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Near-rings that reduce to rings

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It is shown that a near-ring is a ring if it is generated by a group of automorphisms of its additive group that contains all inner automorphisms.

Let N be a near-ring and A a group of automorphisms of the additive group N^+ of N. Let I denote the group of inner automorphisms of N^+.

We prove the following theorem.

THEOREM 1. *If* N *is additively generated by* A *and* $I \leq A$, *then* N *is a ring.*

Proof. Our assumption implies (see [1], p. 76) that N is distributively generated. We shall assume that N satisfies the left distributive law. By [1], Theorem 4.4.3, it then suffices to show that N^+ is abelian. Let α and β be elements of N. We have

$$\alpha = \mu_1 + \mu_2 + \ldots + \mu_n$$

and

$$\beta = \lambda_1 + \lambda_2 + \ldots + \lambda_m ,$$

where μ_i or $-\mu_i$ is in A for $i = 1, \ldots, n$, and λ_j or $-\lambda_j$ is in A for $j = 1, \ldots, m$.

Now

(1) $$\alpha + \beta = \mu_1 + \ldots + \mu_n + \lambda_1 + \ldots + \lambda_m .$$

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Suppose for λ and μ in A, it is shown that $\lambda + \mu = \mu + \lambda$ then it follows that $-\mu + \lambda = \lambda + (-\mu)$ and, by continued application to (1), we conclude that $\alpha + \beta = \beta + \alpha$. Thus it remains to show that for λ and μ in A, $\lambda + \mu = \mu + \lambda$.

The identity automorphism 1, which is an element of A and thus of N, is the unit element of N (see [1], 1.3.1).

We have that there exists γ in I and thus in N such that

$$-1 + \rho + 1 = \rho\gamma \tag{2}$$

for all ρ in N. Let δ be an N-homomorphism of N^+ into N^+. Applying δ to (2) we conclude that

$$(-1)\delta + \rho\delta + 1\delta = -1 + \rho\delta + 1 . \tag{3}$$

Now let δ be the map of N^+ into N^+ defined by $\eta\delta = \lambda^{-1}\mu\eta$ for all η in N. It is easily checked that δ is an N-homomorphism and, on application to (3), we conclude that

$$-\lambda^{-1}\mu + \lambda^{-1}\mu\rho + \lambda^{-1}\mu = -1 + \lambda^{-1}\mu\rho + 1$$

for all ρ in N. If we take $\rho = 1$ then it follows that

$$\lambda^{-1}\mu = -1 + \lambda^{-1}\mu + 1$$

or

$$1 + \lambda^{-1}\mu = \lambda^{-1}\mu + 1 .$$

Now it follows that $\lambda + \mu = \mu + \lambda$ and the proof is complete.

This theorem has a straightforward generalisation which we now state.

THEOREM 2. *If N is a near-ring and $I \leq N$, then the subgroup of N^+ generated by the units of N is abelian.*

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Reference

[1] A. Fröhlich, "Distributively generated near-rings (I. Ideal theory)", *Proc. London Math. Soc.* (3) 8 (1958), 76-94.

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