and $E(x)$ are in B'' and are continuous on the left, $f(x)$ is continuous, and $H(x)$ satisfies the discontinuity condition* det $(I \pm \tfrac{1}{2} jH) \neq 0$, *then the problem*

$$\boldsymbol{dy} = \boldsymbol{dH}(x)\boldsymbol{y} + \boldsymbol{dE}(x)f(x)$$

has the general solution

$$y(x) = Y(x)y_0 + Y(x)\int_a^x Y^{-1}(\xi)dE(\xi)f(\xi)$$

in which $Y(x)$ is any solution of the homogeneous problem

$$\boldsymbol{dY} = \boldsymbol{dH}(x)\boldsymbol{Y}$$

which has an inverse at all points.

Proof. The fact that there is such a matrix $Y(x)$ was established in the preceding paragraphs. We first verify that our solution satisfies the jump part of our j-differential equation. At any point x one finds, on the one hand, that

$$\begin{aligned} jy = jY(x)y_0 + Y(x+)\int_a^{x+} ZdEf - Y(x)\int_a^{x+} ZdEf \\ + Y(x)\int_a^{x+} ZdEf - Y(x)\int_a^{x} ZdEf, \\ = jY(x)y_0 + Y(x)\int_x^{x+} ZdEf + jY(x)\int_a^{x+} ZdEf, \end{aligned}$$

but, on the other hand, that

$$\begin{aligned} jy = jY(x)y_0 + Y(x+)\int_a^{x+} ZdEf - Y(x+)\int_a^{x} ZdEf \\ + Y(x+)\int_a^{x} ZdEf - Y(x)\int_a^{x} ZdEf, \\ = jY(x)y_0 + Y(x+)\int_x^{x+} ZdEf + jY(x)\int_a^{x} ZdEf. \end{aligned}$$

Taking the average of these two evaluations, we have

$$\begin{aligned} jy &= jY(x)y_0 + jY(x)\,.\,\overline{\int_a^x ZdEf} + \overline{Y}(x)\overline{Z}(x)jEf, \\ &= jY(x)y_0 + jH\,.\,\overline{Y(x)\int_a^x ZdEf} - \tfrac{1}{4}jHjY + ZdEf + jEf - \tfrac{1}{4}jYjZjEf \\ &= jH(x)\overline{y(x)} + jEf - \tfrac{1}{4}jH[jY\overline{Z} + \overline{Y}jZ]jEf \\ &= jH(x)\overline{y(x)} + jEf, \end{aligned}$$

by the " supplementary relations " of (3) and because $jY\bar{Z}+\bar{Y}jZ \equiv 0$; the result obtained is that required by the jump part of the j-differential equation.

Finally,

$$\lim_{\delta\to+0}\left\{\frac{Y(x+\delta)-Y(x+)}{\delta}\right\} = Y'^{+}(x+)y_0+\lim\frac{1}{\delta}\left\{Y(x+\delta)\int_a^{x+\delta} ZdEf\right.$$

$$\left.-Y(x+)\int_a^{x+\delta} ZdEf+Y(x+)\int_a^{x+\delta} ZdEf-Y(x+)\int_a^{x+} ZdEf\right\}$$

$$= Y'^{+}(x+)y_0+Y'^{+}(x+)\int_a^{x+} ZdEf+E'^{+}(x+)f(x+)$$

$$= H'^{+}(x+)y(x+)+E'^{+}(x+)f(x+),$$

as required by the derivative part of the differential equation.

Note.—Wall (6, Theorem 1) anticipates our final theorem in the continuous case, as does MacNerney's 1954 paper. The latter gives a product-limit solution (4, Theorem 3.5). In his 1955 Elisha Mitchell Scientific Society paper, MacNerney applies a σ-mean-integral to the linear matric integral problem in B but imposes a continuity restriction (5, p. 192, line 12), on the given function corresponding to our $H(x)$, quite different from our requirement of being in B''.

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