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ON THE DIVISIBILITY OF HOMOGENEOUS DIRECTED GRAPHS

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ABSTRACT Let $\mathcal{T}$ be a finite set of finite tournaments We will give a necessary and sufficient condition for the $\mathcal{T}$-free homogeneous directed graph $H_{\mathcal{T}}$ to be divisible That is, that there is a partition of $H_{\mathcal{T}}$ into two classes such that neither of them contains an isomorphic copy of $H_{\mathcal{T}}$

Introduction. Let T be a finite tournament. A *decomposition* of a tournament T is a partition of the vertex set into three classes L, M, N such that $L \neq \emptyset$ and there are directed edges from x to y and from z to x whenever $x \in L$, $y \in M$ and $z \in N$; we write $T = (L, M, N)$ to indicate such a decomposition. We assume that $1 \leq |L| \leq |T|$. If a finite tournament S together with a partition of S into two classes A and B is given we will indicate this by saying that $[A, B]$ is a *partitioned tournament*. Two partitioned tournaments $[A_0, B_0]$ and $[A_1, B_1]$ are isomorphic if there exists a tournament isomorphism from $[A_0, B_0]$ onto $[A_1, B_1]$ which also preserves the partition. If (L, M, N) is a decomposition of a finite tournament, then $p(L, M, N)$ is the partitioned tournament $[M, N]$.

Let $\mathcal{T}$ be a finite set of finite tournaments, such that no element of $\mathcal{T}$ is a subtournament of an other tournament of $\mathcal{T}$. If $[A, B]$ is a partitioned tournament then:

$\mathcal{T}_{[A,B]} = \{L$: there is a decomposition (L, M, N) of some element of $\mathcal{T}$ such that $p(L, M, N)$ is isomorphic to $[A, B]\}$.

The set of minimal elements of $\mathcal{T}_{[A,B]}$ is called a *derived set* of $\mathcal{T}$. Clearly, there are only finitely many derived sets of $\mathcal{T}$ and every derived set of $\mathcal{T}$ is finite. If $\mathcal{R}$ and $\mathcal{S}$ are two sets of tournaments, we write $\mathcal{R} \prec \mathcal{S}$ iff for every $R \in \mathcal{R}$ there is an $S \in \mathcal{S}$ such that S is a subtournament of R. Observe that the relation $\prec$ is transitive.

$H_{\mathcal{T}}$ is the countable homogeneous directed graph which embeds every finite $\mathcal{T}$-free directed graph [4]. We say $H_{\mathcal{T}}$ is *indivisible*, if for every partition of $H_{\mathcal{T}}$ into two classes, one of the classes contains an isomorphic copy of $H_{\mathcal{T}}$. The main result of this paper is the following theorem:

THEOREM. *If $\mathcal{T}$ is a finite set of finite tournaments, then $H_{\mathcal{T}}$ is indivisible if and only if the set of derived sets of $\mathcal{T}$ is totally ordered under the relation $\prec$.*

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1. **Preliminaries.** A directed graph, or digraph, D is a a set of vertices $V(D)$ together with a set of directed edges or arcs $\vec{E}(D) \subset V(D) \times V(D)$. The arc (a, b) is directed from a to b and is denoted by $\overrightarrow{ab}$. D contains no loops and at most one of $\overrightarrow{ab}$, $\overrightarrow{ba}$ is an arc of D. A *tournament* is a digraph in which every pair of vertices is linked by an arc. A digraph G is a *subdigraph* of D if $V(G) \subset V(D)$ and $\vec{E}(G) = (G \times G) \cap E(D)$. We say that D *embeds* G if it contains an induced subdigraph isomorphic to G. This will be denoted by $G \longrightarrow D$.

If $G \not\longrightarrow D$ then we say that D *omits* G or is *G-free*. If $\mathcal{A}$ is a set of digraphs we say D is *$\mathcal{A}$-free* if D is A-free for every $A \in \mathcal{A}$. A countable digraph D is *homogeneous* if every isomorphism (local automorphism) $\alpha: A \longrightarrow B$ between finite subdigraphs A, B of D extends to an automorphism of D; (see [4], page 313). Lachlan [7] has classified the countable homogeneous tournaments, and Schmerl has classified the countable homogeneous partially ordered sets [11], and independently, Henson [5] and Peretyiatkin [9] have shown the existence of $2^{\aleph_0}$ nonisomorphic homogeneous digraphs. Recently, Cherlin classified all homogeneous directed graphs [1].

A digraph D is *indivisible* if for every partition $V(D) = R \cup B$ there is an isomorphism $f: D \longrightarrow D$ such that either $f(D) \subset R$ or $f(D) \subset B$. (Excellent references for this and related concepts are [4] and [10]). The problem of classifying all countable divisible homogeneous undirected graphs has been completely solved [2] and [6]. In [8] all countable homogeneous undirected graphs have been classified.

Let D be a countable homogeneous digraph. The *age* of D, denoted by $\mathcal{A}(D)$, is the set of all finite digraphs (up to isomorphism) that can be embedded in D. It is well known that $\mathcal{A}(D)$ has the following *amalgamation property*:

(AP) if $A_0, A_1, A_2 \in \mathcal{A}(D)$ and $f_i: A_0 \longrightarrow A_i$ $(i = 1, 2)$ are embeddings then there is a digraph $A \in \mathcal{A}(D)$ and embeddings $g_i: A_i \longrightarrow A$ $(i = 1, 2)$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

In fact a countable class Σ of finite digraphs is the age of some countable homogeneous digraph if and only if Σ satisfies AP and is closed under taking subdigraphs. Furthermore, two homogeneous digraphs with the same age are isomorphic. Another characterizing property of homogeneous digraphs is the following *embedding property*:

(EP) if $A \in \mathcal{A}(D)$ and $x \in A$ then every embedding $f: A - x \longrightarrow D$ extends to an embedding $g: A \longrightarrow D$.

$\mathcal{A}(D)$ is called *indivisible* if for every partition of D into two classes, R and B, one of the classes embeds every member of $\mathcal{A}(D)$. It is a standard compactness argument to show that indivisibility of $\mathcal{A}(D)$ is equivalent to the following Ramsey property [10]. For every $A_1, \ldots, A_n \in \mathcal{A}(D)$ there exists $B \in \mathcal{A}(D)$ such that for every partition $B = B_1 \cup \cdots \cup B_n$ there is an i such that $A_i \longrightarrow B_i$ [10]. Folkman, [3], has proven that the set of K_n-free undirected graphs is indivisible.

We shall consider the following strengthening of AP called the *free amalgamation property*:

(FAP) if $A_0, A_1, A_2 \in \mathcal{A}(D)$ and $f_i: A_0 \longrightarrow A_i$ $(i = 1, 2)$ are embeddings then there is $A \in \mathcal{A}(D)$ and embeddings $g_i: A_i \longrightarrow A$ $(i = 1, 2)$ such that the following are satisfied:

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(a) $g_1 \circ f_1 = g_2 \circ f_2$
(b) if $x_1 \in g_1\big(A_1 - f_1(A_0)\big)$ and $x_2 \in g_2\big(A_2 - f_2(A_0)\big)$ then $x_1 \neq x_2$ and none of $\overrightarrow{x_1x_2}$, $\overrightarrow{x_2x_1}$ is an arc of A.

REMARK. We shall say that $g_1(A_1) \cup g_2(A_2)$ is obtained from A_1 and A_2 by free amalgamation over $f_1(A_0) \approx f_2(A_0)$.

A countable homogeneous digraph whose age has the FAP property will be called a *freely amalgamated homogeneous* (FAP) digraph. Necessarily, every FAP digraph is infinite.

Let $\bar{\mathcal{T}}(D) = \{T : T \text{ is a tournament and } T \in \mathcal{A}(D)\}$. Obviously $\bar{\mathcal{T}}(D)$ is hereditary, *i.e.* is closed under forming subtournaments.

LEMMA 1. *Let Σ be a class of tournaments closed under forming subtournaments. Then there is a unique FAP digraph D such that $\bar{\mathcal{T}}(D) = \Sigma$.*

PROOF. Let Σ^* be the class of finite digraphs A satisfying: Every tournament embeddable into A belongs to Σ. Clearly Σ^* is closed under taking subgraphs and has FAP since the free amalgamation creates no new tournaments. Hence Σ^* is the age of a unique FAP digraph D which also satisfies $\bar{\mathcal{T}}(D) = \Sigma$. The uniqueness of D follows from the fact that every FAP digraph D for which $\Sigma \subset \mathcal{A}(D)$ must satisfy $\Sigma^* \subset \mathcal{A}(D)$. ■

The previous lemma asserts that a FAP digraph is characterised by the set of tournaments which it embeds. This can be re-stated in terms of a set of finite forbidden tournaments as follows.

Let $\mathcal{T}(D) = \{T : T \text{ is a tournament minimal w.r.t. being non-embeddable in } D\}$. Then no member of $\mathcal{T}(D)$ is embeddable into any other member and for every class of tournaments Σ with this property there exists a unique FAP digraph D such that $\Sigma = \mathcal{T}(D)$. Or, in other words, if $\mathcal{T}$ is a set of finite tournaments then there exists exactly one FAP digraph $H_{\mathcal{T}}$ which is $\mathcal{T}$-free. The age of $H_{\mathcal{T}}$ is the set of all finite $\mathcal{T}$-free digraphs. Due to the uniqueness property of homogeneous structures [4], $H_{\mathcal{T}}$ can also be characterised as the unique countable homogeneous digraph whose age consists of all finite $\mathcal{T}$-free digraphs.

FAP digraphs have many useful properties which are not true in general for homogeneous digraphs. One such property which the reader can easily verify is that the extension g in the aforementioned embedding property EP can be chosen in infinitely many ways. Other properties are stated below.

LEMMA 2. *Let D be a homogeneous digraph with the FAP and X a finite subset of D. Then the subdigraph induced by*

$$N(X) = \Big\{y \in D - X : \forall x \in X\big(\overrightarrow{xy} \notin \vec{E}(D) \wedge \overrightarrow{yx} \notin E(D)\big)\Big\} \text{ is isomorphic to } D.$$

PROOF. Let $A \in \mathcal{A}(D)$ and $z \in A$. We show that every embedding $f: A - z \longrightarrow N(X)$ extends to an embedding of A into $N(X)$. Let G be the digraph consisting of the disjoint

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union of X and A with no further arcs. G is obtained by freely amalgamating A and X over the empty set. Therefore $G \in \mathcal{A}(D)$. Let $i: X \longrightarrow D$ be the inclusion map, then $f \cup i$ embeds $G - z$ into D and therefore can be extended to an embedding $g: G \longrightarrow D$. The restriction $g \lceil A$ is the required extension of f. ∎

DEFINITION. A homogeneous digraph D is called *weakly indivisible* if it satisfies the following:

(WI) for every $A \in \mathcal{A}(D)$ and every $X \subset D$, if $A \not\rightarrow X$ then $D \longrightarrow D - X$.

LEMMA 3. *Let D be a FAP digraph. Then D is weakly indivisible.*

PROOF. Let $A \in \mathcal{A}(D)$ and $X \subset D$ be such that $A \not\rightarrow X$. We use induction on the cardinality of A assuming that $|A| \geq 2$. Let $B \in \mathcal{A}(D)$, $z \in B$ and $f: B - z \longrightarrow D - X$ be an embedding. Let $x \in A$. By the induction hypothesis, the statement (WI) holds for $A - x$. Therefore, by Lemma 2, we can assume that there exists an embedding $g: A - x \longrightarrow X \cap N\big(f(B - z)\big)$. Consider the digraph G obtained from A and B by free amalgamation over $\{x\} \approx \{z\}$. Let x_0 be the image of both x and z in G. the map $f \cup g$ is an embedding of $G - x_0$ into D and therefore it extends to an embedding $h: G \longrightarrow D$. Clearly, $h(x_0) \in D - X$. This proves that f can be extended to an embedding of $f': B \longrightarrow D - X$ by letting $f'(z) = h(x_0)$. ∎

Clearly the weak indivisibility of a homogeneous digraph D implies that $\mathcal{A}(D)$ is indivisible. So we have the following theorem which extends Folkman's theorem [3] to the case of digraphs.

THEOREM 4. *Let Σ be a class of tournaments. Then for every set of finite Σ-free digraphs $G_1, \ldots, G_n$ there exists a finite Σ-free digraph H such that for every partiton of H into $H_1 \cup \cdots \cup H_n$, there exists an $i \leq n$ and an embedding $f_i: G_i \longrightarrow H_i$.*

2. **The indivisibility of FAP digraphs.** Let $\mathcal{T}$ be a set of finite tournaments and put $D = H_{\mathcal{T}}$. We shall assume that D is defined on N the set of positive integers. For each vertex $x \in D$ let

$$\Gamma'(x) = \{y : y < x \text{ and } \overrightarrow{xy} \in \vec{E}(D)\},$$
$$\Gamma''(x) = \{y : y < x \text{ and } \overrightarrow{yx} \in \vec{E}(D)\},$$
$$\Gamma(x) = \big(\Gamma'(x), \Gamma''(x)\big).$$

For $A, B \subset N$ we write $A < B$ if $\max A < \min B$. For $m \in N$, we denote by $[m] := \{k \in \omega : k \leq m\}$. Let $\mathcal{F}$ denote the class

$$\mathcal{F} = \{(A, B) : A, B \text{ are finite subsets of } \omega \text{ and } A \cap B = \phi\}.$$

For $(A, B), (A', B') \in \mathcal{F}$ we write $(A, B) \subset (A', B')$ if $A \subset A'$ and $B \subset B'$. For a set A of positive integers, we also write A to denote the digraph induced by D on A. An embedding $f: (A, B) \longrightarrow (A', B')$ will always mean an embedding $f: A \cup B \longrightarrow A' \cup B'$ which sends A into A' and B into B'. (A, B), (A', B') are isomorphic, $(A, B) \approx (A', B')$, if there is an embedding f $(A,B) \longrightarrow (A',B')$ such that the restrictions of f to A and B are graph-isomorphisms

For $(A,B) \in \mathcal{F}$ we define

$$\mathcal{C}(A,B) = \left\{x \in \omega \quad \forall a \in A\, \forall b \in B\left(\overrightarrow{xa} \in \vec{E}(D) \wedge \overrightarrow{bx} \in \vec{E}(D)\right)\right\}$$

Note that $\mathcal{C}(A,B) \cap (A \cup B) = \emptyset$

LEMMA 5 *For all $(A,B) \in \mathcal{F}$, $\mathcal{C}(A,B)$ is an FAP digraph*

PROOF If $(A,B) = (\emptyset,\emptyset)$ then $\mathcal{C}(A,B) = D$ by definition Let $(A,B) \neq (\emptyset,\emptyset)$ Let $M,N \subset \mathcal{C}(A,B)$ and f $M \longrightarrow N$ be an isomorphism Let ι be the identity map on $A \cup B$ Then $f \cup \iota$ $A \cup B \cup M \longrightarrow A \cup B \cup N$ is an isomorphism Then there is an automorphism σ $D \longrightarrow D$ which extends $f \cup \iota$ Since σ fixes both A and B, it must map $\mathcal{C}(A,B)$ onto itself The restriction $\sigma\lceil\mathcal{C}(A,B)$ extends f to an automorphism of $\mathcal{C}(A,B)$ This means that $\mathcal{C}(A,B)$ is a homogeneous digraph To show that $\mathcal{C}(A,B)$ is an FAP digraph, let M,N,N',L be finite subgraphs of $\mathcal{C}(A,B)$ where N,N' are isomorphic and N is a subgraph of M and N' is a subgraph of L To amalgamate M and L freely over $N \approx N'$ we simply amalgamate $M \cup A \cup B$ and $L \cup A \cup B$ freely over $A \cup B \cup N \approx A \cup B \cup N'$ and then discard the elements of $A \cup B$ ■

We define a preorder on $\mathcal{F}$ by letting $(A,B) \prec (A',B')$ if there exists an embedding $\mathcal{C}(A',B') \longrightarrow \mathcal{C}(A,B)$ Obviously $(A,B) \subset (A',B')$ implies that $(A,B) \succ (A',B')$ We now state

THEOREM 6 *If D is indivisible then $\prec$ is a total preorder on $\mathcal{F}$*

PROOF Assume for a contradiction that $(A,B),(A',B') \in \mathcal{F}$ are not $\prec$ comparable Then $(A,B),(A',B') \in \mathcal{F}$ and $\mathcal{C}(A,B),\mathcal{C}(A',B')$ are both non-empty FAP digraphs by Lemma 5 For any pair of pairs $(E,F),(E',F') \in \mathcal{F}$ we write $(E,F) <_1 (E',F')$ if $\max(E \cup F)\Delta(E' \cup F') \in E' \cup F'$ Here Δ is the symmetric difference operator Observe that $<_1$ imposes a total order on the elements of $\mathcal{F}$ We shall define a partition $D = D_1 \cup D_2 \cup D_3$ as follows Let $x \in D$, then

(a) $x \in D_1$ if there exists $(E,F) \subset \Gamma(x)$ such that $(E,F) \approx (A,B)$ and $(E,F) <_1 (E',F')$ for every $(E',F') \subset \Gamma(x)$ satisfying $(E',F') \approx (A',B')$

(b) $x \in D_2$ if there exists $(E',F') \subset \Gamma(x)$ such that $(E',F') \approx (A',B')$ and $(E',F') <_1 (E,F)$ for every $(E,F) \subset \Gamma(x)$ with $(E,F) \approx (A,B)$

(c) $x \in D_3$ otherwise

We shall show that none of D_1, D_2 and D_3 embeds D First we observe that every embedding σ $D \longrightarrow D_\iota$ can be assumed to be orderpreserving The reason for this is that every isomorphism f $C - x \longrightarrow D$, where $x \in C \in \mathcal{A}(D)$, can be extended to an isomorphism g $C \longrightarrow D$ in infinitely many ways This implies that for every embedding σ $D \longrightarrow D_\iota$, we can define another order-preserving embedding σ_1 $D \longrightarrow \sigma(D)$ which we might as well consider instead of σ

(a) D_1 contains no isomorphic copy of D Assume that σ $D \longrightarrow D_1$ is an embedding Let $y \in \mathcal{C}(A',B')$ be such that $y > \max(A' \cup B')$ and put $z = \sigma(y)$ Then $z \in \mathcal{C}(M,N)$

where $(M,N) = \sigma(A',B') = \big(\sigma(A'),\sigma(B')\big)$. Since $z \in D_1$ there must exist $(E,F) \subset \Gamma(z)$ such that $(E,F) \approx (A,B)$ and $(E,F) <_1 (M,N)$. There are only finitely many such (E,F) which we enumerate by $(E_1,F_1),\ldots,(E_k,F_k)$. This defines a partition $\mathcal{C}(M,N) = C_1 \cup \cdots \cup C_k$ where $y \in C_j$ if $\min\{i : y \in \mathcal{C}(E_i,F_i)\} = j$.

Since the age of $\mathcal{C}(M,N)$ is indivisible, Lemma 3 and Lemma 5, there is a class, say C_j which embeds every element of $\mathcal{A}\big(\mathcal{C}(M',N')\big) = \mathcal{A}\big(\mathcal{C}(A,B)\big)$. Therefore $\mathcal{A}\big(\mathcal{C}(A',B')\big) \subset \mathcal{A}\big(\mathcal{C}(A,B)\big)$. This implies that $\mathcal{C}(A,B) \longrightarrow \mathcal{C}(A',B')$ by the homogeneity of $\mathcal{C}(A',B')$. Hence we arrived at a contradiction to our assumptions. The proof that D_2 contains no isomorphic copy of D is similar.

(b) D_3 contains no isomorphic copy of D. This is easy to see since D_3 does not contain a vertex y such that $(A,B) \longrightarrow \Gamma(y) \cap D_3$. ∎

Before we proceed to discuss the sufficiency of the condition in Theorem 6, we investigate this condition further in terms of the set of tournaments forbidden in D.

We wish to describe $\mathcal{C}(A,B)$ in terms of (A,B). From Lemma 5, $\mathcal{C}(A,B)$ is a FAP and therefore is characterized by its set of forbidden tournaments $\mathcal{T}\big(\mathcal{C}(A,B)\big)$. Let $T \in \mathcal{T}(D)$ and assume that (K,M,N) is a decomposition of T. If $(M,N) \longrightarrow (A,B)$ then, clearly $K \not\longrightarrow \mathcal{C}(A,B)$. It is also true that every tournament P for which $\mathcal{C}(A,B)$ is P-free must arise in this way. Let $L(A,B) = \{K : \exists T \in \mathcal{T}(D)$ with decomposition (K,M,N) such that $(M,N) \longrightarrow (A,B)$ is an embedding$\}$. Then $\mathcal{T}\big(\mathcal{C}(A,B)\big)$ is exactly the set of minimal (w.r.t. embedding) tournaments in $L(A,B)$.

LEMMA 7. *$(A,B) \prec (A',B')$ if and only if for each $L \in L(A,B)$ there exists $L' \in L(A',B')$ such that there is an embedding $L' \longrightarrow L$.*

PROOF. There is an embedding $\mathcal{C}(A',B') \longrightarrow \mathcal{C}(A,B)$ if and only if every tournament L in $L(A,B)$ satisfies $L \not\longrightarrow \mathcal{C}(A',B')$, that is, there exists $L' \in L(A',B')$ such that $L' \longrightarrow L$. ∎

Observe now that if the derived sets of $\mathcal{T}$ are totally ordered, then the intersection of a set of derived sets is again a derived set. But this means that for $(A,B) \in \mathcal{F}$, $L(A,B)$ is a derived set of $\mathcal{T}$. Furthermore every derived set is equal to some $L(A,B)$. Hence we have observed: The derived sets of $\mathcal{T}$ form a total order under $\prec$ if and only if $\prec$ is a total preorder of the pairs (A,B) of $\mathcal{F}$. If $D = H_{\mathcal{T}}$ is indivisible, then the set of derived sets of $\mathcal{T}$ form a total order under $\prec$. This together with Lemma 6 and Theorem 7 establishes the theorem stated in the introduction.

3. The proof of the sufficiency of the condition.

Assume now that the relation $\prec$ is a total preorder on $\mathcal{F}$. Let $\sim$ denote the equivalence relation defined on $\mathcal{F}$ by $(A,B) \sim (A',B')$ if and only if $\mathcal{C}(A,B) \cong \mathcal{C}(A',B')$. Then $(\mathcal{F}|\sim,\prec)$ is a linear order. We prove the converse of Theorem 6 under the assumption that this linear order is finite.

THEOREM 7. *Let $(\mathcal{F}|\sim,\prec)$ be a finite total order. Then D is indivisible.*

PROOF. Assume that the vertices of D are colored red and blue. We must show that one of the two color classes contains an isomorphic copy of D. By the hypothesis there is an integer $n \geq 1$ and a function $\rho\colon \mathcal{F} \longrightarrow \{1,2,\ldots,n\}$ such that $\rho(A,B) < \rho(A',B')$ if and only if $(A,B) \prec (A',B')$ but $(A',B') \not\prec (A,B)$. We also assume that the range of ρ is an initial interval of ω with length n. We call ρ the *rank function* for $\mathcal{F}$. Let $(A,B) \in \mathcal{F}$ and let $m = \max(A \cup B)$. If $\rho(A - \{m\}, B - \{m\}) < \rho(A,B)$ then (A,B) will be called *rank-critical*. Let

$$\mathcal{H} = \{(A,B,\alpha) : (A,B) \in \mathcal{F},\ \alpha \in D,\ \max(A \cup B) < \alpha\}.$$

For $E = (A,B,\alpha) \in \mathcal{H}$, we let $\Pi_1(E) = A$, $\Pi_2(E) = B$ and $\Pi_3(E) = \alpha$. We also define $\rho(E) = \rho(A,B)$, $\mu(E) = \min(A \cup B)$ and $\mathcal{C}(E) = \{y \in D : y > \alpha$ and $\Gamma'(y) \cap [\alpha] = A$ and $\Gamma''(y) \cap [\alpha] = B\}$. For $E' = (A',B',\alpha') \in \mathcal{H}$, we define $E \cup E' = (A \cup A', B \cup B', \max\{\alpha, \alpha'\})$, and write $E < E'$ whenever $\alpha < \mu(E')$. For each $x \in D$, let $\varphi(x)$ denote the following formula:

$$(\exists E_1 \in \mathcal{H})\big(x = \mu(E_1) \wedge \rho(E_1) = 1\big) \text{ such that}$$
$$(\forall F_1 \in \mathcal{H})\big(E_1 < F_1 \wedge \rho(E_1 \cup F_1) = 1\big)$$
$$\vdots$$
$$(\exists E_i \in \mathcal{H})\big(F_{i-1} < E_i \wedge \rho(E_1 \cup F_1 \cup E_2 \cup \cdots \cup E_i) = i\big) \text{ such that}$$
$$(\forall F_i \in \mathcal{H})\big(E_i < F_i \wedge \rho(E_1 \cup F_1 \cup E_2 \cup F_2 \cup \cdots \cup E_i \cup F_i) = i\big)$$
$$\vdots$$
$$(\exists E_n \in \mathcal{H})\big(F_{n-1} < E_n \wedge \rho(E_1 \cup F_1 \cup \cdots \cup E_n) = n\big) \text{ such that}$$
$$(\forall F_n \in \mathcal{H})\big(E_n < F_n \wedge \rho(E_1 \cup F_1 \cup \cdots \cup E_n \cup F_n) = n\big)$$
$$\text{the set } (\mathcal{C}(E_1 \cup F_1 \cup \cdots \cup F_n) \text{ contains infinitely many blue vertices.}$$

Let $\psi(x)$ denote the formula obtained from $\varphi(x)$ by interchanging the quantifiers $\exists$, $\forall$ and replacing the word 'blue' by 'red'. It is clear that for each $x \in D$ at least one of $\varphi(x)$ and $\psi(x)$ holds. x is called a *blue generator* if $\varphi(x)$ holds and otherwise x will be called a *red generator*. The proof will be divided into two cases.

CASE 1. There are infinitely many blue generators.

We shall construct a sequence $B_1 < \sigma(1) < B_2 < \sigma(2) < \cdots$ of elements of D such that

(1) $B_1, B_2, \ldots$ are blue generators;
(2) the vertices $\sigma(1), \sigma(2), \ldots$ are colored blue;
(3) the map $\sigma\colon D \longrightarrow D$ sending k into $\sigma(k)$ is an embedding.

Certain sets will be 'squeezed' between the elements of the above sequence; these are exactly $\Pi_1(E_i)$, $\Pi_2(E_i)$ for the triples E_i obtained from the formulas $\varphi(B_j)$. We shall always use sets of the form $\{\sigma(i_1), \ldots, \sigma(i_\ell)\}$ as $\Pi_1(F_i)$, $\Pi_2(F_i)$ in these formulas. The elements $\Pi_3(E_i)$, $\Pi_3(F_i)$ will be called *constraints* and will be used to ensure that the constructed sets and elements are disjoint. Let us describe this construction in more detail.

Assume we have just chosen B_k. We then introduce $\Pi_1\big(E_1(B_k)\big)$, $\Pi_2\big(E_1(B_k)\big)$ into our construction. We call $\Pi_3\big(E_1(B_k)\big)$ the *current constraint*. Any element or set to be subsequently included in the construction must be larger than the current constraint. Before we construct $\sigma(k)$ we consider all pairs $(C, H) \in \mathcal{F}$ satisfying

(4) $\max(C \cup H) < k$;

(5) either $(C \cup \{k\}, H)$ or $(C, H \cup \{k\})$ is rank-critical and let t_{j+1} be k.

Let (C, H) be such a pair, say $(C \cup \{k\}, H)$ is *rank-critical*. Let $t_1 = \min(C \cup H)$ and denote by $t_2 < \cdots < t_j$ those vertices $t \in C \cup H$ for which $(C \cap [t], H \cap [t])$ is rank-critical. Put

(6) $C_i = \{x \in C : t_i \leq x < t_{i+1}\}$ and

(7) $H_i = \{x \in H : t_i \leq x < t_{i+1}\}$ for all i with $1 \leq i \leq j$.

We call $t_1, \ldots, t_j$ the *critical vertices* of (C, H) and $(C_1, H_1), \ldots, (C_j, H_j)$ the partition of (C, H) corresponding to them. The construction is such that for suitable $\beta_i \in D$, the triples $F_i = \big(\sigma(C_i), \sigma(H_i), \beta_i\big)$, $1 \leq i \leq j-1$, have been used in the formula $\varphi(B_{t_1})$ to create triples $E_i = E_i(B_{t_1}; F_1, \ldots, F_{i-1})$, $1 \leq i \leq j-1$. The sets $\Pi_1(E_i)$, $\Pi_2(E_i)$ were included in the construction such that they lie in the interval between B_{t_i} and $\sigma(t_i)$. Now choose β_j larger than the current constraint. Put $F_j = \big(\sigma(C_j); \sigma(H_j), \beta_j\big)$. From the formula $\varphi(B_{t_1})$ we obtain $E_{j+1} = E_{j+1}(B_{t_1}, F_1, \ldots, F_j)$. We then include $\Pi_1(E_{j+1})$, $\Pi_2(E_{j+1})$ in the construction and take $\Pi_3(E_{j+1})$ as the new constraint. In order to be able to apply $\varphi(B_{t_1})$ we have assumed that

(8) $\rho(E_1 \cup F_1 \cup \cdots \cup E_i \cup F_i) = \rho(E_1 \cup F_1 \cup \cdots \cup E_i)$, $1 \leq i \leq j$;

(9) $\rho(E_1 \cup F_1 \cup \cdots \cup F_i \cup E_{i+1}) = \rho(E_1 \cup F_1 \cup \cdots \cup E_i) + 1$, $1 \leq i \leq j-1$.

We shall show later that condition (8) always holds. However, if (9) is not satisfied then some modification is needed. To demonstrate this, let us assume that $\rho(C \cup \{k\}, H) = \rho(C, H) + r$ where $r \geq 2$. The triple E_{j+1} above satisfies

$$\rho(E_1 \cup F_2 \cup \cdots \cup F_j \cup E_{j+1}) = \rho(E_1 \cup F_1 \cup \cdots \cup F_j) + 1 = \rho(C, D) + 1.$$

The second equality follows from (8). We want to replace E_{j+1} by a triple E^*_{j+1} which satisfies

$$\rho(E_1 \cup F_1 \cup \cdots \cup F_j \cup E^*_{j+1}) = \rho(C \cup \{k\}, H).$$

We recursively choose arbitrary triples $F'_{j+1}, \ldots, F'_{j+r-1}$ and apply $\varphi(B_{t_1})$ to get $E'_{j+2}, \ldots, E'_{j+r}$ where

$$E_{j+1} < F'_{j+1} < E'_{j+2} < \cdots < F'_{j+r-1} < E'_{j+r},$$

$$\rho(E_1 \cup F_1 \cup \cdots \cup E_{j+1} \cup F'_{j+1} \cup E'_{j+2} \cup \cdots \cup E'_{j+\ell} \cup F'_{j+\ell})$$
$$= \rho(E_1 \cup F_1 \cup \cdots \cup E_{j+1} \cup F'_{j+1} \cup E'_{j+2} \cup \cdots \cup E'_{j+\ell}).$$

That is, after having found $E'_{j+\ell}$ we choose $F'_{j+\ell}$ so that $\Pi_1(F'_{j+\ell}) = \Pi_2(F'_{j+\ell}) = \emptyset$ and $\Pi_3(F'_{j+\ell}) > \Pi_3(E'_{j+\ell})$. Then the existential quantifier in line $2(j+\ell)$ of the formula $\varphi(B_{t_1})$ will produce $E'_{j+\ell+1}$. We then put $E^*_{j+1} = E_{j+1} \cup F'_{j+1} \cup \cdots \cup F'_{j+r-1} \cup E'_{j+r}$. We include $\Pi_1(E^*_{j+1})$, $\Pi_2(E^*_{j+1})$ in our construction and consider $\Pi_3(E^*_{j+1})$ as the new constraint. We repeat this procedure for each (C, H) satisfying (4) and (5). We then proceed to construct $\sigma(k)$.

If $\Gamma(k) = (\emptyset, \emptyset)$ then we choose a blue vertex $y \in D$ such that $\big(\Gamma'(y) \cap [\beta], \Gamma''(y) \cap [\beta]\big) = (\emptyset, \emptyset)$ where β is some element in D larger than the current constraint. The existence of such a y follows from Lemma 2. Then we put $\sigma(k) = y$.

Assume now that $\Gamma(k) = (C, H) \neq (\emptyset, \emptyset)$. Let $t_1 < \cdots < t_j$ be the critical points of (C, H) and (C_i, H_i), $1 \leq i \leq j$, be the corresponding partition of (C, H). It follows from the above construction that the triples $F_i = \big(\sigma(C_i), \sigma(H_i), \beta_i\big)$, $1 \leq i \leq j-1$, have been used in $\varphi(B_{t_1})$ to induce triples $E_i = E_i(B_{t_1}, F_1, \ldots, F_{i-1})$, $1 \leq i \leq j$. Assume that

$$\rho(E_1 \cup F_1 \cup \cdots \cup E_j \cup F_j) = \rho(E_1 \cup F_1 \cup \cdots \cup E_j).$$

Then, choose an element $\beta_j \in D$ larger than the current constraint. Put $F_j = (C_j, H_j, \beta_j)$. The last line of $\varphi(B_{t_1})$ says we can choose a blue element $y \in D$, $y > \beta_j$ such that $y \in \mathcal{C}(E_1 \cup F_1 \cup \cdots \cup F_j)$. Observe that this implies that

$$\big(\Gamma'(y) \cap \{\sigma(1), \ldots, \sigma(k-1)\}, \Gamma''(y) \cap \{\sigma(1), \ldots, \sigma(k-1)\}\big) = \big(\sigma(C), \sigma(H)\big).$$

Then we put $\sigma(k) = y$ and continue the construction by choosing a blue generator $B_{k+1} > \sigma(k)$. Thus we have to prove that

$$\rho(E_1 \cup F_1 \cup \cdots \cup E_i \cup F_i) = \rho(E_1 \cup F_1 \cup \cdots \cup E_i), \quad 1 \leq i \leq j,$$

or, equivalently, $\mathcal{C}(E_1 \cup F_1 \cup \cdots \cup E_i)$ can be embedded into $\mathcal{C}(E_1 \cup F_1 \cup \cdots \cup E_i \cup F_i)$. We prove this by induction on i. Arguing by contradiction, we assume that there is a decomposable tournament $(L, M, N) \in \mathcal{T}(D)$ such that $L \longrightarrow \mathcal{C}(E_1 \cup F_1 \cup \cdots \cup E_i)$ but $L \not\rightarrow \mathcal{C}(E_1 \cup F_1 \cup \cdots \cup E_i \cup F_i)$, and such that

$$(M, N) \not\rightarrow \big(\Pi_1(E_1 \cup F_1 \cup \cdots \cup E_i), \Pi_2(E_1 \cup F_1 \cup \cdots \cup E_i)\big),$$

and

$$(M, N) \longrightarrow \big(\Pi_1(E_1 \cup F_1 \cup \cdots \cup E_i \cup F_i), \Pi_2(E_1 \cup F_1 \cup \cdots \cup F_i)\big).$$

Let $\{x_1, \ldots, x_n\}$ be the image of (M, N) under this last embedding where $x_1 < \cdots < x_n$. Thus we have $x_n \in \Pi_1(F_i) \cup \Pi_2(F_i)$. By induction,

$$\rho(E_1 \cup F_1 \cup \cdots \cup E_i) = \rho(C_1 \cup \cdots \cup C_i,\ H_1 \cup \cdots \cup H_i),$$

which implies that $\{x_1, \ldots, x_n\} \not\subseteq \sigma(C_1 \cup \cdots \cup C_i \cup H_1 \cup \cdots \cup H_i)$. It follows that some x_h belongs to

$$\Pi_1(E_1 \cup E_2 \cup \cdots \cup E_i) \cup \Pi_2(E_1 \cup E_2 \cup \cdots \cup E_i).$$

Let j be the maximum of such h and assume say, that $x_j \in \Pi_1(E_\ell) \cup \Pi_2(E_\ell)$. Since (M, N) is a tournament, each of $x_{j+1}, \ldots, x_n$ is connected by an arc to x_j. According to the above description of the construction we have $\overrightarrow{x_h x}$, $\overrightarrow{y x_h} \in \vec{E}(D)$ for every $j < h \leq n$ and every

$$x \in \Pi_1(E_1 \cup F_1 \cup \cdots \cup F_{\ell-1} \cup E_\ell),$$

$$y \in \Pi_2(E_1 \cup F_1 \cup \cdots \cup F_{\ell-1} \cup E_\ell).$$

It also follows that each of $x_{j+1}, \ldots, x_n$ is connected by an arc to $\sigma(t_\ell)$ where the direction of this arc depends only on $\sigma(t_\ell)$. We note that $\sigma(t_\ell)$ is different from $x_1, \ldots, x_n$ since there is no arc between $\sigma(t_\ell)$ and E_ℓ. (An arc from $\sigma(w)$ to E_ℓ means there is an arc from w to t_ℓ.) Let us assume that $t_\ell \in C_\ell$ (the case $t_\ell \in H_\ell$ is similar). Thus we have

$$\{x_{j+1}, \ldots, x_n\} \subset \mathcal{C}\big(\sigma(C_1 \cup \cdots \cup C_{\ell-1} \cup \{t_\ell\}), \sigma(H_1 \cup \cdots \cup H_{\ell-1})\big),$$

and

$$\{x_{j+1}, \ldots, x_n\} \subset \mathcal{C}(E_1 \cup F_1 \cup \cdots \cup E_\ell).$$

By induction, we have

$$\begin{aligned} \rho(E_1 \cup F_1 \cup \cdots \cup E_\ell) &= \rho(C_1 \cup \cdots \cup C_\ell, H_1 \cup \cdots \cup H_\ell) \\ &= \rho(C_1 \cup \cdots \cup C_{\ell-1} \cup \{t_\ell\}, H_1 \cup \cdots \cup H_{\ell-1}) \end{aligned}$$

since t_ℓ was a critical vertex. However, this is a contradiction, since

$$L \cup \{x_{j+1}, \ldots, x_n\} \longrightarrow \mathcal{C}(C_1 \cup \cdots \cup C_{\ell-1} \cup \{t_\ell\}, H_1 \cup \cdots \cup H_{\ell-1})$$

but

$$L \cup \{x_{j+1}, \ldots, x_n\} \not\to \mathcal{C}(E_1 \cup F_1 \cup \cdots \cup E_\ell).$$

This completes the proof for the first case.

CASE 2. There are infinitely many red generators.

In this case we construct a sequence $R_1 < \sigma(1) < R_2 < \sigma(2) < \cdots$ where $R_1, R_2, \ldots$ are red generators and $\sigma(1), \sigma(2), \ldots$ are red vertices forming an isomorphic copy of D. Here we use sets of the form $\{\sigma(i_1), \ldots, \sigma(i_j)\}$ as $\Pi_1(E_i), \Pi_2(E_i)$ in the formulas $\psi(R_j)$ to create the F_i's. The details of this construction are essentially the same as in the previous case and indeed can be obtained by systematically replacing symbols in the proof of Case 1 by appropriate other symbols. Essentially in the same way as formula $\psi(X)$ can be obtained from $\varphi(X)$ by formal negation and then replacing the phrase all but finitely many by the phrase infinitely many. The details will therefore be omitted.

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