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A Class of Abstract Linear Representations for Convolution Function Algebras over Homogeneous Spaces of Compact Groups

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Abstract. This paper introduces a class of abstract linear representations on Banach convolution function algebras over homogeneous spaces of compact groups. Let G be a compact group and H a closed subgroup of G. Let μ be the normalized G-invariant measure over the compact homogeneous space G/H associated with Weil's formula and $1 \leq p < \infty$. We then present a structured class of abstract linear representations of the Banach convolution function algebras $L^p(G/H, \mu)$.

1 Introduction

The mathematical theory of Banach convolution algebras plays significant and classical roles in abstract harmonic analysis, representation theory, functional analysis, operator theory, and C^*-algebras, see [1–3, 10, 15, 21, 22] and the references therein. Over the last decades, some new aspects and applications of Banach convolution algebras have achieved significant popularity in time-frequency (Gabor) analysis and coorbit theory, see [4–6, 11] and the references therein.

The following paper introduces the structured class of linear representations over the Banach function algebras related to homogeneous spaces (coset spaces) of compact groups. In a nutshell, homogeneous spaces are group-like structures with many interesting applications in mathematical physics, differential geometry, geometric analysis, and coherent state (covariant) transforms, see [16–20].

Section 2 is devoted to fixing notations and provides a summary of classical harmonic analysis over compact groups and homogeneous spaces (left coset spaces) of compact groups. Let G be a compact group and H a closed subgroup of G. Let μ be the normalized G-invariant measure over the homogeneous space G/H associated with Weil's formula and $1 \leq p < \infty$. In section 3 we study abstract harmonic analysis over the Banach function spaces related to homogeneous spaces of compact groups. Then we introduce the abstract notion of generalized convolution and involution for L^p-function spaces over homogeneous spaces of compact groups. We also study properties of these convolutions and involutions. Finally, we shall introduce a class of structured linear representations over function sub-algebras of the Banach convolution function algebras $L^p(G/H, \mu)$ and we address properties of these representations.

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2 Preliminaries and Notations

Let X be compact Hausdorff space. By $\mathcal{C}(X)$ we mean the space of all continuous complex valued functions on X. If μ is a positive Radon measure on X, then for each $1 \leq p < \infty$, the Banach space of equivalence classes of μ-measurable complex valued functions $f: X \to \mathbb{C}$ such that

$$\|f\|_{L^p(X,\mu)} = \Big(\int_X |f(x)|^p d\mu(x) \Big)^{1/p} < \infty$$

is denoted by $L^p(X,\mu)$. It contains $\mathcal{C}(X)$ as a $\|\cdot\|_{L^p(X,\mu)}$-dense subspace.

Let G be a compact group with the probability Haar measure dx. For $p \geq 1$ the notation $L^p(G)$ stands for the Banach function space $L^p(G,dx)$. The standard convolution for $f, g \in L^p(G)$ is defined via

$$f *_G g(x) = \int_G f(y) g(y^{-1}x)\, dy \quad (x \in G).$$

The involution for $f \in L^p(G)$, is defined by $f^{*^G}(x) = \overline{f(x^{-1})}$ for $x \in G$. Then the Banach function space $L^p(G)$ equipped with the above convolution and involution is a Banach $*$-algebra, that is,

$$\|f *_G g\|_p \leq \|f\|_p \|g\|_p, \tag{2.1}$$

$$(f *_G g)^{*^G} = g^{*^G} *_G f^{*^G}, \tag{2.2}$$

for all $f, g \in L^p(G)$, see [7,15,22] and the references therein.

Any continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G determines a non-degenerate $*$-representation of the Banach $*$-algebra $L^p(G)$ on the Hilbert space $\mathcal{H}_\pi$ via the linear map $f \mapsto \pi(f)$ given by the following operator valued integral [7, Theorem 3.9]:

$$\pi(f) = \int_G f(x)\pi(x) dx. \tag{2.3}$$

It is also shown that each non-degenerate $*$-representation of the Banach $*$-algebra $L^p(G)$ on a Hilbert space $\mathcal{H}$ arises from a unique continuous unitary representation of G on the Hilbert space $\mathcal{H}$ via (2.3)[7, Theorem 3.11].

Let H be a closed subgroup of G with the probability Haar measure dh. The left coset space G/H is interpreted as a locally compact homogeneous space, and G acts on it from the left. The map $q: G \to G/H$ given by $x \mapsto q(x) := xH$ is the surjective canonical map. The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces have been quite well studied by several authors, see [7,15,22] and the references therein. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f \in \mathcal{C}(G)$ and $T_H(f)(xH) = \int_H f(xh)\, dh$. Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$ for all Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x = \mu$ for all $x \in G$. The homogeneous space G/H has a normalized G-invariant measure μ that satisfies Weil's formula

$$\int_{G/H} T_H(f)(xH)\, d\mu(xH) = \int_G f(x)\, dx, \tag{2.4}$$

and hence the linear map T_H is norm-decreasing, that is,

$$\|T_H(f)\|_{L^1(G/H,\mu)} \leq \|f\|_{L^1(G)},$$

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for all $f \in L^1(G)$, see [22, §8.2].

For a function $\varphi \in L^p(G/H,\mu)$ and $z \in G$, the left action of z on φ is defined by $L_z\varphi(xH) = \varphi(z^{-1}xH)$ for $xH \in G/H$. Then it can be readily checked that $L_z: L^p(G/H,\mu) \to L^p(G/H,\mu)$ is a unitary operator.

3 Classical Harmonic Analysis over Function Spaces on Homogeneous Spaces of Compact Groups

Throughout this paper we assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure dh, and μ is the normalized G-invariant measure on the compact homogeneous space G/H satisfying (2.4) with respect to the probability Haar measures of G and H. Henceforth, we may say μ is the normalized G-invariant measure over the compact homogeneous space G/H, at times.

The following proposition shows that the linear map $T_H: \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous [8, 9, 12–14].

Proposition 3.1 *The linear map $T_H: \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.*

The next theorem [13,14] proves that the linear map T_H is norm-decreasing in other L^p-spaces when $p > 1$.

Theorem 3.2 *Let μ be the normalized G-invariant measure on G/H, and $p \geq 1$. The linear map $T_H: \mathcal{C}(G) \to \mathcal{C}(G/H)$ satisfies $\|T_H(f)\|_{L^p(G/H,\mu)} \leq \|f\|_{L^p(G)}$ for all $f \in \mathcal{C}(G)$. Hence, it has a unique extension to a norm-decreasing linear map from $L^p(G)$ onto $L^p(G/H,\mu)$.*

As an immediate consequence of Theorem 3.2 we deduce the following corollary.

Corollary 3.3 *Let μ be the normalized G-invariant measure on G/H, and $p \geq 1$. Let $\varphi \in L^p(G/H,\mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^p(G)$ with*

$$\|\varphi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)}. \tag{3.1}$$

Proof Indeed, using Weil's formula, we can write

$$\begin{aligned}\|\varphi_q\|^p_{L^p(G)} &= \int_G |\varphi_q(x)|^p\,dx = \int_{G/H} T_H(|\varphi_q|^p)(xH)\,d\mu(xH)\\ &= \int_{G/H}\Big(\int_H |\varphi_q(xh)|^p\,dh\Big)\,d\mu(xH),\end{aligned}$$

and since H is compact and dh is normalized, we get

$$\begin{aligned}\int_{G/H}\Big(\int_H |\varphi_q(xh)|^p\,dh\Big)\,d\mu(xH) &= \int_{G/H}\Big(\int_H |\varphi(xhH)|^p\,dh\Big)\,d\mu(xH)\\ &= \int_{G/H}\Big(\int_H |\varphi(xH)|^p\,dh\Big)\,d\mu(xH)\end{aligned}$$

$$= \int_{G/H} |\varphi(xH)|^p \Big(\int_H dh \Big) d\mu(xH)$$
$$= \int_{G/H} |\varphi(xH)|^p \, d\mu(xH),$$

which implies (3.1). ■

The next proposition shows that the linear operator $T_H \colon L^2(G) \to L^2(G/H,\mu)$ is a partial isometric linear map.

Proposition 3.4 *Let μ be the normalized G-invariant measure on G/H. Then*

$$T_H \colon L^2(G) \to L^2(G/H,\mu)$$

is a partial isometric linear map.

The following corollaries are straightforward consequences of Proposition 3.4. Let $\mathcal{J}^2(G,H) := \{f \in L^2(G) : T_H(f) = 0\}$ and let $\mathcal{J}^2(G,H)^\perp$ be the orthogonal complement of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.

Corollary 3.5 *Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$ respectively. Then for each $f \in L^2(G)$ and for almost everywhere $x \in G$ we have*

$$P_{\mathcal{J}^2(G,H)^\perp}(f)(x) = T_H(f)(xH), \qquad P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH).$$

Corollary 3.6 *Let μ be the normalized G-invariant measure on G/H.*

(i) $\mathcal{J}^2(G,H)^\perp = \{\psi_q = \psi \circ q : \psi \in L^2(G/H,\mu)\}$.
(ii) *For $f \in \mathcal{J}^2(G,H)^\perp$ and $h \in H$, we have $R_h f = f$.*
(iii) *For $\psi \in L^2(G/H,\mu)$, we have $\|\psi_q\|_{L^2(G)} = \|\psi\|_{L^2(G/H,\mu)}$.*
(iv) *For $f, g \in \mathcal{J}^2(G,H)^\perp$, we have $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$.*

Remark 3.7. Invoking Corollary 3.6, one can regard the Hilbert space $L^2(G/H,\mu)$ as a closed subspace of $L^2(G)$, *i.e.*, the closed subspace consists of all $f \in L^2(G)$ that satisfy $R_h f = f$ for all $h \in H$. Then Theorem 3.2 and Proposition 3.4 guarantee that the linear map $T_H \colon L^2(G) \to L^2(G/H,\mu) \subset L^2(G)$ is an orthogonal projection onto $L^2(G/H,\mu)$.

4 Banach Convolution Algebras over Homogeneous Spaces of Compact Groups

In this section we present the abstract structure of function $*$-algebras over homogeneous space (left coset spaces) of compact groups.

Let $\mathcal{C}(G : H) := \{f \in \mathcal{C}(G) : R_h f = f \ \forall h \in H\}$. Then one can define

$$A(G : H) := \{f \in \mathcal{C}(G) : L_h f = f \text{ for } h \in H\},$$
$$A(G/H) := \{\varphi \in \mathcal{C}(G/H) : L_h \varphi = \varphi \ \text{ for } h \in H\}.$$

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For $1 \leq p < \infty$, we define

$$A^p(G:H) := \{f \in L^p(G) : L_h f = f \text{ for } h \in H\},$$
$$A^p(G/H,\mu) := \{\varphi \in L^p(G/H,\mu) : L_h\varphi = \varphi \text{ for } h \in H\},$$

where $L_z f(x) := f(z^{-1}x)$ and $R_z f(x) := f(xz)$, for $z, x \in G$.

It is easy to see that $A^p(G/H,\mu)$ is the topological closure of $A(G/H)$ in

$$L^p(G/H,\mu)$$

and hence it is a closed linear subspace of $L^p(G/H,\mu)$. One can also readily check that $A^p(G:H)$ is the topological closure of $A(G:H)$ in $L^p(G)$ and hence it is a closed linear subspace of $L^p(G)$.

Remark 4.1. Let G be a compact group and let H be a closed normal subgroup of G. Let μ be the normalized G-invariant measure over the left coset space G/H and $1 \leq p \leq \infty$. Let $\varphi \in \mathcal{C}(G/H)$ and $t \in H$. Then, for $xH \in G/H$, we have $t^{-1}xH = xH$. Hence we can write $L_t\varphi(xH) = \varphi(t^{-1}xH) = \varphi(xH)$. Thus we deduce that $\varphi \in A(G/H)$. Therefore, $A(G/H) = \mathcal{C}(G/H)$ and also $A^p(G/H,\mu) = L^p(G/H,\mu)$ if H is normal in G.

We continue by listing some basic observations.

Proposition 4.2 *Let μ be the normalized G-invariant measure on G/H. Then*

(i) *T_H maps $\mathcal{C}(G:H)$ onto $\mathcal{C}(G/H)$.*
(ii) *T_H maps $A(G:H)$ onto $A(G/H)$.*
(iii) *T_H maps $A^p(G:H)$ onto $A^p(G/H,\mu)$.*

Proof (i) This is straightforward.

(ii) Let $f \in A(G:H)$, $x \in G$, and $t \in H$. Then we have

$$L_t T_H(f)(xH) = T_H(f)(t^{-1}xH) = \int_H f(t^{-1}xh)\,dh = \int_H f(xh)\,dh = T_H(f)(xH),$$

which implies that $T_H(f) \in A(G/H)$. Let $\psi \in A(G/H)$. Then $\psi_q \in A(G:H)$ and $T_H(\psi_q) = \psi$. Hence, we deduce that T_H maps $A(G:H)$ onto $A(G/H)$.

(iii) Using (i) and since $A(G:H)$ is dense $L^p(G:H)$ and $A(G/H)$ is dense in

$$A^p(G/H)$$

as well, we conclude that T_H maps $A^p(G:H)$ onto $A^p(G/H,\mu)$. ■

Proposition 4.3 *Let G be a compact group and H be a closed subgroup of G. Let μ be the normalized G-invariant measure on G/H and $f, g \in L^1(G)$.*

(i) *For almost everywhere $x \in G$ we have*

$$T_H(f *_G g)(xH) = \int_{G/H}\Big(\int_H f(yt)\Big(\int_H g(t^{-1}y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH).$$

(ii) *For $g \in A^1(G:H)$ and almost everywhere $x \in G$ we have*

$$T_H(f *_G g)(xH) = \int_{G/H} T_H(f)(yH)T_H(g)(y^{-1}xH)\,d\mu(yH).$$

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Proof (i) Let $f, g \in L^1(G)$ and $x \in G$. We can write

$$T_H(f *_G g)(xH) = \int_H f *_G g(xh)\,dh = \int_H \Big(\int_G f(y)g(y^{-1}xh)\,dy\Big)\,dh.$$

Then, using Weil's formula, we get

$$\begin{aligned} T_H(f *_G g)(xH) &= \int_H \Big(\int_G f(y)g(y^{-1}xh)\,dy\Big)\,dh \\ &= \int_H \Big(\int_{G/H}\Big(\int_H f(yt)g((yt)^{-1}xh)\,dt\Big)\,d\mu(yH)\Big)\,dh \\ &= \int_H \Big(\int_{G/H}\Big(\int_H f(yt)g(t^{-1}y^{-1}xh)\,dt\Big)\,d\mu(yH)\Big)\,dh \\ &= \int_{G/H}\Big(\int_H f(yt)\Big(\int_H g(t^{-1}y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH). \end{aligned}$$

(ii) Now suppose that $g \in A^1(G:H)$. Thus $L_t g = g$ for all $t \in H$. Then using (i) and the fact that H is compact, we have

$$\begin{aligned} T_H(f *_G g)(xH) &= \int_{G/H}\Big(\int_H f(yt)\Big(\int_H g(t^{-1}y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH) \\ &= \int_{G/H}\Big(\int_H f(yt)\Big(\int_H g(y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH) \\ &= \int_{G/H}\Big(\int_H f(yt)\,dt\Big)\Big(\int_H g(y^{-1}xh)\,dh\Big)\,d\mu(yH) \\ &= \int_{G/H} T_H(f)(yH)T_H(g)(y^{-1}xH)\,d\mu(yH). \end{aligned}$$ ∎

For $\psi \in \mathcal{C}(G/H)$, let $J\psi: G/H \to \mathbb{C}$ be given by $J\psi(xH) := \int_H \psi(hxH)\,dh$, for all $xH \in G/H$. Then $J: \mathcal{C}(G/H) \to \mathcal{C}(G/H)$ given by $\psi \mapsto J\psi$ is a linear operator.

Remark 4.4. Let G be a compact group and let H be a closed normal subgroup of G. Then for all $x \in G$ and $h \in H$, we have $hxH = xH$. Hence, for $\psi \in \mathcal{C}(G/H)$ we get

$$J\psi(xH) = \int_H \psi(h^{-1}xH)\,dh = \int_H \psi(xH)\,dh = \psi(xH).$$

Thus we deduce that the linear operator $J: \mathcal{C}(G/H) \to \mathcal{C}(G/H)$ is the identity operator if H is normal in G.

The following theorem presents basic properties of the linear operator J in the framework of abstract harmonic analysis.

Theorem 4.5 *Let μ be the normalized G-invariant measure over G/H.*

(i) *For each $1 \le p < \infty$ and $\psi \in \mathcal{C}(G/H)$ we have $\|J\psi\|_{L^p(G/H,\mu)} \le \|\psi\|_{L^p(G/H,\mu)}$.*
(ii) *J maps $\mathcal{C}(G/H)$ onto $A(G/H)$.*
(iii) *J is a projection onto $A(G/H)$.*

Proof (i) Let $1 \le p < \infty$ and $\psi \in \mathcal{C}(G/H)$. Using compactness of H we get

$$\begin{aligned} \|J\psi\|^p_{L^p(G/H,\mu)} &= \int_{G/H} |J\psi(xH)|^p\,d\mu(xH) = \int_{G/H}\Big|\int_H \psi(hxH)\,dh\Big|^p\,d\mu(xH) \\ &\le \int_{G/H}\int_H |\psi(hxH)|^p\,dh\,d\mu(xH). \end{aligned}$$

Again using compactness of H and replacing x by $h^{-1}x$, we get

$$\begin{aligned}\int_{G/H}\int_H |\psi(hxH)|^p\,d\mu(xH)\,dh &= \int_H\Big(\int_{G/H}|\psi(hxH)|^p\,d\mu(xH)\Big)\,dh\\ &= \int_H\Big(\int_{G/H}|\psi(xH)|^p\,d\mu(h^{-1}xH)\Big)\,dh\\ &= \int_H\Big(\int_{G/H}|\psi(xH)|^p\,d\mu(xH)\Big)\,dh\\ &= \|\psi\|^p_{L^p(G/H,\mu)}.\end{aligned}$$

(ii) Let $\psi \in \mathcal{C}(G/H)$ and $t \in H$. Then we have

$$L_tJ\psi(xH) = J\psi(t^{-1}xH) = \int_H \psi(ht^{-1}xH)\,dh = \int_H \psi(hxH) = J\psi(xH),$$

for all $x \in G$. This implies that $J\psi \in A(G/H)$. Now suppose that $\psi \in A(G/H)$. Then we have $J\psi(xH) = \int_H \psi(hxH)\,dh = \int_H \psi(xH)\,dh = \psi(xH)$, for all $x \in G$. Thus $J\psi = \psi$. Hence, we deduce that J maps $\mathcal{C}(G/H)$ onto $A(G/H)$.

(iii) Let $\psi \in \mathcal{C}(G/H)$ and $x \in G$. Then using the fact that $J\psi \in A(G/H)$, we have

$$J(J\psi)(xH) = \int_H J\psi(hxH)\,dh = \int_H J\psi(xH)\,dh = J\psi(xH),$$

which implies that $J(J\psi) = J\psi$. Hence, we deduce that $J \circ J = J$. Also, since the range of the linear operator J is precisely $A(G/H)$, we conclude that J is a linear projection onto $A(G/H)$. ∎

Then we deduce the following consequences.

Corollary 4.6 *Let μ be the normalized G-invariant measure on G/H and $1 \le p < \infty$.*

(i) *The linear operator $J\colon \mathcal{C}(G/H) \to A(G/H)$ has a unique extension to a bounded linear operator $J_p\colon L^p(G/H,\mu) \to A^p(G/H,\mu)$, satisfying*

$$\|J_p\psi\|_{L^p(G/H,\mu)} \le \|\psi\|_{L^p(G/H,\mu)}.$$

(ii) *The linear operator J_p maps $L^p(G/H,\mu)$ onto $A^p(G/H,\mu)$.*

(iii) *The linear operator J_p is a projection onto $A^p(G/H)$.*

Remark 4.7. Let G be a compact group and let H be a closed normal subgroup of G. Let $1 \le p < \infty$. Then the extended linear operator $J_p\colon L^p(G/H,\mu) \to A^p(G/H,\mu)$ is the identity operator.

Definition 4.8 Let G be a compact group, H a closed subgroup of G, and μ the normalized G-invariant measure over G/H. For $\varphi, \psi \in \mathcal{C}(G/H)$, let $\varphi *_{G/H} \psi\colon G/H \to \mathbb{C}$ be given by

$$\varphi *_{G/H} \psi(xH) = \int_{G/H} \varphi(yH)J\psi(y^{-1}xH)\,d\mu(yH), \tag{4.1}$$

for all $xH \in G/H$.

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Remark 4.9. Let G be a compact group and H a closed normal subgroup of G. And let $\varphi, \psi \in \mathcal{C}(G/H)$ and $x \in G$. Invoking Remark 4.4, the linear map J is the identity operator and hence we have

$$\begin{aligned}\varphi *_{G/H} \psi(xH) &= \int_{G/H} \varphi(yH) J\psi(y^{-1}xH)\, d\mu(yH)\\ &= \int_{G/H} \varphi(yH)\psi(y^{-1}xH)\, d\mu(yH)\\ &= \int_{G/H} \varphi(yH)\psi(y^{-1}HxH)\, d\mu(yH)\\ &= \int_{G/H} \varphi(yH)\psi((yH)^{-1}xH)\, d\mu(yH),\end{aligned}$$

for all $xH \in G/H$. Hence, we deduce that the convolution defined by (4.1) coincides with the canonical convolution over the quotient group G/H if H is normal in G, see [1, 22].

The following results state interesting properties of the convolution $*_{G/H}$.

Proposition 4.10 *Let μ be the normalized G-invariant measure over G/H; let $\varphi, \psi \in \mathcal{C}(G/H)$. Then we have*

(i) $(\varphi *_{G/H} \psi)_q = \varphi_q *_G \psi_q$,
(ii) $\varphi *_{G/H} \psi = T_H(\varphi_q *_G \psi_q)$,
(iii) $L_z(\varphi *_{G/H} \psi) = (L_z\varphi) *_{G/H} \psi$.

Proof (i) Let $x \in G$. Then using Weil's formula, we have

$$\begin{aligned}\varphi_q *_G \psi_q(x) &= \int_G \varphi_q(y)\psi_q(y^{-1}x)\, dy = \int_G \varphi(yH)\psi(y^{-1}xH)\, dy\\ &= \int_{G/H}\Big(\int_H \varphi(yhH)\psi((yh)^{-1}xH)\, dh\Big)\, d\mu(yH)\\ &= \int_{G/H}\Big(\int_H \varphi(yH)\psi(h^{-1}y^{-1}xH)\, dh\Big)\, d\mu(yH)\\ &= \int_{G/H} \varphi(yH)\Big(\int_H \psi(h^{-1}y^{-1}xH)\, dh\Big)\, d\mu(yH)\\ &= \int_{G/H} \varphi(yH)J\psi(y^{-1}xH)\, d\mu(yH) = \varphi *_{G/H} \psi(xH),\end{aligned}$$

which implies that $(\varphi *_{G/H} \psi)_q = \varphi_q *_G \psi_q$.

(ii) Let $x \in G$. Invoking the definition of $*_G$ and since H is compact, we can write

$$\begin{aligned}T_H(\varphi_q *_G \psi_q)(xH) &= \int_H \varphi_q *_G \psi_q(xh)\, dh = \int_H\Big(\int_G \varphi_q(y)\psi_q(y^{-1}xh)\, dy\Big)\, dh\\ &= \int_H\Big(\int_G \varphi(yH)\psi(y^{-1}xhH)\, dy\Big)\, dh\\ &= \int_H\Big(\int_G \varphi(yH)\psi(y^{-1}xH)\, dy\Big)\, dh = \int_G \varphi(yH)\psi(y^{-1}xH)\, dy,\end{aligned}$$

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Thus, using (i), we get

$$T_H(\varphi_q *_G \psi_q)(xH) = \int_G \varphi(yH)\psi(y^{-1}xH)\,dy = \varphi_q *_G \psi_q(x) \\ = (\varphi *_{G/H} \psi)_q(x) = \varphi *_{G/H} \psi(xH),$$

implying that $\varphi *_{G/H} \psi = T_H(\varphi_q *_G \psi_q)$.

(iii) Let $z \in G$. Then we can write

$$L_z(\varphi *_{G/H} \psi)(xH) = \varphi *_{G/H} \psi(z^{-1}xH) = \int_{G/H} \varphi(yH)J\psi(y^{-1}z^{-1}xH)\,d\mu(yH) \\ = \int_{G/H} \varphi(yH)J\psi((zy)^{-1}xH)\,d\mu(yH).$$

Replacing y by $z^{-1}y$ and using the fact that μ is G-invariant, we get

$$\begin{aligned}\int_{G/H} \varphi(yH)J\psi((zy)^{-1}xH)\,d\mu(yH) &= \int_{G/H} \varphi(z^{-1}yH)J\psi(y^{-1}xH)\,d\mu(z^{-1}yH) \\ &= \int_{G/H} \varphi(z^{-1}yH)J\psi(y^{-1}xH)\,d\mu(yH) \\ &= \int_{G/H} L_z\varphi(yH)J\psi(y^{-1}xH)\,d\mu(yH) \\ &= (L_z\varphi) *_{G/H} \psi(xH).\end{aligned}$$

■

Proposition 4.11 *Let μ be the normalized G-invariant measure over G/H; let $f, g \in \mathcal{C}(G)$ with $f \in \mathcal{C}(G{:}H)$. Then we have $T_H(f *_G g) = T_H(f) *_{G/H} T_H(g)$. In particular, for all $\varphi \in \mathcal{C}(G/H)$ and $g \in \mathcal{C}(G)$, we have $T_H(\varphi_q *_G g) = \varphi *_{G/H} T_H(g)$.*

Proof Let $f, g \in \mathcal{C}(G)$ with $f \in \mathcal{C}(G{:}H)$. Then using Proposition 4.3, for $x \in G$, we get

$$\begin{aligned}T_H(f *_G g)(xH) &= \int_{G/H}\Big(\int_H f(yt)\Big(\int_H g(t^{-1}y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH) \\ &= \int_{G/H} f(y)\Big(\int_H\Big(\int_H g(t^{-1}y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH) \\ &= \int_{G/H} T_H(f)(yH)\Big(\int_H\Big(\int_H g(t^{-1}y^{-1}xh)\,dh\Big)\,dt\Big)\,d\mu(yH) \\ &= \int_{G/H} T_H(f)(yH)\Big(\int_H T_H(g)(t^{-1}y^{-1}xH)\,dt\Big)\,d\mu(yH) \\ &= \int_{G/H} T_H(f)(yH)J(T_H(g))(y^{-1}xH)\,d\mu(yH) \\ &= T_H(f) *_{G/H} T_H(g)(xH).\end{aligned}$$

Now let $\varphi \in \mathcal{C}(G/H)$. Then $f := \varphi_q \in \mathcal{C}(G{:}H)$. Thus we get

$$T_H(\varphi_q *_G g) = T_H(\varphi_q) *_{G/H} T_H(g) = \varphi *_{G/H} T_H(g).$$

■

Remark 4.12. Let H be a closed normal subgroup of a compact group G. Let μ be the normalized G-invariant measure over G/H. Then μ is a Haar measure over the

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quotient group G/H. Then using Proposition 4.3, for $x \in G$ and $f, g \in \mathcal{C}(G)$, we can write

$$\begin{aligned} T_H(f *_G g)(xH) &= \int_{G/H} \Big(\int_H f(yt) \Big(\int_H g(t^{-1}y^{-1}xh)\, dh \Big)\, dt \Big)\, d\mu(yH) \\ &= \int_{G/H} \Big(\int_H f(yt)\, dt \Big) \Big(\int_H g(y^{-1}xh)\, dh \Big)\, d\mu(yH) \\ &= \int_{G/H} T_H(f)(xH) T_H(g)(y^{-1}xH) d\mu(yH) \\ &= T_H(f) *_{G/H} T_H(g). \end{aligned}$$

This property of convolution over quotient groups has appeared in [22] as well.

Henceforth, we call $\varphi *_{G/H} \psi$ the *convolution* of φ and ψ. It is easy to check that the map $*_{G/H}: \mathcal{C}(G/H) \times \mathcal{C}(G/H) \to \mathcal{C}(G/H)$ given by $(\varphi, \psi) \mapsto \varphi *_{G/H} \psi$ is bilinear. Also, it can be readily seen that the linear space $\mathcal{C}(G/H)$ with respect to $*_{G/H}$ as multiplication is an associative algebra. It should be mentioned that the associativity of the convolution $*_{G/H}$ follows from Proposition 4.10 (i) and (ii).

The next result shows that the associative algebra $\mathcal{C}(G/H)$ with respect to the norm $\|\cdot\|_{L^p(G/H,\mu)}$ is a normed algebra, for all $1 \le p < \infty$.

Theorem 4.13 *Let μ be the normalized G-invariant measure on G/H and $1 \le p < \infty$. Then, for all $\varphi, \psi \in \mathcal{C}(G/H)$, we have*

$$\|\varphi *_{G/H} \psi\|_{L^p(G/H,\mu)} \le \|\varphi\|_{L^p(G/H,\mu)} \|\psi\|_{L^p(G/H,\mu)}.$$

Proof Let $\varphi, \psi \in \mathcal{C}(G/H)$ and $1 \le p < \infty$. Then, using (2.1), (3.1), and Proposition 4.10, we have

$$\begin{aligned} \|\varphi *_{G/H} \psi\|_{L^p(G/H,\mu)} &= \|(\varphi *_{G/H} \psi)_q\|_{L^p(G)} = \|\varphi_q *_G \psi_q\|_{L^p(G)} \\ &\le \|\varphi_q\|_{L^p(G)} \|\psi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)} \|\psi\|_{L^p(G/H,\mu)}. \end{aligned}$$ ∎

Then we can present the following interesting result.

Theorem 4.14 *Let μ be the normalized G-invariant measure on G/H and $1 \le p < \infty$. The convolution map $*_{G/H}: \mathcal{C}(G/H) \times \mathcal{C}(G/H) \to \mathcal{C}(G/H)$ given by (4.1) has a unique extension to $*^p_{G/H}: L^p(G/H,\mu) \times L^p(G/H,\mu) \to L^p(G/H,\mu)$, in which the Banach function space $L^p(G/H,\mu)$ equipped with the extended convolution is a Banach algebra.*

Proof Invoking density of $\mathcal{C}(G/H)$ in $L^p(G/H,\mu)$ and continuity of the convolution $*_{G/H}$ via Theorem 4.13, one can uniquely extend the convolution map

$$*_{G/H}: \mathcal{C}(G/H) \times \mathcal{C}(G/H) \to \mathcal{C}(G/H)$$

given by (4.1) to the convolution map

$$*^p_{G/H}: L^p(G/H,\mu) \times L^p(G/H,\mu) \to L^p(G/H,\mu)$$

such that

$$\|\varphi *_{G/H} \psi\|_{L^p(G/H,\mu)} \le \|\varphi\|_{L^p(G/H,\mu)} \|\psi\|_{L^p(G/H,\mu)},$$

for all $\varphi, \psi \in L^p(G/H, \mu)$, which equivalently implies that the Banach function space $L^p(G/H, \mu)$ equipped with the extended convolution is a Banach convolution function algebra. ∎

We deduce the following corollary concerning the explicit construction of $*^p_{G/H}$.

Corollary 4.15 *Let μ be the normalized G-invariant measure on G/H and $1 \leq p < \infty$. Then, for all $\varphi, \psi \in L^p(G/H, \mu)$, we have*

$$\varphi *^p_{G/H} \psi(xH) = \int_{G/H} \varphi(yH) J_p \psi(y^{-1}xH)\, d\mu(yH),$$

for almost everywhere $xH \in G/H$.

The next result lists some of the properties of the convolution $*^p_{G/H}$.

Proposition 4.16 *Let μ be the normalized G-invariant measure over G/H. Also, let $\varphi, \psi \in L^p(G/H, \mu)$. Then we have*

(i) $(\varphi *^p_{G/H} \psi)_q = \varphi_q *_G \psi_q$,

(ii) $\varphi *^p_{G/H} \psi = T_H(\varphi_q *_G \psi_q)$,

(iii) $L_z(\varphi *^p_{G/H} \psi) = (L_z\varphi) *^p_{G/H} \psi$.

Then we have the following corollary concerning the subspaces $A^p(G/H, \mu)$.

Corollary 4.17 *Let μ be the normalized G-invariant measure on G/H and $1 \leq p < \infty$. Then $A^p(G/H, \mu)$ is a right ideal of the Banach function algebra $L^p(G/H, \mu)$. In particular, $A^p(G/H, \mu)$ is a Banach function sub-algebra of $L^p(G/H, \mu)$*

Remark 4.18. Let G be a compact group and H a closed normal subgroup of G. Let μ be the normalized G-invariant measure on G/H and $1 \leq p < \infty$. Then automatically μ is precisely a Haar measure of the compact quotient group G/H. Also, let $\varphi, \psi \in L^p(G/H, \mu)$. Invoking Remark 4.7 and Remark 4.9, the linear map J_p is the identity operator and hence we have

$$\varphi *^p_{G/H} \psi(xH) = \int_{G/H} \varphi(yH)\psi(y^{-1}HxH)\, d\mu(yH),$$

for almost everywhere $xH \in G/H$. Thus we deduce that the extended convolution $*^p_{G/H}$ coincides with the canonical convolution over the quotient group G/H if H is normal in G.

Definition 4.19 Let G be a compact group and H a closed subgroup of G. For $\varphi \in \mathcal{C}(G/H)$, let $\varphi^{*_{G/H}}: G/H \to \mathbb{C}$ be given by

$$\varphi^{*_{G/H}}(xH) = \int_H \overline{\varphi(h^{-1}x^{-1}H)}\, dh, \tag{4.2}$$

for all $xH \in G/H$.

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Let $xH = yH$ for $x, y \in G$. Then we have $y = xt$ for some $t \in H$. Hence we can write

$$\int_H \overline{\varphi(h^{-1}x^{-1}H)}\,dh = \int_H \overline{\varphi(h^{-1}t^{-1}x^{-1}H)}\,dh = \int_H \overline{\varphi(h^{-1}y^{-1}H)}\,dh,$$

which implies that $\varphi^{*_{G/H}}(xH) = \varphi^{*_{G/H}}(yH)$. This guarantees that $\varphi^{*_{G/H}}$ is a well-defined function over G/H.

Henceforth we call $\varphi^{*_{G/H}}$ an involution of φ. It is easy to check that the map

$$^{*_{G/H}}\colon \mathcal{C}(G/H) \to \mathcal{C}(G/H)$$

given by $\varphi \mapsto \varphi^{*_{G/H}}$ is conjugate linear.

Remark 4.20. Let G be a compact group and H a closed normal subgroup of G. Let $\varphi \in \mathcal{C}(G/H)$ and $x \in G$. Then, for all $x \in G$ and $h \in H$, we have $hxH = xH$. Hence for $\psi \in \mathcal{C}(G/H)$, we get

$$\varphi^{*_{G/H}}(xH) = \int_H \overline{\varphi(h^{-1}x^{-1}H)}\,dh = \int_H \overline{\varphi(x^{-1}H)}\,dh = \overline{\varphi(x^{-1}H)}.$$

Thus, we deduce that the involution defined by (4.2) coincides with the canonical involution over the compact quotient group G/H if H is normal in G.

Proposition 4.21 *Let $\varphi \in \mathcal{C}(G/H)$, $1 \le p < \infty$, and let μ be the normalized G-invariant measure on G/H. Then we have*

(i) $\varphi^{*^{G/H} *^{G/H}} = J\varphi$,

(ii) $\varphi^{*_{G/H}} = T_H(\varphi_q^{*_G})$,

(iii) $\|\varphi^{*_{G/H}}\|_{L^p(G/H,\mu)} \le \|\varphi\|_{L^p(G/H,\mu)}$.

Proof (i) Let $x \in G$ and $h \in H$. Then we have

$$\overline{\varphi^{*_{G/H}}(h^{-1}x^{-1}H)} = \int_H \varphi(t^{-1}xhH)\,dt = \int_H \varphi(t^{-1}xH)\,dt.$$

Thus we get

$$\varphi^{*^{G/H} *^{G/H}}(xH) = \int_H \overline{\varphi^{*_{G/H}}(h^{-1}x^{-1}H)}\,dh = \int_H \varphi(t^{-1}xH)\,dt = J\varphi(xH).$$

(ii) Let $x \in G$. Then we have

$$\begin{aligned} T_H(\varphi_q^{*_G})(xH) &= \int_H \varphi_q^{*_G}(xh)\,dh \\ &= \int_H \overline{\varphi_q(h^{-1}x^{-1})}\,dh = \int_H \overline{\varphi(h^{-1}x^{-1}H)}\,dh = \varphi^{*_{G/H}}(xH). \end{aligned}$$

(iii) Using (ii), compactness of H, and Weil's formula, we have

$$\begin{aligned} \|\varphi^{*_{G/H}}\|_{L^p(G/H,\mu)} &= \int_{G/H} |\varphi^{*_{G/H}}(xH)|^p\,d\mu(xH) \\ &= \int_{G/H} |T_H(\varphi_q^{*_G})(xH)|^p\,d\mu(xH) \\ &\le \int_{G/H} T_H(|\varphi_q^{*_G}|^p)(xH)\,d\mu(xH) \\ &= \int_G |\varphi_q^{*_G}(x)|^p\,dx = \|\varphi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)}. \end{aligned}$$

■

Corollary 4.22 *Let $\varphi \in A(G/H)$, $1 \le p < \infty$, and let μ be the normalized G-invariant measure on G/H. Then we have*

(i) $\varphi^{*^{G/H}*^{G/H}} = \varphi$,
(ii) $\|\varphi^{*_{G/H}}\|_{L^p(G/H,\mu)} = \|\varphi\|_{L^p(G/H,\mu)}$,
(iii) $(\varphi^{*_{G/H}})_q = \varphi_q^{*_G}$.

Then we can deduce the following result.

Proposition 4.23 *Let $\varphi, \psi \in \mathcal{C}(G/H)$. Then we have*

$$(\varphi *_{G/H} \psi)^{*^{G/H}} = \psi^{*^{G/H}} *_{G/H} \varphi^{*^{G/H}}.$$

Proof Using Propositions 4.10, 4.21, and (2.2) we have

$$(\varphi *_{G/H} \psi)^{*^{G/H}} = T_H\big((\varphi *_{G/H} \psi)_q^{*^G}\big) = T_H\big((\varphi_q *_G \psi_q)^{*^G}\big) = T_H(\psi_q^{*^G} *_G \varphi_q^{*^G}).$$

Since $\varphi_q^{*^G} \in A(G:H)$, using Proposition 4.3, we can write

$$\begin{aligned} T_H(\psi_q^{*^G} *_G \varphi_q^{*^G})(xH) &= \int_{G/H} T_H(\psi_q^{*^G})(yH) T_H(\varphi_q^{*^G})(y^{-1}xH)\, d\mu(yH) \\ &= \int_{G/H} T_H(\psi_q^{*^G})(yH) J T_H(\varphi_q^{*^G})(y^{-1}xH)\, d\mu(yH) \\ &= T_H(\psi_q^{*^G}) *_{G/H} T_H(\varphi_q^{*^G})(xH) = \psi^{*^{G/H}} *_{G/H} \varphi^{*^{G/H}}(xH), \end{aligned}$$

for $x \in G$, which completes the proof. ■

Then we can summarize our recent results as follows.

Corollary 4.24 *Let μ be the normalized G-invariant measure over the homogeneous space G/H and $p \ge 1$. The normed space $(A(G/H), \|\cdot\|_{L^p(G/H,\mu)})$ equipped with the convolution $*_{G/H}$ and the involution $^{*_{G/H}}$ is a normed $*$-algebra.*

The following proposition presents properties of involution over L^p-spaces.

Proposition 4.25 *Let μ be the normalized G-invariant measure on G/H and $1 \le p < \infty$. The involution map $^{*^{G/H}}: \mathcal{C}(G/H) \to \mathcal{C}(G/H)$ given by (4.2) has a unique extension to $^{*^{G/H}}: L^p(G/H,\mu) \to L^p(G/H,\mu)$ which, for all $\varphi \in L^p(G/H,\mu)$, satisfies*

(i) $\varphi^{*^{G/H}*^{G/H}} = J_p\varphi$,
(ii) $\varphi^{*_{G/H}} = T_H(\varphi_q^{*_G})$,
(iii) $\|\varphi^{*_{G/H}}\|_{L^p(G/H,\mu)} \le \|\varphi\|_{L^p(G/H,\mu)}$.

Proof Let $\varphi \in L^p(G/H,\mu)$. Invoking the density of $\mathcal{C}(G/H)$ in $L^p(G/H,\mu)$, let $\{\varphi_n\} \in \mathcal{C}(G/H)$ with $\varphi = \lim_n \varphi_n$. Then we define $\varphi^{*^{G/H}} := \lim_n \varphi_n^{*^{G/H}}$. Then $^{*^{G/H}}: L^p(G/H,\mu) \to L^p(G/H,\mu)$ is well defined and satisfies (i)–(iii). ■

Corollary 4.26 *Let μ be a G-invariant measure on G/H and $1 \le p < \infty$. Then we have $\varphi^{*_{G/H}}(xH) = \int_H \overline{\varphi(h^{-1}x^{-1}H)}\,dh$, for almost all $x \in G$.*

Corollary 4.27 *Let μ be the normalized G-invariant measure on G/H and $\varphi \in A^p(G/H,\mu)$ with $1 \le p < \infty$. Then we have*

(i) $\varphi^{*^{G/H}*^{G/H}} = \varphi$,
(ii) $\|\varphi^{*_{G/H}}\|_{L^p(G/H,\mu)} = \|\varphi\|_{L^p(G/H,\mu)}$,
(iii) $(\varphi^{*_{G/H}})_q = \varphi_q^{*_G}$.

The next result summarizes our recent results in terms of the Banach $*$-algebras.

Theorem 4.28 *Let μ be the normalized G-invariant measure on G/H and $1 \le p < \infty$. The Banach function algebra $A^p(G/H,\mu)$ equipped with the extended involution is a Banach function $*$-algebra.*

We finish this section by the following interesting observations.

Proposition 4.29 *Let μ be the normalized G-invariant measure over the compact homogeneous space G/H, $p \ge 1$ and $\varphi \in L^p(G/H,\mu)$. Then*

$$(\varphi^{*^{G/H}})_q = \big((J_p\varphi)_q\big)^{*^G}. \tag{4.3}$$

Proof Let μ be the normalized G-invariant measure over the compact homogeneous space G/H and $\varphi \in L^p(G/H,\mu)$. Then for $x \in G$, we have

$$\begin{aligned}((J_p\varphi)_q)^{*^G}(x) &= \overline{(J_p\varphi)_q(x^{-1})} = \overline{J_p\varphi(x^{-1}H)}\\ &= \Big(\int_H \varphi(hx^{-1}H)\,dh\Big)^{-} = \int_H \overline{\varphi(hx^{-1}H)}\,dh\\ &= \int_H \overline{\varphi(h^{-1}x^{-1}H)}\,dh = \varphi^{*^{G/H}}(xH) = (\varphi^{*^{G/H}})_q(x),\end{aligned}$$

which completes the proof. ∎

Corollary 4.30 *Let μ be the normalized G-invariant measure over the compact homogeneous space G/H, $p \ge 1$, and $\varphi \in A^p(G/H,\mu)$. Then, $(\varphi^{*^{G/H}})_q = \varphi_q^{*^G}$.*

5 Abstract Representations of Convolution Function Algebras over Homogeneous Spaces of Compact Groups

In this section we present a classical study for a class of abstract linear representations on Banach convolution function algebras over homogeneous spaces of compact groups. It is still assumed that G is a compact group and H is a closed subgroup of G. Also, μ is the normalized G-invariant measure over the compact homogeneous space G/H associated with Weil's formula and $1 \le p < \infty$. We then introduce a class of structured abstract linear representations of the Banach function sub-algebras of $L^p(G/H,\mu)$.

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For a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, define

$$T_H^\pi := \int_H \pi(h)\, dh, \tag{5.1}$$

where the operator valued integral (5.1) is considered in the weak sense. In other words, $\langle T_H^\pi \zeta, \xi\rangle = \int_H \langle \pi(h)\zeta, \xi\rangle\, dh$, for $\zeta, \xi \in \mathcal{H}_\pi$. The function $h \mapsto \langle \pi(h)\zeta, \xi\rangle$ is bounded and continuous on H and H is compact. Thus the right integral is the ordinary integral of a function in $L^1(H)$. Hence, T_H^π is a bounded operator on $\mathcal{H}_\pi$ with $\|T_H^\pi\| \le 1$.

Let $\mathcal{K}_\pi^H := \{\zeta \in \mathcal{H}_\pi : \pi(h)\zeta = \zeta \text{ for all } h \in H\}$. Then $\mathcal{K}_\pi^H$ is a closed subspace of $\mathcal{H}_\pi$ and we have $\mathcal{R}(T_H^\pi) = \mathcal{K}_\pi^H$, where $\mathcal{R}(T_H^\pi) = \{T_H^\pi \zeta : \zeta \in \mathcal{H}_\pi\}$.

Next we present basic properties of the linear operator T_H^π.

Proposition 5.1 *Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then*

(i) *The linear operator T_H^π is a partial isometric (orthogonal) projection;*

(ii) *The linear operator T_H^π is the identity operator if and only if $\pi(h) = I$ for all $h \in H$.*

Proof (i) Using compactness of H, it can be easily checked that $(T_H^\pi)^* = T_H^\pi$. As well, we achieve that

$$\begin{aligned} T_H^\pi T_H^\pi &= \Big(\int_H \pi(h)\, dh\Big)\Big(\int_H \pi(t)\, dt\Big) = \int_H \pi(h)\Big(\int_H \pi(t)\, dt\Big)\, dh \\ &= \int_H \Big(\int_H \pi(h)\pi(t)\, dt\Big)\, dh = \int_H \Big(\int_H \pi(ht)\, dt\Big)\, dh = \int_H T_H^\pi dt = T_H^\pi. \end{aligned}$$

(ii) Let $\pi(h) = I$ for all $h \in H$. Thus, it is straightforward to see that $T_H^\pi = I$. Conversely, assume that $T_H^\pi = I$. Then for $t \in H$, we can write

$$\begin{aligned} \pi(t) = \pi(t) I = \pi(t) T_H^\pi = \pi(t)\Big(\int_H \pi(h) dh\Big) &= \int_H \pi(t)\pi(h)\, dh \\ = \int_H \pi(th)\, dh = \int_H \pi(h)\, dh = T_H^\pi = I. \end{aligned}$$

■

Let $\pi: G \to \mathcal{U}(\mathcal{H}_\pi)$ be a continuous unitary representation of G on the Hilbert space $\mathcal{H}_\pi$ with $T_H^\pi \neq 0$. For $xH \in G/H$, define $\Gamma_\pi(xH) := \pi(x) T_H^\pi$. Thus, we have

$$\langle \zeta, \Gamma_\pi(xH)\xi\rangle = \langle \zeta, \pi(x) T_H^\pi \xi\rangle,$$

for all $\zeta, \xi \in \mathcal{H}_\pi$.

Then we have

$$\Gamma_\pi(xH) = \pi(x) \int_H \pi(h)\, dh = \int_H \pi(x)\pi(h)\, dh = \int_H \pi(xh)\, dh.$$

For $\varphi \in L^1(G/H, \mu)$, define the linear operator $\Gamma_\pi(\varphi)$ on $\mathcal{H}_\pi$ via

$$\Gamma_\pi(\varphi) := \int_{G/H} \varphi(xH)\Gamma_\pi(xH)\, d\mu(xH), \tag{5.2}$$

The operator-valued integral (5.2) is also considered in the weak sense, *i.e.*,

$$\langle \Gamma_\pi(\varphi)\zeta, \xi\rangle = \int_{G/H} \varphi(xH)\langle \Gamma_\pi(xH)\zeta, \xi\rangle\, d\mu(xH),$$

for all $\zeta, \xi \in \mathcal{H}_\pi$.

In other words, for the continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G with $T_H^\pi \neq 0$ and $\zeta, \xi \in \mathcal{H}_\pi$, we have $\langle \Gamma_\pi(\varphi)\zeta, \xi\rangle = \int_{G/H} \varphi(xH)\langle \pi(x)T_H^\pi\zeta, \xi\rangle\, d\mu(xH)$.

Thus for $\zeta, \xi \in \mathcal{H}_\pi$, we get

$$\begin{aligned}
|\langle \Gamma_\pi(\varphi)\zeta, \xi\rangle| &= \Big| \int_{G/H} \varphi(xH)\langle \pi(x)T_H^\pi\zeta, \xi\rangle\, d\mu(xH)\Big| \\
&\le \int_{G/H} |\varphi(xH)||\langle \pi(x)T_H^\pi\zeta, \xi\rangle|\, d\mu(xH) \\
&\le \int_{G/H} |\varphi(xH)|\|\pi(x)T_H^\pi\zeta\|\|\xi\|\, d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)|\|T_H^\pi\zeta\|\|\xi\|\, d\mu(xH) \\
&\le \int_{G/H} |\varphi(xH)|\|\zeta\|\|\xi\|\, d\mu(xH) = \|\zeta\|\|\xi\|\|\varphi\|_{L^1(G/H,\mu)}.
\end{aligned}$$

Therefore, $\Gamma_\pi(\varphi)$ is a bounded linear operator on $\mathcal{H}_\pi$ satisfying

$$\|\Gamma_\pi(\varphi)\| \le \|\varphi\|_{L^1(G/H,\mu)}. \tag{5.3}$$

The next results present basic properties of the linear operators $\Gamma_\pi(\varphi)$ with $\varphi \in L^1(G/H, \mu)$.

Proposition 5.2 *Let μ be the normalized G-invariant measure on the compact homogeneous space G/H. Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$, $f \in L^1(G)$, and $\varphi \in L^1(G/H, \mu)$. Then*

(i) $\Gamma_\pi(T_H(f)) = \pi(f)T_H^\pi$,
(ii) $\Gamma_\pi(T_H(f))T_H^\pi = \Gamma_\pi(T_H(f))$,
(iii) *If $\pi(R_h f) = \pi(f)$ for all $h \in H$, we have $\Gamma_\pi(T_H(f)) = \pi(f)$,*
(iv) $\Gamma_\pi(\varphi) = \pi(\varphi_q)$.

Proof (i) Let $\zeta, \xi \in \mathcal{H}_\pi$. Invoking the definition of the linear operator $\Gamma_\pi(T_H(f))$ and using Weil's formula in the weak sense, we can write

$$\begin{aligned}
\langle \Gamma_\pi(T_H(f))\zeta, \xi\rangle &= \int_{G/H} T_H(f)(xH)\langle \Gamma_\pi(xH)\zeta, \xi\rangle\, d\mu(xH) \\
&= \int_{G/H} T_H(f)(xH)\langle \pi(x)T_H^\pi\zeta, \xi\rangle\, \mu(xH) \\
&= \int_{G/H} T_H(f.g_{\zeta,\xi})(xH)\, d\mu(xH) \\
&= \int_G f(x)\langle \pi(x)T_H^\pi\zeta, \xi\rangle\, dx \\
&= \Big\langle \Big(\int_G f(x)\pi(x)\, dx\Big) T_H^\pi\zeta, \xi\Big\rangle = \langle \pi(f)T_H^\pi\zeta, \xi\rangle,
\end{aligned}$$

where $g_{\zeta,\xi}\colon G \to \mathbb{C}$ is given by $g_{\zeta,\xi}(x) := \langle \pi(x)T_H^\pi\zeta, \xi\rangle$ for $x \in G$. Since $\zeta, \xi \in \mathcal{H}_\pi$ was arbitrary, we deduce that $\Gamma_\pi(T_H(f)) = \pi(f)T_H^\pi$.

(ii) Let $f \in L^1(G)$. Then using (i), and since T_H^π is a projection, we get

$$\Gamma_\pi(T_H(f))T_H^\pi = \pi(f)T_H^\pi T_H^\pi = \pi(f)T_H^\pi = \Gamma_\pi(T_H(f)).$$

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(iii) Let $f \in L^1(G)$ with $\pi(R_h f) = \pi(f)$ for all $h \in H$. Then using (i), we get

$$\Gamma_\pi(T_H(f)) = \pi(f)T_H^\pi = \pi(f)\Big(\int_H \pi(h)dh\Big) = \int_H \pi(f)\pi(h)\,dh$$
$$= \int_H \pi(f)\pi(h^{-1})\,dh = \int_H \pi(R_h f)\,dh = \int_H \pi(f)\,dh = \pi(f).$$

(iv) Let $\varphi \in L^1(G/H,\mu)$. Then we have $\varphi_q \in L^1(G:H)$. Hence, $R_h\varphi_q = \varphi_q$ for all $h \in H$. Thus, we get $\pi(R_h\varphi_q) = \pi(\varphi_q)$ for all $h \in H$. Therefore, using (iii), we can write $\Gamma_\pi(\varphi) = \Gamma_\pi(T_H(\varphi_q)) = \pi(\varphi_q)$. ■

The next proposition presents the connection of Γ_π with $*^{G/H}$.

Proposition 5.3 *Let μ be the normalized G-invariant measure on G/H. Let $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$ and $p \geq 1$. Then for $\varphi \in L^p(G/H,\mu)$, we have*

$$\Gamma_\pi(\varphi^{*^{G/H}}) = \Gamma_\pi(J_p\varphi)^*. \tag{5.4}$$

Proof Let $\varphi \in L^p(G/H,\mu)$. Then using (4.3), we have

$$\Gamma_\pi(\varphi^{*^{G/H}}) = \pi\big((\varphi^{*^{G/H}})_q\big) = \pi\big(((J_p\varphi)_q)^{*^G}\big) = \pi((J_p\varphi)_q)^* = \Gamma_\pi(J_p\varphi)^*. \quad ■$$

The following result presents an interesting commutation relation of Γ_π with J and T_H^π.

Proposition 5.4 *Let μ be the normalized G-invariant measure on the compact homogeneous space G/H. Let $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$ and $p \geq 1$. Then $\Gamma_\pi \circ J_p = T_H^\pi \circ \Gamma_\pi$.*

Proof Let $\varphi \in L^p(G/H,\mu)$. Then we have

$$\Gamma_\pi(J_p\varphi) = \Gamma_\pi\Big(\int_H L_h\varphi\,dh\Big) = \int_H \Gamma_\pi(L_h\varphi)\,dh = \int_H \pi(h)\Gamma_\pi(\varphi)\,dh$$
$$= \Big(\int_H \pi(h)\,dh\Big)\Gamma_\pi(\varphi) = T_H^\pi\Gamma_\pi(\varphi). \quad ■$$

The following theorem shows that the map $\varphi \mapsto \Gamma_\pi(\varphi)$, defines a representation of the Banach algebra $L^p(G/H,\mu)$.

Theorem 5.5 *Let μ be the normalized G-invariant measure on G/H and $p \geq 1$. Also let $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then $\Gamma_\pi\colon L^p(G/H,\mu) \to \mathcal{B}(\mathcal{H}_\pi)$ given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a bounded linear representation of the Banach algebra $L^p(G/H,\mu)$ on the Hilbert space $\mathcal{H}_\pi$ satisfying*

$$\bigcap_{\varphi\in L^1(G/H,\mu)} \ker(\Gamma_\pi(\varphi)) = \ker T_H^\pi. \tag{5.5}$$

Proof Let $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. It is easy to see that the map $\varphi \mapsto \Gamma_\pi(\varphi)$ is linear. Also, the linear map $\Gamma_\pi\colon L^p(G/H,\mu) \to \mathcal{B}(\mathcal{H}_\pi)$ is bounded. Indeed, using (5.3) for $\varphi \in L^p(G/H,\mu)$, we can write

$$\|\Gamma_\pi(\varphi)\| \leq \|\varphi\|_{L^1(G/H,\mu)} \leq \|\varphi\|_{L^p(G/H,\mu)}.$$

Let $\varphi, \psi \in L^p(G/H, \mu)$. Then we have

$$\Gamma_\pi(\varphi *^p_{G/H} \psi) = \pi((\varphi *^p_{G/H} \psi)_q) = \pi(\varphi_q *_G \psi_q)$$
$$= \pi(\varphi_q)\pi(\psi_q) = \Gamma_\pi(\varphi)\Gamma_\pi(\psi),$$

which shows that the map $\Gamma_\pi: L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ is a bounded linear representation. Let $\zeta \in \ker T_H^\pi$ and let $\varphi \in L^p(G/H, \mu)$ be arbitrary. Also, let $f \in L^p(G)$ with $\varphi = T_H(f)$. Then we have $\Gamma_\pi(\varphi)\zeta = \Gamma_\pi(T_H(f))\zeta = \pi(f)T_H^\pi\zeta = 0$, which implies that $\zeta \in \ker(\Gamma_\pi(\varphi))$. Hence, $\ker T_H^\pi \subseteq \bigcap_{\varphi \in L^p(G/H,\mu)} \ker(\Gamma_\pi(\varphi))$. Conversely, let $\zeta \in \bigcap_{\varphi \in L^p(G/H,\mu)} \ker(\Gamma_\pi(\varphi))$. Then $\Gamma_\pi(\varphi)\zeta = 0$, for all $\varphi \in L^p(G/H, \mu)$. Thus for $f \in L^p(G)$, we can write $\pi(f)T_H^\pi\zeta = \Gamma_\pi(T_H(f))\zeta = 0$. Therefore, $\pi(f)T_H^\pi\zeta = 0$, for all $f \in L^p(G)$. Since the $*$-representation $\pi: L^p(G) \to \mathcal{B}(\mathcal{H}_\pi)$ is non-degenerate, we get $T_H^\pi\zeta = 0$ and hence $\zeta \in \ker T_H^\pi$. This implies that $\bigcap_{\varphi \in L^1(G/H,\mu)} \ker(\Gamma_\pi(\varphi)) \subseteq \ker T_H^\pi$. Thus, we conclude (5.5). ∎

The next corollary presents a criterion that guarantees the representation

$$\Gamma_\pi: L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$$

to be non-degenerate.

Corollary 5.6 *Let μ be the normalized G-invariant measure on G/H and $p \geq 1$. And let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then*

$$\Gamma_\pi: L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$$

given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a non-degenerate representation of the Banach algebra $L^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$ if and only if $\pi(h) = I$ for all $h \in H$. In this case we have $\Gamma_\pi(L_h\varphi) = \Gamma_\pi(\varphi)$, for all $h \in H$ and $\varphi \in L^p(G/H, \mu)$.

Proof Invoking (5.5), the representation $\Gamma_\pi: L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is non-degenerate if and only if the linear operator T_H^π is injective. Since T_H^π is an orthogonal projection, we deduce that T_H^π is injective if and only if $T_H^\pi = I$. Then Proposition 5.1 guarantees that T_H^π is injective if and only if $\pi(h) = I$ for all $h \in H$. Therefore, we conclude that the representation $\Gamma_\pi: L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is non-degenerate if and only if $\pi(h) = I$ for all $h \in H$. In this case, for $h \in H$ and $\varphi \in L^p(G/H, \mu)$, we can write

$$\Gamma_\pi(L_h\varphi) = \pi((L_h\varphi)_q) = \pi(L_h\varphi_q) = \pi(h)\pi(\varphi_q) = \pi(\varphi_q) = \Gamma_\pi(\varphi),$$

which completes the proof. ∎

The following theorem shows that the map $\varphi \mapsto \Gamma_\pi(\varphi)$ defines a representation of the Banach $*$-algebra $A^p(G/H, \mu)$.

Theorem 5.7 *Let μ be the normalized G-invariant measure over G/H and $p \geq 1$; let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then*

$$\Gamma_\pi: A^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$$

given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a bounded $*$-representation of the Banach $*$-algebra $A^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$.

Proof Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then using Theorem 5.5, the mapping $\Gamma_\pi : L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a bounded representation of the Banach algebra $L^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$. Thus, the restriction of Γ_π to the closed sub-algebra $A^p(G/H, \mu)$ of $L^p(G/H, \mu)$ is also a bounded representation of the Banach $*$-algebra $A^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$. Now let $\varphi \in A^p(G/H, \mu)$. Then we have $J_p\varphi = \varphi$. Thus, using (5.4), we get

$$\Gamma_\pi(\varphi^{*^{G/H}}) = \Gamma_\pi(J_p\varphi)^* = \Gamma_\pi(\varphi)^*,$$

which guarantees that $\Gamma_\pi : A^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a bounded $*$-representation of the Banach $*$-algebra $A^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$. ■

The next result also presents a criterion which guarantees the representation

$$\Gamma_\pi : A^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$$

to be non-degenerate.

Corollary 5.8 *Let μ be the normalized G-invariant measure on G/H and $p \geq 1$. Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then*

$$\Gamma_\pi : A^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$$

given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a non-degenerate $*$-representation of the Banach $*$-algebra $A^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$ if and only if $\pi(h) = I$ for all $h \in H$.

Proof By Theorem 5.7, the map $\Gamma_\pi : A^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ given by $\varphi \mapsto \Gamma_\pi(\varphi)$ is a $*$-representation of the Banach $*$-algebra $A^p(G/H, \mu)$ on the Hilbert space $\mathcal{H}_\pi$. Then using (5.5), we can write

$$\ker T_H^\pi = \bigcap_{\varphi \in L^p(G/H,\mu)} \ker(\Gamma_\pi(\varphi)) \subseteq \bigcap_{\varphi \in A^p(G/H,\mu)} \ker(\Gamma_\pi(\varphi)).$$

Thus, if the $*$-representation $\Gamma_\pi : A^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ is non-degenerate, then we deduce that $\ker T_H^\pi = \{0\}$ and hence T_H^π is injective. Therefore, $\pi(h) = I$ for all $h \in H$. Conversely, suppose that $\pi(h) = I$ for all $h \in H$. Then $T_H^\pi = I$ and hence $\ker T_H^\pi = \{0\}$. Now let $\zeta \in \bigcap_{\varphi \in A^p(G/H,\mu)} \ker(\Gamma_\pi(\varphi))$. Thus, using Proposition 5.4 for $\varphi \in L^p(G/H, \mu)$, we can write $\Gamma_\pi(\varphi)\zeta = T_H^\pi \Gamma_\pi(\varphi)\zeta = \Gamma_\pi(J_p\varphi)\zeta = 0$, since $J_p\varphi \in A^p(G/H, \mu)$. Thus, $\Gamma_\pi(\varphi)\zeta = 0$, for all $\varphi \in L^p(G/H, \mu)$. Using Corollary 5.6, the representation $\Gamma_\pi : L^p(G/H, \mu) \to \mathcal{B}(\mathcal{H}_\pi)$ is non-degenerate and hence we conclude that $\zeta = 0$, which completes the proof. ■

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