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HYPERBOLIC KNOT COMPLEMENTS WITHOUT CLOSED EMBEDDED TOTALLY GEODESIC SURFACES

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Abstract

It is conjectured that a hyperbolic knot complement does not contain a closed embedded totally geodesic surface. In this paper, we show that there are no such surfaces in the complements of hyperbolic 3-bridge knots and double torus knots. Some topological criteria for a closed essential surface failing to be totally geodesic are given. Roughly speaking, sufficiently 'complicated' surfaces cannot be totally geodesic.

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1. Introduction

The following conjecture was proposed by Menasco and Reid in [9].

CONJECTURE 1.1 ([9, Conjecture 1], [5, Problem 1.76]). *A hyperbolic knot in the 3-sphere does not have a closed, totally geodesic surface embedded in its complement.*

We remark that there are cusped hyperbolic 3-manifolds containing a closed embedded totally geodesic surface. Thus, it seems that Conjecture 1.1 applies only to 'simple' 3-manifolds like the 3-sphere. This conjecture was solved for alternating knots [7], almost alternating knots [2], toroidally alternating knots [1], closed 3-braids [6], Montesinos knots [10] and tunnel number one knots [9]. In fact, except for tunnel number one knots, any closed incompressible surface in these knot complements is meridionally compressible. This implies that there exists an accidental parabolic element, and so the surface is not even quasi-Fuchsian. For tunnel number one knots, the conjecture was solved algebraically in [9].

In this paper we give another topological settlement of the conjecture for tunnel number one knot and prove the following result.

THEOREM 1.2. *Hyperbolic 3-bridge knots and double torus knots do not have a closed totally geodesic surface embedded in their complements.*

Moreover, we give slightly stronger results for free genus one knots. We also give two topological criteria for a closed essential surface failing to be totally geodesic in the complement of a hyperbolic knot K. Roughly speaking, sufficiently 'complicated' surfaces cannot be totally geodesic.

For a closed surface S in $S^3 - K$, we define the *order* $o(S; K)$ of S for K as follows [11]. Let $i_* : H_1(S) \to H_1(S^3 - K)$ be the homomorphism induced by the inclusion map $i : S \to S^3 - K$. Since $\mathrm{Im}(i_*)$ is a subgroup of $H_1(S^3 - K) = \mathbb{Z}\langle\text{meridian}\rangle$, there is an integer m such that $\mathrm{Im}(i_*) = m\mathbb{Z}$. Then we define $o(S; K) = |m|$.

THEOREM 1.3. *Let K be a hyperbolic knot in S^3 of genus g, S a closed essential surface in $S^3 - K$ having order o. If $o > 3g - 3$, then S is not totally geodesic.*

The *canonical genus* $g_c(K)$ of K is defined to be the minimum among genera of Seifert surfaces built by Seifert's algorithm on all diagrams of K. Let V_1 denote the volume of a regular ideal tetrahedron in $\mathbb{H}^3$ and $V_{g'}$ that of a regular truncated tetrahedron with dihedral angles $\pi/3g'$ in $\mathbb{H}^3$.

THEOREM 1.4. *Let K be a hyperbolic knot in S^3 of canonical genus g, S a closed essential surface of genus g' in $S^3 - K$. If $g' V_{g'} \geq 60 g V_1$, then S is not totally geodesic.*

2. Partial settlements

Throughout this paper, we denote by K a knot in S^3. We always denote the regular neighbourhood of K by $N(K)$ and the exterior of K, which is the space obtained by removing the interior of $N(K)$ from S^3, by $E(K)$.

We start with giving some definitions. Let S be a closed essential surface in $E(K)$. A nontrivial simple closed curve l on S is called an *accidental peripheral* when there is an annulus A in $E(K)$ such that $A \cap S = \partial A \cap S = l$ and $A \cap \partial N(K) = \partial A - l$. We call such an essential surface S with an accidental peripheral an *accidental surface*. Let M_1 and M_2 denote the closure of components of $S^3 - S$, where M_1 contains K. If S is accidental, then we can construct an essential annulus A_1 in $M_1 - K$ by pasting two copies of an accidental annulus A for S and an annulus $\partial N(K) - A$. But the converse does not hold generally. An example is given in [3, Figure 2]. The following gives us a topological necessary condition for a closed essential surface isotopic to a totally geodesic surface.

LEMMA 2.1. *If K is hyperbolic and S is totally geodesic in $S^3 - K$, then S is not accidental and neither $M_1 - K$ nor M_2 contains an essential annulus.*

REMARK. This lemma is well known, but we give a proof for the convenience of the reader. We remark that this necessary condition is rather far from complete, see [3].

PROOF. Suppose that S is totally geodesic. Then the representation of $\pi_1(S)$ induced by a faithful discrete representation of $\pi_1(S^3 - K)$ is Fuchsian. Therefore, it does not contain an accidental parabolic element. Assume $M_1 - K$ or M_2 contains an essential annulus. Then its double contains an essential torus. However, when S is totally geodesic, both doubles of $M_1 - K$ and M_2 are hyperbolic, and hence atoroidal. It is a contradiction. □

For tunnel number one knots, Conjecture 1.1 was solved in [9, Corollary 4] by an algebraic method. The next theorem gives us a purely topological proof.

THEOREM 2.2. *If K is a tunnel number one knot in S^3, S a closed essential surface in $E(K)$, M_1 and M_2 the closure of components of $S^3 - S$, where M_1 contains K, then S is accidental or M_2 contains an essential annulus.*

PROOF. Let $V_1 \cup V_2$ be a genus two Heegaard splitting of $E(K)$, where V_1 is a compression body and V_2 is a handlebody. By an isotopy of S, we may assume that $S \cap V_1$ consists of mutually parallel essential separating disks and non-separating disks in V_1, where the labels are consecutive in V_1. Suppose that $|S \cap V_1|$ is minimal among all surfaces isotopic to S. Note that $|S \cap V_1| \neq 0$ because of the incompressibility of S. If $S \cap V_2$ is a single disk, then S is a 2-sphere and hence compressible. Otherwise, since $S \cap V_2$ is ∂-compressible in V_2, a ∂-compression of $S \cap V_2$ yields a band b in V_1. By the minimality of $|S \cap V_1|$, b forms an incompressible non-∂-parallel annulus A in V_1 together with some disk of $S \cap V_1$. If A is non-separating in V_1, then there is an accidental annulus for A in V_1. If A is separating in V_1, then either there is an accidental annulus for A in V_1, or a sub-annulus A_0 of $\partial V_1 - A$ gives an essential annulus in M_1 or M_2. In particular, if A_0 is in M_1, then the next ∂-compression of $S \cap V_2$ yields a separating annulus parallel to A, and there is an essential annulus in M_2. □

It follows from Lemma 2.1 and Theorem 2.3 that hyperbolic 3-bridge knot complements do not contain a closed embedded totally geodesic surface.

THEOREM 2.3. *Let K be a 3-bridge knot in S^3, S a closed essential surface in $E(K)$, M_1 and M_2 the closure of components of $S^3 - S$, where M_1 contains K. Then S is meridionally compressible or M_2 contains an essential annulus.*

PROOF. Suppose that S is meridionally incompressible. Let $(B_1, T_1) \cup (B_2, T_2)$ be a 3-bridge tangle decomposition of (S^3, K), D_i $(i = 1, 2)$ a disjoint union of disks in B_i which are co-bounded by the strings of T_i and arcs in ∂B_i. By an isotopy of S, we may assume that $S \cap D_1 = \emptyset$, $S \cap B_1$ consists of essential disks in $B_1 - T_1$, $S \cap D_2$ consists of essential arcs on $S \cap B_2$, and $(S \cap B_2) - D_2$ consists of open disks. Under the above conditions, we assume that $|S \cap B_1|$ is minimal among all surfaces isotopic to S. By the incompressibility of S, $|S \cap B_1| \geq 1$. We perform a ∂-compression of $S \cap B_2$ along an outermost disk in D_2, and obtain an incompressible annulus A_1 in $B_1 - T_1$ which consists of a disk of $S \cap B_1$ and a resultant band b. It follows from the meridional incompressibility of S that $S \cap B_1$ consists of mutually parallel disks which divide (B_1, T_1) into the trivial 1-string tangle (B_{11}, T_{11}) and the trivial 2-string tangle (B_{22}, T_{22}), and the band b runs over (B_{22}, T_{22}). Note that $|S \cap B_1|$ is an even integer greater than 1 since S separates S^3 into M_1 $(\supset K)$ and M_2. If we perform the ∂-compression of $S \cap B_2$ along the next outermost arc, then we obtain an incompressible annulus A_2 in $B_1 - T_1$ which is parallel to A. Hence, there exists an annulus A_0 in $B_1 - T_1$ connecting A_1 and A_2. We note that A_0 is contained in M_2. If A_0 is ∂-parallel in M_2, then it contradicts the minimality of $|S \cap B_1|$. Since $S \cap D_2$ are essential arcs in $S \cap B_2$, A_i $(i = 1, 2)$ is incompressible in M_2. Therefore, A_0 is incompressible in M_2, thus A_0 is an essential annulus in M_2. □

A knot K in S^3 is said to be a *double torus* if K is contained in a genus two Heegaard surface of S^3. It is known that 2-bridge knots, pretzel knots, tunnel number one knots, and free genus one knots are double torus. Combined with previous results, the next theorem completes the proof of Theorem 1.2.

THEOREM 2.4. *Let K be a non-cable double torus knot with respect to a genus two Heegaard splitting $(F; V_1, V_2)$ of S^3, S a closed essential surface in $E(K)$. Then S is accidental or $M_1 - K$ or M_2 contains an essential annulus.*

REMARK. The condition 'non-cable' cannot be omitted. It is easy to see that cable knots with tunnel number one companion knots are double torus. In general, an accidental surface for the companion knot is not accidental for a cable knot if it is disjoint from its companion solid torus.

PROOF. We divide the proof into two cases, whether $F - K$ is incompressible in $S^3 - K$ or not.

Suppose that $F - K$ is compressible in $S^3 - K$ and let D be a compressing disk for $F - K$ in $S^3 - K$. We may assume that by exchanging it if necessary, D is a non-separating disk in V_1. Put $V_1' = cl(V_1 - N(D))$ and $F' = \partial V_1'$. Then V_1' is a solid torus and its core, say γ, is a tunnel number one knot. Therefore, K is a tunnel number one knot or a cable knot with a companion knot γ. In the former case, Theorem 2.3 gives the conclusion. In the latter case, if γ is non-trivial, then this contradicts the assumption of K. Hence γ is trivial and K is a torus knot. Since any incompressible closed surface in a torus knot complement must be isotopic to a peripheral torus, this case does not occur.

Next, suppose that $F - K$ is incompressible in $S^3 - K$, and assume that $|S \cap F|$ is minimal up to isotopy of S. Then each loop of $S \cap F$ is essential in both S and $F - K$. If there are mutually parallel loops of $S \cap F$ in F or there is a loop of $S \cap F$ which is parallel to K in F, then M_i contains an essential annulus or S is accidental. Otherwise, $S \cap F$ consists of a single loop which separates F into two once punctured tori $F_i (\subset M_i)$, and K is non-separating in F_i. Since the loop $S \cap F$ is essential in S, F_1 is incompressible in M_1. Let D be a compressing disk for S in S^3, hence in M_1. We may assume that by modifying D if necessary, $D \cap F_1$ consists of essential arcs in F_1. Note that by the incompressibility of S in $S^3 - K$, $D \cap F_1 \neq \emptyset$. Let α be an outermost arc of $D \cap F_1$ in D, and let δ be the corresponding outermost disk in D. We perform a ∂-compression of F_1 along δ in M_1, and get an annulus F_1' disjoint from F_1. Since F_1 is incompressible in M_1, F_1' is also incompressible in M_1, hence in $M_1 - K$. If F_1' is essential in $M_1 - K$, then we have the desired conclusion. So, suppose that F_1' is parallel to a sub-annulus A_1 in ∂M_1, and let V be the solid torus co-bounded by F_1' and A_1. Now, to recover F_1, we attach a band b corresponding to the ∂-compressing disk δ, to F_1'. If b is contained in V, then we can find a compressing disk for F_1 in M_1, a contradiction. Otherwise, F_1 is parallel to a sub-surface in ∂M_1, thus in S. This implies that K is isotopic to a loop in S, hence S is accidental. □

A Seifert surface F for K in S^3 is said to be *free* if the fundamental group $\pi_1(S^3 - F)$ is free. We define the *free genus* of K as the minimal genus over all free Seifert surfaces for K. For free genus one knots, we also obtain the following theorem.

THEOREM 2.5. *If K is a free genus one knot in S^3, S a closed essential surface in $E(K)$, M_1 and M_2 the closure of components of $S^3 - S$, where M_1 contains K, then S is accidental or M_2 contains an essential annulus.*

PROOF. Let F be a genus one free Seifert surface for K. We may assume that $F \cap S$ consists of essential loops in both F and S, and assume that $|F \cap S|$ is minimal. Since $\pi_1(S^3 - F)$ is a free group, $F \cap S \neq \emptyset$. If there is a loop of $F \cap S$ which is parallel to K in F, then S is accidental. Otherwise, all loops of $F \cap S$ are non-separating and mutually parallel in F. Since S separates S^3, hence F, there is an annulus component

A of $F - S$ in M_2. Then, the minimality of $|F \cap S|$ assures us that A is essential in M_2. □

3. Criteria for failing to be totally geodesic

Theorem 1.3 follows from the next theorem and Lemma 2.1.

THEOREM 3.1. *Let K be a knot in S^3 of genus g, S a closed essential surface in $E(K)$ having order o. If $o > 3g - 3$, then S is accidental or $M_1 - K$ or M_2 contains an essential annulus.*

PROOF. Suppose that $o > 3g - 3$ holds. Let F be a minimal genus Seifert surface for K. We assume that $|S \cap F|$ is minimal among all minimal genus Seifert surfaces. Then each curve of $S \cap F$ is essential in both S and F.

Let us show that $|S \cap F| \geq o$. To do this, we construct a finite connected graph, say G_S, by identifying a connected component of $S - (S \cap F)$ with a vertex, and a curve of $S \cap F$ with an edge. We fix an orientation of each edge of G_s induced from orientations of S, F and S^3. If G_S is a tree, then each edge is a cutting edge, and so each curve of $S \cap F$ is separating on S. But this contradicts the fact that $o > 0$. Hence G_S has at least one cycle. Let c be a cycle of G_S with an arbitrary orientation. We give the weight $+1$ to an edge of c when the orientation of the edge coincides with that of c, and -1 otherwise. Then $|S \cap F| = |E(G_S)| \geq |o(c)|$ holds, where $o(c)$ is the sum of the weights of edges of c. On the other hand, a loop on S can be constructed for c such that the linking number of it and K is equal to $o(c)$. This implies that $|o(c)| \geq o$, and hence $|S \cap F| \geq o$.

By the above argument and the assumption, $|S \cap F| \geq o > 3g - 3$ holds. In the case that $n := |S \cap F| > 3g - 2$, we find an accidental annulus for S or a sub-annulus bounded by two curves of $S \cap F$ on F, since there are at most $3g - 2$ mutually non-parallel and non-∂-parallel simple closed curves on F. This sub-annulus is essential in M_i, since otherwise, we can reduce $|S \cap F|$. In the remaining case, that is, $n = 3g - 2$, we consider the graph G_F constructed from F in the same way as G_S. Suppose that the curves of $S \cap F$ are mutually non-parallel and non-∂-parallel on F. Then that curves give a pants decomposition of F. This decomposition admits a checkerboard-coloring, since each pair of pants is contained in M_1 or M_2. Let us note that G_F has the following properties:

(1) There is a vertex $v_0 \in V(G_F)$ having degree 2;
(2) Every $v \in V(G_F) - \{v_0\}$ has degree 3.

Then the next lemma gives us a contradiction. Consequently, $S \cap F$ has some mutually parallel curves or a ∂-parallel curve in F, and hence we can find an accidental annulus for S or an essential annulus in M_i. □

LEMMA 3.2. *Let G be a finite graph satisfying the following two properties*:

(1) *There is a vertex $v_0 \in V(G)$ having degree* 2;
(2) *Every $v \in V(G) - \{v_0\}$ has degree* 3.

Then G is not a bipartite graph.

PROOF. Suppose that there is a 2-coloring $f : V(G) \to \{B, W\}$, and assume that $f(v_0) = B$. Let V_w be the set $\{v \in V(G) | f(v) = W\}$ and let V_b be the set $\{v \in V(G) | f(v) = B\}$. Then the number of edges which are incident to the vertices of V_b is $3(|V_b| - 1) + 2$, and the number of those for V_w is $3|V_w|$. However, since every edge of G connects a vertex of V_b and one of V_w, these values must be equal. Hence a contradiction. □

REMARK. In the proof of Theorem 3.1, we note that $|S \cap F| = o > 0$ if and only if $S \cap F$ consists of mutually parallel non-separating loops in S.

Finally we prove Theorem 1.4.

PROOF OF THEOREM 1.4. On one hand, it was shown in [4] that the volume of the complement of a hyperbolic knot is less than $120g V_1$, where g is the canonical genus of the knot and V_1 is the volume of a regular ideal tetrahedron in $\mathbb{H}^3$. On the other hand, it was shown in [8] that the minimal volume of complete hyperbolic 3-manifolds with a totally geodesic boundary of genus g' is $g' V_{g'}$, where $V_{g'}$ is the volume of a regular truncated tetrahedron with dihedral angles $\pi/3g'$ in $\mathbb{H}^3$. Recall that K is a hyperbolic knot in S^3 of canonical genus g and S is a closed essential surface of genus g' in $E(K)$. Now, suppose that S is totally geodesic in $S^3 - K$. Let M_1 and M_2 be the closure of components of $S^3 - S$, where M_1 contains K. Then both $M_1 - K$ and M_2 are hyperbolic 3-manifolds with totally geodesic boundary, and so $\mathrm{vol}(S^3 - K) = \mathrm{vol}(M_1 - K) + \mathrm{vol}(M_2)$. As a consequence, we get $60g V_1 > g' V_{g'}$. □

REMARK. Since V_1 is approximately 1.01494 and $V_{g'}$ is computed by using the formula given in [8], we can see that the inequality $60g V_1 > g' V_{g'}$ fails if $g = 1$ and $g' > 16$. Therefore, there is no closed embedded totally geodesic surface of genus $g' > 16$ in a hyperbolic knot complement of canonical genus one. Note that $V_{g'} \nearrow V_0$ as $g' \to \infty$, where V_0 is the volume of a regular ideal octahedron in $\mathbb{H}^3$ and is approximately 3.6639.

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