Canad. Math. Bull. Vol. 31 (4), 1988

SEMIBOUNDED EXTENSIONS OF SINGULAR ORDINARY DIFFERENTIAL OPERATORS

BY
ALLAN M. KRALL

ABSTRACT. The self-adjoint extensions of the singular differential operator $Ly = [(py')' + qy]/w$, where $p < 0$, $w > 0$, $q \geqq mw$, are characterized under limit-circle conditions. It is shown that as long as the coefficients of certain boundary conditions define points which lie between two lines, the extension they help define has the same lower bound.

Introduction. Since the advent of studying differential operators in a Hilbert space (see 7), there has been a great deal of interest in the characterization of self-adjoint, semibounded operators (e.g. [1, 2, 3, 7, 8]. A self-adjoint operator, of course, has a real spectrum, and a semibound for such an operator also serves as a semibound for its spectrum. The Friedrichs extension [1, 8] of a symmetric, semibounded operator is the self-adjoint extension with the greatest lower (or least upper) bound.

If we consider the Sturm-Liouville operator

$$\ell y = [(py')' + qy]/w$$

where p, q, w are real valued and measurable on $[a, b]$, p^{-1}, q and w are integrable, $p < 0$, $w > 0$ and $q \geqq mw$ for some fixed m, then when y is in the domain of the minimal operator L_m,

$$\begin{aligned}\langle L_m y, y\rangle &= \int_a^b [(py')' + qy]\bar{y}dx \\ &= (-p)y'\bar{y}|_a^b + \int_a^b [-p|y'|^2 + q|y|^2]dx \\ &\geqq m\int_a^b |y|^2 w dx = m\langle y, y\rangle,\end{aligned}$$

and so L_m is bounded below by m.

(We remind the reader that the minimal and maximal operators are characterized in the following way: Let D_m consist of those elements y satisfying

1. y is in $L^2(a, b; w)$

Received by the editors February 12, 1987, and, in revised form, October 28, 1987.
AMS Subject Classification (1980): 34B20 (primary); 34B25 (secondary).

https://doi.org/10.4153/CMB-1988-063-x Published online by Cambridge University Press

2. y is differentiable and py' is absolutely continuous on compact subintervals of (a, b)

3. $\ell y = [(py')' + qy]/w$ is in $L^2(a, b; w)$

4. At regular boundaries y and y' are 0. At singular boundaries y and y' vanish in a neighborhood of the boundary.

Then let L'_m be defined by setting $L'_m y = \ell y$ for all y in D_m. The minimal operator, L_m, is the closure of L'_m. If both a and b are regular points, L_m and L'_m coincide.

The domain, D_M, of the maximal operator, L_M, consists of those elements y satisfying 1, 2 and 3 above. L_M is defined by setting $L_M y = \ell y$ for all y in D_M. It is well known that $L^*_m = L_M$ and $L^*_M = L_m$. (For further information we cite [6, pp. 60-70].)

In the regular case self-adjoint extensions are characterized by boundary conditions: Let L be a self-adjoint extension of L_m whose domain is restricted by

$$h_0 y(a) - h_1 p(a) y'(a) = 0, \quad h_0^2 + h_1^2 = 1$$

$$H_0 y(b) + H_1 p(b) y'(b) = 0, \quad H_0^2 + H_1^2 = 1.$$

Then the terms $(-p) y' \bar{y} |_a^b$ cannot be thrown away. Setting

$$h_1 y(a) + h_0 p(a) y'(a) = \phi$$

and

$$-H_1 y(b) + H_0 p(b) y'(b) = \psi,$$

where ϕ and ψ are arbitrary, the parametric forms of the boundary conditions,

$$y(a) = -h_1\phi, \quad p(b)y'(b) = h_0\phi$$

$$y(b) = H_1\psi, \quad p(b)y'(b) = H_0\psi$$

are found, and

$$\langle Ly, y\rangle = H_0 H_1 |\psi|^2 + h_0 h_1 |\phi|^2 + \int_a^b [(-p)|y'|^2 + q|y|^2] dx.$$

If $H_0 H_1 \geqq 0$ and $h_0 h_1 \geqq 0$, then we also have $\langle Ly, y\rangle \geqq m\langle y, y\rangle$.

THEOREM. *The regular Sturm-Liouville operator L, defined on $L^2[a, b; w]$ by*

$$Ly = [(py')' + qy]/w$$

where p, q and w are real valued and measurable on $[a, b]$, p^{-1}, q and w are integrable, $p < 0$, $w > 0$ and $q \geqq mw$, and whose domain is constrained by boundary conditions

$$h_0 y(a) - h_1 p(a) y'(a) = 0$$

$$H_0 y(b) + H_0 p(b) y'(b) = 0$$

is bounded below by m, just as the minimal operator generated by

$$\ell y = [(py')' + qy]/w,$$

if and only if $h_0h_1 \geqq 0$, $H_0H_1 \geqq 0$.

The purpose of this article is to extend this result to singular problems by using newly formulated singular boundary conditions [4]. The technique is somewhat more complicated, but is otherwise surprisingly similar.

SINGULAR EXTENSIONS. The only change in the assumptions is that we require that p^{-1}, q and w be locally integrable rather than integrable over all of $[a, b]$. The problem remains the same: What is to be done with the terms $(-p)y'\bar{y}|_a^b$?

For the sake of simplicity we temporarily assume that a is still a regular point. At a we assume a boundary condition $h_0 y(a) - h_1 p(a)y'(a) = 0$ as before. The singularity at b is assumed to be limit-circle. That is, all solutions of $(py')' + qy = \lambda wy$ are in $L^2(a, b; w)$ for all values of λ. The alternative, the limit point case, in which for each λ, $\operatorname{Im} \lambda \neq 0$, there is only one independent solution in $L^2(a, b; w)$ requires no constraint at b. The operator satisfying the boundary condition at a is self-adjoint, and, if $h_0h_1 \geqq 0$, is also bounded below by m.

We make the fundamental assumption that the energy integral

$$\int_a^b [-p|y'|^2 + q|y|^2]dx$$

is finite for all y in the domain of the maximal operator.

This is essentially a physical assumption, since in many applications the two terms under the integral represent kinetic and potential energy. The assumption also is encountered when "left-definite" boundary value problems are considered. We shall not consider them in this paper.

LEMMA. *For all y, z in the domain of the maximal operator* $\lim_{x\to b} (py')\bar{z}$ *exists.*

PROOF. Define

$$\{y, y\} = \int_a^{b'} [(py')' + qy]\bar{y}dx - \int_a^{b'} [(-p)|y'|^2 + q|y|^2]dx.$$

This equals $(py')\bar{y}|_a^{b'}$. As $b' \to b$, the limit clearly exists. By use of the polarization identity,

$$\{y, z\} = (1/4)[\{y + z, y + z\} - \{y - z, y - z\} + i\{y + iz, y + iz\} - i\{y - iz, y - iz\}],$$

which equals $(py')\bar{z}|_a^{b'}$, $(py')\bar{z}|_a^b$ is defined, since as $b' \to b$ the limit exists as the limit of the four terms on the right, each of which has a limit.

We need to define what are singular boundary conditions. Let u, v be solutions of

$$(py')' + qy = 0$$

which also satisfy

$$p(uv' - u'v) \equiv 1.$$

(These can be found by standard methods beginning at any point in (a, b), then extended throughout all of (a, b).) Let b_u, b_v be given as solutions of

$$\begin{pmatrix} u & v \\ pu' & pv' \end{pmatrix}\begin{pmatrix} b_v \\ -b_u \end{pmatrix} = \begin{pmatrix} y \\ py' \end{pmatrix}$$

for any y in the domain of the maximal operator. Then at $x = b'$,

$$b_u = p(u'y - uy')$$

$$b_v = p(v'y - vy').$$

It is well known [4], [5] that as $b' \to b$, b_u, b_v have finite limits. We denote these by $B_u(y)$ and $B_v(y)$. These are singular boundary conditions.

The most general singular boundary condition at b is a linear combination of these:

$$H_0 B_u(y) + H_1 B_v(y) = 0, \quad H_0^2 + H_1^2 = 1.$$

It is also known [5] that the operator whose domain is restricted by the regular boundary condition at a and by the condition above is self-adjoint if and only if all the coefficients h_0, h_1, H_0, H_1 are real. We denote that operator by $\mathscr{L}$.

Let us now consider $(py')\bar{y}(b')$ as $b' \to b$.

$$(py')\bar{y} = (\bar{y}, p\bar{y}')\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ py' \end{pmatrix}$$

$$= (\bar{b}_v, -\bar{b}_u)\begin{pmatrix} u & pu' \\ v & pv' \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u & v \\ pu' & pv' \end{pmatrix}\begin{pmatrix} b_v \\ -b_u \end{pmatrix}$$

$$= (\bar{b}_v, -\bar{b}_u)\begin{pmatrix} pu'u & pv'u \\ pu'v & pv'v \end{pmatrix}\begin{pmatrix} b_v \\ -b_u \end{pmatrix}$$

By the lemma all the terms in the central matrix have limits. The vectors do likewise. So as $b' \to b$

$$(py')\bar{y}(b) = (B_v(y), -B_u(y))\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} B_v(y) \\ -B_u(y) \end{pmatrix}.$$

Note further that

$$(pu'v + pv'u)^2 - 4(pu'v)(pv'u) = [p(uv' - v'u)]^2 \equiv 1.$$

Hence as $b' \to b$, we find $(\beta + \gamma)^2 - 4\alpha\delta = 1$. They are not all zero.

Now let

$$H_0 B_u(y) + H_1 B_v(y) = 0$$
$$-H_1 Bu(y) + H_0 B_v(y) = \psi$$

where ψ is arbitrary. Then

$$\begin{pmatrix} B_v(y) \\ -B_u(y) \end{pmatrix} = \begin{pmatrix} H_0 & H_1 \\ H_1 & -H_0 \end{pmatrix} \begin{pmatrix} \psi \\ 0 \end{pmatrix}$$

and

$$(py')\bar{y}(b) = (\bar{\psi}, 0) \begin{pmatrix} H_0 & H_1 \\ H_1 & -H_0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} H_0 & H_1 \\ H_1 & -H_0 \end{pmatrix} \begin{pmatrix} \psi \\ 0 \end{pmatrix}$$
$$= [\alpha H_0^2 + (\beta + \gamma) H_0 H_1 + \delta H_1^2] |\psi|^2.$$

This is greater than or equal to zero if and only if H_0 and H_1 are chosen such that

$$\alpha H_0^2 + (\beta + \gamma) H_0 H_1 + \delta H_1^2 \geqq 0.$$

THEOREM. *Let $\mathscr{L}$ be an extension of the minimal operator generated by $[(py')' + qy]/w$ on $L^2(a, b; w)$ whose domain is restricted by*

$$h_0 y(a) - h_1 p(a) y'(a) = 0, \quad h_0^2 + h_1^2 = 1,$$
$$H_0 B_u(y) + H_1 B_v(y) = 0, \quad H_0^2 + H_1^2 = 1.$$

$\mathscr{L}$ is bounded below by m if and only if

$$h_0 h_1 \geqq 0$$

and

$$\alpha H_0^2 + (\beta + \gamma) H_0 H_1 + \delta H_1^2 \geqq 0.$$

The equation above represents two lines in the (H_0, H_1)-plane. If $\alpha \geqq 0$, the area in which (H_0, H_1) consists of those regions containing the H_0 axis. If $\alpha < 0$, (H_0, H_1) must be outside those regions. If $\delta > 0$, (H_0, H_1) must be in those regions containing the H_1 axis. If $\delta < 0$, (H_0, H_1) must be outside those regions. If $\alpha = \delta = 0$, then when $\beta + \gamma > 0$, (H_0, H_1) must be in the first or third quadrants. If $\beta + \gamma < 0$, (H_0, H_1) must be in the second or fourth quadrants.

The regular case at b may be considered as a limit-circle case. We choose u and v so that

$$u(b) = 1, \quad p(b) u'(b) = 0,$$
$$v(b) = 0, \quad p(b) v'(b) = 1.$$

https://doi.org/10.4153/CMB-1988-063-x Published online by Cambridge University Press

Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is the same matrix found in the introduction.

The results can be extended to cover the limit-circle case at a as well. We use $C_u(y)$ and $C_v(y)$ to indicate the limits of b_u and b_v as $x \to a$. The most general linear boundary condition at a is

$$h_0 C_u(y) - h_1 C_v(y) = 0, \quad h_0^2 + h_1^2 = 1.$$

If h_0, h_1, H_0, H_1 are real, then the extension of the minimal operator $\mathscr{L}$ whose domain is restricted by

$$H_0 B_u(y) + H_1 B_v(y) = 0,$$
$$h_0 C_u(y) - h_1 C_v(y) = 0,$$

is self-adjoint. Setting

$$h_0 C_u(y) - h_1 C_v(y) = 0$$
$$h_1 C_u(y) + h_0 C_v(y) = \phi,$$

where ϕ is arbitrary, yields

$$\begin{pmatrix} C_v(y) \\ -C_u(y) \end{pmatrix} = \begin{pmatrix} h_0 & -h_1 \\ -h_1 & -h_0 \end{pmatrix} \begin{pmatrix} \phi \\ 0 \end{pmatrix}$$

and

$$-(py')\bar{y}(a) = -(\phi, 0) \begin{pmatrix} h_0 & -h_1 \\ -h_1 & -h_0 \end{pmatrix} \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \begin{pmatrix} h_0 & -h_1 \\ -h_1 & -h_0 \end{pmatrix} \begin{pmatrix} \phi \\ 0 \end{pmatrix}$$
$$= -[\epsilon h_0^2 - (\zeta + \eta) h_0 h_1 + \theta h_1^2] |\phi|^2,$$

where $\begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix}$ is the limit of $\begin{pmatrix} pu'u & pv'v \\ pu'v & pv'v \end{pmatrix}$ as $x \to a$.

THEOREM. *Let $\mathscr{L}$ be an extension of the minimal operator generated by $[(py')' + qy]/w$ on $L^2(a, b; w)$ whose domain is restricted by*

$$h_0 C_u(y) - h_1 C_v(y) = 0, \quad h_0^2 + h_1^2 = 1,$$
$$H_0 B_u(y) + H_1 B_v(y) = 0, \quad H_0^2 + H_1^2 = 1.$$

$\mathscr{L}$ is bounded below by m if and only if

$$\epsilon h_0^2 - (\zeta + \eta) h_0 h_1 + \theta h_1^2 \leqq 0,$$
$$\alpha H_0^2 + (\beta + \gamma) H_0 H_1 + \delta H_1^2 \geqq 0.$$

A word of warning is in order. The Legendre operator $\ell y = ((x^2 - 1)y')'$, defined on $L^2(-1, 1)$ does not fit the criteria assumed here. The energy integral

$$\int_{-1}^{1} (1 - x^2)|y'|^2 dx$$

can be infinite. $y = (1/2)\ell n[(1 - x)/(1 + x)]$ is in the domain of the maximal operator, but has infinite energy.

REMARKS. We would like to thank Hans G. Kaper, Man Kam Kwong and Anton Zettl for introducing us to this problem and for providing helpful preprints [2], [3] to serve as a guide.

REFERENCES

1. K. O. Friedrichs, *Spektral theorie halbbeschrankter operatoren*, Math. Ann. **109** (1934), pp. 465-487, pp. 685-713.

2. H. G. Kaper and M. K. Kwong, *Regularizing transformations for certain singular Sturm-Liouville boundary value problems*, SIAM J. Math. Anal. **15** (1984), pp. 957-963.

3. H. G. Kaper, M. K. Kwong and A. Zettl, *Characterizations of the Friedrichs extensions of the singular Sturm-Liouville expressions*, J. Math. Anal. Appl. **17** (1986), pp. 772-777.

4. A. M. Krall, J. K. Shaw and D. B. Hinton, *Boundary conditions for differential systems in intermediate limit situations*, Proc. Conf. Diff. Eq., I. W. Knowles and R. T. Lewis Eds., Elsevier Science Publ. B.V. (North-Holland) 1984, pp. 301-306.

5. A. M. Krall and L. L. Littlejohn, *Orthogonal polynomials and singular Sturm-Liouville systems, I*, Rocky Mt. J. Math., to appear.

6. M. A. Naimark, *Linear differential operators, part II*, Frederick Ungar Publ. Co., New York, 1968.

7. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Frederick Ungar Publ. Co., New York, 1955.

8. M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc., 1932.

MCALLISTER BUILDING
THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802