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OSCILLATION IN DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. We obtain sufficient conditions for the oscillation of all solutions of the linear delay differential equation with positive and negative coefficients

$$\dot{y}(t) + P(t)y(t-\tau) - Q(t)y(t-\sigma) = 0$$

where

$$P, Q \in C[[t_0, \infty), \mathbb{R}^+] \text{ and } \tau, \sigma \in [0, \infty).$$

Extensions to neutral differential equations and some applications to the global asymptotic stability of the trivial solution are also given.

1. **Introduction.** Consider the linear delay differential equation with positive and negative coefficients

$$\dot{y}(t) + P(t)y(t-\tau) - Q(t)y(t-\sigma) = 0 \tag{1}$$

where

$$P, Q \in C[[t_0, \infty), \mathbb{R}^+] \text{ and } \tau, \sigma \in [0, \infty). \tag{2}$$

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of Eq. (1) and for the global asymptotic stability of the trivial solution. Extensions to neutral differential equations are also given.

The oscillation of Eq. (1) when the coefficients P and Q are positive constants or when P and Q are positive and asymptotically constants has been investigated in [1], [5] and [6]. Also the asymptotically behavior of the oscillatory solutions of Eq. (1) and of the neutral equation

$$\frac{d}{dt}[y(t) - py(t-\tau)] + q(t)y(t-\sigma) = 0$$

has been studied in [7].

Let $m = \max\{\tau, \sigma\}$. By a solution of Eq. (1) we mean a function $y \in C[[t_1 - m, \infty), \mathbb{R}]$, for some $t_1 \geq t_0$, such that y is continuously differentiable on $[t_1, \infty)$ and such that Eq. (1) is satisfied for $t \geq t_1$. As in customery, a solution of Eq. (1) is said to *oscillate* if it has arbitrarily large zeroes. Otherwise the solution is called *nonoscillatory*.

2. **Oscillation and stability of Eq. (1).** In this section we study the oscillation and the asymptotic behavior of all solutions of Eq.(1).

The following lemma will be useful in the proofs of the main results.

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LEMMA 1. *Assume that (2) holds,*

(3) $\tau \geq \sigma$,

(4) $P(t) \geq Q(t+\sigma-\tau), P(t) \not\equiv Q(t+\sigma-\tau)$ *on* $[t_1+\tau-\sigma, \infty)$, *for any* $t_1 \geq t_0$

and

(5) $$(\tau-\sigma)Q(t) \leq 1 \ \textit{for} \ t \geq t_0.$$

Let $y(t)$ *be an eventually positive solution of Eq. (1) and set*

(6) $$z(t) = y(t) - \int_{t-\tau}^{t-\sigma} Q(s+\sigma)y(s)\, ds \ \textit{for} \ t \geq t_0+\tau-\sigma.$$

Then eventually $z(t)$ *is a nonincreasing and positive function.*

PROOF. Assume that $t_1 \geq t_0$ is such that

$$y(t) > 0 \ \text{for} \ t \geq t_1.$$

Then

(7) $$\dot{z}(t) = -[P(t) - Q(t+\sigma-\tau)]y(t-\tau) \leq 0 \ \text{for} \ t \geq t_1+\tau$$

and so $z(t)$ is nonincreasing for $t \geq t_1$. Clearly, either

(8) $$\lim_{t\to\infty} z(t) = -\infty$$

or

(9) $$\lim_{t\to\infty} z(t) = \ell \in \mathbb{R}.$$

First assume that $y(t)$ is unbounded function. Then there exists a sequence of points $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = \infty \ \text{and} \ y(t_n) = \max_{s\leq t_n} y(s) \ \text{for} \ n = 1, 2, \ldots.$$

From (6) we now see that for n sufficiently large

$$\begin{aligned} z(t_n) &= y(t_n) - \int_{t_n-\tau}^{t_n-\sigma} Q(s+\sigma)y(s)\, ds \\ &\geq [1 - \int_{t_n-\tau}^{t_n-\sigma} Q(s+\sigma)\, ds]y(t_n) \\ &\geq 0. \end{aligned}$$

As $z(t)$ is nonincreasing, it follows that $z(t) > 0$ and the proof is complete when $y(t)$ is unbounded.

Next assume that $y(t)$ is a bounded function. Then (8) cannot hold and so (9) holds.

Set

$$\mu = \limsup_{t \to \infty} y(t)$$

and let $\{\xi_n\}$ be a sequence of points such that

$$\lim_{n \to \infty} \xi_n = \infty \text{ and } \lim_{n \to \infty} y(\xi_n) = \mu.$$

Then for $\varepsilon > 0$ and sufficiently small and for n sufficiently large,

$$\begin{aligned} z(\xi_n) &= y(\xi_n) - \int_{\xi_n - \tau}^{\xi_n - \sigma} Q(s+\sigma) y(s)\, ds \\ &\geq y(\xi_n) - (\mu + \varepsilon). \end{aligned}$$

By taking limits as $n \longrightarrow \infty$ we see that

$$\ell \geq -\varepsilon.$$

As ε is arbitrary, it follows that $\ell \geq 0$ and so $z(t)$ is positive. The proof is complete.

The next theorem gives sufficient conditions so that every nonoscillary solution of Eq. (1) tends to zero as $t \longrightarrow \infty$.

THEOREM 1. *Assume that (2)–(5) hold and that one of the following two conditions is satisfied:*
(H_1) There exists a positive constant a such that

$$P(t) - Q(t + \sigma - \tau) \geq a \text{ for } t \geq t_0 + \tau - \sigma \tag{10}$$

(H_2) There exists a positive constant $b \in (0, 1)$ such that

$$(\tau - \sigma) Q(t) \leq 1 - b \text{ for } t \geq t_0 \tag{11}$$

and

$$\int_{t_0 + \tau - \sigma}^{\infty} [P(s) - Q(s + \sigma - \tau)]\, ds = \infty. \tag{12}$$

Then every nonoscillatory solution of Eq. (1) tends to zero as $t \longrightarrow \infty$.

PROOF. It suffices to prove that every eventually positive solution $y(t)$ of Eq. (1) tends to zero as $t \longrightarrow \infty$. By Lemma 1 it follows that the function $z(t)$, which is defined by (6), is nonincreasing and eventually positive. Hence

$$\lim_{t \to \infty} z(t) = \ell \in [0, \infty). \tag{13}$$

By integrating both sides of (7) from t_1 to infinity, for t_1 sufficiently large, we find

$$\ell - z(t_1) = \int_{t_1}^{\infty} [P(s) - Q(s + \sigma - \tau)] y(s - \tau)\, ds \tag{14}$$

From this and either (10) or (12) it follows that

$$\liminf_{t\to\infty} y(t) = 0.$$

Also (6) implies that $z(t) \le y(t)$ and so $\ell = 0$. If $\tau = \sigma$, then $z(t) = y(t)$ and because $\ell = 0$,

$$\lim_{t\to\infty} y(t) = 0. \tag{15}$$

In the remainder of the proof we will assume that $\tau > \sigma$. First we assume that (10) holds. Then from (14) we see that $y \in L^1[t_1, \infty)$. As $Q(t)$ is bounded, it follows that $Q(s+\sigma-\tau)y(s) \in L^1[t_1,\infty)$ and so

$$\lim_{t\to\infty} \int_{t-\tau}^{t-\sigma} Q(s+\sigma-\tau)y(s)\,ds = 0. \tag{16}$$

From this and the fact that $\ell = 0$ we see that (15) holds. The proof when (10) holds is complete.

Next assume that (11) and (12) hold. We first claim that $y(t)$ is bounded. Otherwise there exists a sequence of points $\{\xi_n\}$ such that

$$\lim_{n\to\infty} \xi_n = \infty,\; y(\xi_n) = \max_{s\le\xi_n} y(s) \;\text{ for }\; n = 1,2,\dots \;\text{ and }\; \lim_{n\to\infty} y(\xi_n) = \infty.$$

Then by (6) and (11)

$$\begin{aligned} z(\xi_n) &= y(\xi_n) - \int_{\xi_n-\tau}^{\xi_n-\sigma} Q(s+\sigma)y(s)\,ds \\ &\ge y(\xi_n) - \frac{1-b}{\tau-\sigma}\int_{\xi_n-\tau}^{\xi_n-\sigma} y(s)\,ds \\ &\ge y(\xi_n) - (1-b)y(\xi_n) \\ &= by(\xi_n) \longrightarrow \infty \;\text{ as } n\longrightarrow\infty \end{aligned}$$

which contradicts the fact that $\ell = 0$ and establishes our claim that $y(t)$ is bounded. Set

$$\mu = \limsup_{t\to\infty} y(t)$$

and let $\{s_n\}$ be a sequence of points such that

$$\lim_{n\to\infty} s_n = \infty \;\text{ and }\; \lim_{n\to\infty} y(s_n) = \mu.$$

Then for $\varepsilon > 0$ and sufficiently small and for n sufficiently large, it follows from (6) and (11) that

$$\begin{aligned} z(s_n) &= y(s_n) - \int_{s_n-\tau}^{s_n-\sigma} Q(s+\sigma)y(s)\,ds \\ &\ge y(s_n) - (\mu+\varepsilon)(1-b). \end{aligned}$$

By taking limits as $n \longrightarrow \infty$ and by using the fact that $\ell = 0$ we see that

$$0 \ge \mu b - \varepsilon(1-b).$$

As $\varepsilon > 0$ can be taken arbitrarily small, it follows that $\mu = 0$ and the proof is complete.

The next result provides sufficient conditions for the oscillation of all solutions of Eq. (1).

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THEOREM 2. *Assume that (2)–(5) hold and that either*

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}[P(s)-Q(s+\sigma-\tau)]\,ds>\frac{1}{e} \tag{17}$$

or

$$\limsup_{t\to\infty}\int_{t-\tau}^{t}[P(s)-Q(s+\sigma-\tau)]\,ds>1 \tag{18}$$

Then every solution of Eq. (1) oscillates.

PROOF. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $y(t)$. By Lemma 1 it follows that the function $z(t)$, which is defined by (6), is an eventually positive function. Also by (7) and the fact that eventually

$$0<z(t)\le y(t),$$

we see that eventually,

$$\dot z(t)+[P(t)-Q(t+\sigma-\tau)]z(t-\tau)\le 0 \tag{19}$$

It is well known however, see [8], that under condition (17) or (18), the inequality (19) cannot have an eventually positive solution. This contradicts the fact that $z(t)$ is eventually positive and completes the proof.

In [7] it was shown that if (2) holds, if

$$P(t)-Q(t+\sigma-\tau)\ne 0 \text{ for } t\ge t_0$$

and if

$$2\limsup_{t\to\infty}\left|\int_{t-\tau}^{t-\sigma}|Q(s+\sigma)|\,ds\right|+\limsup_{t\to\infty}\int_{t-\tau}^{t}|P(s)-Q(s+\sigma-\tau)|\,ds<1, \tag{20}$$

then every oscillatory solution of Eq. (1) tends to zero as $t\longrightarrow\infty$. By combining this result with Theorems 1 and 2 we obtain immediately the following stabiltiy results.

COROLLARY 1. *Assume (2), (3), (5) and (20) hold. Also suppose that*

$$P(t)-Q(t+\sigma-\tau)>0 \text{ for } t\ge t_0 \tag{21}$$

and that either (10) or (11) and (12) hold. Then the trivial solution of Eq. (1) is globally asymptotically stable.

COROLLARY 2. *In addition to the hypotheses of Theorem 2, assume that (20) and (21) hold. Then the trivial solution of Eq. (1) is globally asymptotically stable.*

EXAMPLE 1. The delay differential equation

$$\dot y(t)+2\sin^2 t\,y(t-\pi)-\frac{1}{2}\cos^2 t\,y(t-\frac{\pi}{2})=0$$

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satisfies (17) and (2)–(5). Therefore by Theorem 2, every solution of this equation oscillates.

EXAMPLE 2. The delay differential equation

$$\dot{y}(t) + \frac{2}{t}y(t-2) - \frac{1}{t}y(t-1) = 0$$

satisfies the hypotheses of Corollary 1 with $t_0 = 2$. Hence every solution tends to zero as $t \longrightarrow \infty$.

EXAMPLE 3. The delay differential equation

$$\dot{y}(t) + \frac{1}{5\pi}(3 + \cos t)\, y(t-\pi) - \frac{1}{5\pi}(1 + \sin t)\, y(t - \frac{\pi}{2}) = 0$$

satisfies the hypotheses of Theorem 2 and Corollary 2. Hence every solution oscillates and tends to zero as $t \longrightarrow \infty$.

3. **Oscillation and stability of neutral equations.** In this section we extend the results of Section 2 to the neutral delay differential equation with positive and negative coefficients

$$\frac{d}{dt}[y(t) - R(t)y(t-\rho)] + P(t)y(t-\tau) - Q(t)y(t-\tau) = 0 \tag{22}$$

where

$$P, Q, R \in C[[t_0, \infty), \mathbb{R}^+], \rho \in (0, \infty) \text{ and } \tau, \sigma \in [0, \infty). \tag{23}$$

Let $m = \max\{\rho, \tau, \sigma\}$. By a *solution* of Eq. (22) we mean a function $y \in C[[t_1 - m, \infty), \mathbb{R}]$ for some $t_1 \geq t_0$, such that $y(t) - R(t)y(t-\rho)$ is continuously differentiable for $t \geq t_1$ and such that Eq. (22) is satisfied for $t \geq t_1$. Such a y is also called a solution on $[t_1, \infty)$.

Let t_1 be a given initial point and let $\phi \in [[t_1 - m, t_1], \mathbb{R}]$ be a given initial function. Then one can show, by the method of steps, that Eq. (22) has a unique solution on $[t_1, \infty)$ satisfying the initial condition

$$y(t) = \phi(t) \text{ for } t_1 - m \leq t \leq t_1.$$

The oscillation of neutral delay differential equations with positive and negative constant coefficients was studied in [3], and [4].

The following lemma which is extracted from [2] is needed in the proof of Theorem 3.

LEMMA 2 ([2]). *Let F, G, R: $[t_0, \infty) \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be such that*

$$F(t) = G(t) - R(t)G(t-c), \; t \geq t_0 + \max\{0, c\}.$$

Assume that there exists a positive number r such that

$$0 \leq R(t) \leq r.$$

Suppose also that $G(t) > 0$ for $t \geq t_0$, $\liminf_{t\longrightarrow\infty} G(t) = 0$ and that $\lim_{t\longrightarrow\infty} F(t) = L \in \mathbb{R}$, exists. Then $L = 0$.

The next lemma and the following two theorems are the duals of Lemma 1 and Theorems 1 and 2, respectively, for neutral equations and will be stated without proofs.

LEMMA 3. *Assume that (23) holds,*

(24) $\tau \geq \sigma,$

(25) $P(t) \geq Q(t+\sigma-\tau)$

and $P(t) - Q(t+\sigma-\tau) \not\equiv 0$ *on* $[t_1+\tau-\sigma, \infty)$, *for any* $t_1 \geq t_0$

(26) $0 \leq R(t) \leq r \leq 1$ *for* $t > t_0$ *and some* $r \in [0,1]$

and

(27) $$(\tau-\sigma)Q(t) \leq 1 - r \text{ for } t \geq t_0.$$

Let $y(t)$ *be an eventually positive solution of Eq. (22) and set*

$$v(t) = y(t) - R(t)y(t-\rho) - \int_{t-\tau}^{t-\sigma} Q(s+\sigma)\, ds.$$

Then eventually $v(t)$ *is nonincreasing and positive.*

THEOREM 3. *Assume that (23), (24) and (25) hold,* $Q(t)$ *is bounded,*

$$R(t) \leq r \leq 1 \text{ for } t > t_0 \text{ and some } r \in [0,1)$$

and

$$(\tau-\sigma)Q(t) \leq 1 - r \text{ for } t \geq t_0.$$

Suppose also that one of the following two conditions is satisfied:
(H_1) There exists a positive constant a such that

$$P(t) - Q(t+\sigma-\tau) \geq a \text{ for } t \geq t_0 + \tau - \sigma$$

(H_2) There exists a positive constant b such that

$$(\tau-\sigma)Q(t) \leq 1 - r - b \text{ for } t \geq t_0$$

and

$$\int_{t_0+\tau-\sigma}^{\infty} [P(s) - Q(s+\sigma-\tau)]\, ds = \infty.$$

Then every nonoscillatory solution of Eq. (22) tends to zero as $t \longrightarrow \infty$.

COROLLARY 3. *Assume that the hypotheses of Theorem 3 are satisfied. Then every unbounded solution of Eq. (22) oscillates.*

THEOREM 4. *Assume that conditions (23)–(27) hold and that either (17) or (18) is satisfied. Then every solution of Eq. (22) oscillates.*

Next, we will obtain an oscillation result for the neutral delay differential equation

(28) $$\frac{d}{dt}[y(t) - R(t)y(t-\rho)] + py(t-\tau) - qy(t-\sigma) = 0$$

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where

$$R \in C^1[[t_0,\infty),\mathbb{R}^+], \rho \in (0,\infty) \text{ and } p,q,\tau,\sigma \in [0,\infty). \tag{29}$$

THEOREM 5. *Assume that (29) holds,*

$p > q, \tau \geq \sigma, \tau > \rho,$

$0 \leq R(t) \leq r < 1$ *and* $\dot{R}(t) \geq 0$ *for* $t \geq t_0$ *and some* $r \in [0,1), q(\tau-\sigma) \leq 1-r$

and

$$\frac{(\tau-\rho)(p-q)}{1-\lim_{t\to\infty} R(t)} > \frac{1}{e}. \tag{30}$$

Then every solution of Eq. (28) oscillates.

PROOF. Assume, for the sake of contradiciton, that Eq. (28) has an eventually positive solution $y(t)$. Set

$$v(t) = y(t) - R(t)y(t-\rho) - q\int_{t-\tau}^{t-\sigma} y(s)\,ds.$$

Then it follows by direct substitution into Eq. (28) that $v(t)$ is a differentiable solution of the neutral equation

$$\dot{v}(t) - R(t-\tau)\dot{v}(t-\rho) + pv(t-\tau) - qv(t-\sigma) = 0.$$

Also, by Lemma 3 and Theorem 3, $v(t)$ is eventually nonincreasing and positive and $\lim_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} v(t) = 0$. Set

$$w(t) = v(t) - R(t-\tau)v(t-\rho) - q\int_{t-\tau}^{t-\sigma} v(s)\,ds.$$

Then

$$\lim_{t\to\infty} w(t) = 0$$

and

$$\dot{w}(t) = -(p-q)v(t-\tau) - \dot{R}(t-\tau)v(t-\rho) \leq 0$$

which imply that eventually,

$$w(t) > o. \tag{31}$$

Now observe that

$$w(t) \leq v(t) - R(t-\tau)v(t-\rho) \leq [1-R(t-\tau)]v(t-\rho)$$

and so

$$-\frac{p-q}{1-R(t+\rho-2\tau)}w(t+\rho-\tau) \geq -(p-q)v(t-\tau) \geq \dot{w}(t).$$

Hence eventually,

$$\dot{w}(t) + \frac{p-q}{1-R(t+\rho-2\tau)} w(t-(\tau-\rho)) \leq 0. \tag{32}$$

But from (30) we see that

$$\liminf_{t\to\infty} \int_{t-(\tau-\rho)}^{t} \frac{p-q}{1-R(s+\rho-2\tau)}\, ds > \frac{1}{e}$$

and so (32) cannot have an eventually positive solution. This contradicts (31) and the proof is complete.

Concerning the asymptotic behaviour of the oscillatory solutions of Eq. (22) we can establish the following extension of the result preceeding Corollary 1. This result is a slight extension of a similar result in [7] and its proof will be omitted.

THEOREM 6. *Assume that*

$$P, Q, R \in C[[t_0,\infty), \mathbb{R}\,] \text{ and } \tau, \sigma, \rho \in [0,\infty).$$

Suppose that

$$P(t) - Q(t+\sigma-\tau) \neq 0 \text{ for } t \geq t_0 + \tau - \sigma \tag{33}$$

and that

$$2\limsup_{t\to\infty}\left[|R(t)| + \left|\int_{t-\tau}^{t-\sigma} |P(s+\sigma)|\, ds\right|\right] + \limsup_{t\to\infty}\int_{t-\tau}^{t} |P(s) - Q(s+\tau-\sigma)|\, ds < 1. \tag{34}$$

then every oscillatory solution of Eq. (22) tends to zero as $t \longrightarrow \infty$.

By combining Theorem 6 with Theorem 3, 4 and 5 we obtain immediately the following stability results for neutral equations.

COROLLARY 4. *In addition to the hypotheses of Theorem 3 assume that (33) and (34) hold. Then every solution of Eq. (22) tends to zero as* $t \longrightarrow \infty$.

COROLLARY 5. *In addition to the hypotheses of Theorem 4 assume that (33) and (34) hold. Then every solution of Eq. (22) oscillates and tends to zero as* $t \longrightarrow \infty$.

COROLLARY 6. *In addition to the hypotheses of Theorem 5 assume that*

$$(3p-q)\tau - 2p\sigma + 2\lim_{t\to\infty} R(t) < 1.$$

Then every solution of Eq. (28) oscillates and tends to zero as $t \longrightarrow \infty$.

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