# ON ALMOST UNIFORM CONVERGENCE OF FAMILIES OF FUNCTIONS

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In [5] Tolstov showed by a counterexample that Egoroff's theorem on almost uniform convergence cannot be extended to families of functions $\{f_t(x)\}$, with $t$ a continuous real parameter.[1] However, Frumkin [2] proved that this is possible provided that some sets of measure zero (depending on $t$) are disregarded when each particular $f_t(x)$ is considered.[2] This interesting result was obtained by using the rather involved machinery of Kantorovitch's semi-ordered spaces and $L^p$ spaces. In the present note we intend to give a simpler and more general proof. Indeed, it will be seen that only a slight modification of the standard proof of Egoroff's theorem is necessary to obtain Frumkin's theorem in a more general form. We shall establish the following result.

THEOREM. Suppose that, for each real $t$, $f_t(x)$ is a measurable extended real-valued function on a set $E$ $(mE < \infty)$ in an arbitrary measure space,[3] and that, for some $t_o$ $(|t_o| < \infty)$,

[1] Another counterexample was given by J. D. Weston [6]. Both examples are based on the axiom of choice.

[2] The exact formulation of Frumkin's result is as in the theorem which we state below, with the additional restrictions that $m$ is Lebesgue measure on a line interval, all $f_t(x)$ are a.e. finite, and $0 \leq x, t \leq 1$ (so that the case $t_o = \pm\infty$ is excluded).

[3] By a measure space we mean a triple $(S, M, m)$ where $M$ is a $\sigma$-field of subsets of $S$ (i.e., a set-family closed under countable unions and complementation, with $S \in M$), and $m$ is a measure (i.e., a non-negative completely additive set function) defined on $M$.

$f_{t_o}(x)$ is a.e. finite, and $\lim_{t \to t_o} f_t(x) = f_{t_o}(x)$ a.e. on $E$. Then, given $\eta > 0$, there is a measurable set $D \subseteq E$, with $m(E-D) < \eta$, and such that, for every $\varepsilon > 0$, there is a real $q > 0$ such that $\operatorname{essup}_{x \in D} |f_t(x) - f_{t_o}(x)| < \varepsilon$ whenever $|t-t_o| < q$. This holds also for $t_o = +\infty$ ($t_o = -\infty$), with "$|t-t_o| < q$" replaced by "$t > q$" ("$t < -q$", respectively).

Proof. We consider first the case $t_o = +\infty$, and write $f(x)$ instead of $f_{t_o}(x)$. No generality is lost by assuming that $f(x)$ is finite and that $f_t(x) \to f(x)$ everywhere on $E$ (drop a set of measure 0 ! ). With this assumption, we define, for each real $t$ and each positive integer $n$, the (measurable) set

$$(1) \qquad E_t^n = \{x \in E \,|\, |f_t(x) - f(x)| < 2^{-n}\}$$

i.e., the set of all $x \in E$ such that $|f_t(x) - f(x)| < 2^{-n}$. Also, for $k = 1, 2, \ldots$, we put

$$(2) \qquad D_k^n = \bigcap_{t \geq k} E_t^n,$$

the intersection being over all real $t \geq k$, so that $D_k^n$ may not be measurable. Clearly, $D_k^n \subseteq D_{k+1}^n$, $k = 1, 2, \ldots$. Moreover, the convergence $f_t(x) \to f(x)$ easily implies that

$$(3) \qquad E = \bigcup_{k=1}^{\infty} D_k^n, \quad n = 1, 2, \ldots.$$

We now extend the measure $m$ to a (finite) outer measure $m^*$ on all subsets of $E$, setting $m^*A$ = g.l.b. of the measures of all measurable subsets of $E$ which contain $A$. Then, as is well known,[4)] every set $A \subseteq E$ has a measurable cover, i.e.

[4)] Cf. Halmos [3], p. 50 ff.

a measurable superset $H \supseteq A$ such that $m^*A = mH$ and such that all measurable subsets of $H-A$ have measure zero. Moreover, formula (3) combined with the fact that $D^n_k$ increases with $k$, implies that, for each $n$, $\lim_{k\to\infty} m^*D^n_k = mE < \infty$.[5)] Hence, given $\eta > 0$, one can find, for each $n$, a positive integer $k_n$ such that $mE - m^*D^n_{k_n} < \eta/2^{-n}$. Let $D = \bigcap_{n=1}^{\infty} \bar{D}^n_{k_n}$ where $\bar{D}^n_{k_n}$ is the measurable cover of $D^n_{k_n}$. Then, as is readily seen, $m(E-D) < \eta$. Moreover, for each $n$ and $t$, the set $Z^n_t = \bar{D}^n_{k_n} - E^n_t$ is measurable; it is contained in $\bar{D}^n_{k_n} - D^n_{k_n}$ whenever $t \geq k_n$; thus $mZ^n_t = 0$ for $t \geq k_n$. By (1), we have $|f_t(x) - f(x)| < 2^{-n}$ for $x \in E^n_t$, hence for $x \in \bar{D}^n_{k_n} - Z^n_t$ and certainly, for $x \in D - Z^n_t$, with $t \geq k_n$ (so that $mZ^n_t = 0$). Therefore we have $\operatorname{essup}_{x \in D} |f_t(x) - f(x)| \leq 2^{-n}$ for all $n$ and $t \geq k_n$; from this, however, the assertion made in the theorem (for the case $t_o = +\infty$) immediately follows.

The same proof holds also in the cases $t_o = -\infty$ and $|t_o| < \infty$. The only difference is in that, in case $t_o = -\infty$, the intersection in formula (2) is extended over all real numbers $t \leq -k$, while in the finite case it extends over all $t$ such that $|t - t_o| \leq 1/k$.

Thus the proof is complete. It is (admittedly) the standard proof of Egoroff's theorem, but due to this very fact it is considerably shorter and simpler than Frumkin's method. It is also more general since it needs no special restrictions on the measure $m$ (except that $mE < \infty$), does not need the assumption that <u>all</u> the functions $f_t(x)$ are a.e. finite, and admits also

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5) Cf. Munroe [4], p. 94, or Halmos [3], p. 53, Ex. 4.

the case $t_o = \pm\infty$.[6] Moreover, this proof can be extended, with practically no changes, to the Moore-Smith convergence of families $\{f_t(x)\}$ where t is not necessarily a real parameter but runs over some directed set possessing a countable cofinal subset, whereas Frumkin's proof does not permit such a generalization.

## REFERENCES


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[6] These restrictions are necessary in Frumkin's proof since, otherwise, he cannot use $L^p$ spaces.



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