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EXISTENCE OF REGULAR COVERINGS ASSOCIATED WITH LEAVES OF CODIMENSION ONE FOLIATIONS

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§ 1. Statement of results

In this paper we are concerned with transversely orientable codimension one foliations. Let $\mathscr{F}$ be a C^r-foliation as above in a smooth manifold M, $r \geqq 1$, and let F_0 be a closed leaf of $\mathscr{F}$. A neighborhood U of F_0 is called a *bicollar* of F_0 in this paper if there is a normal line bundle $\nu: U \to F_0$ with respect to a fixed Riemannian metric on M such that each fibre of ν is transverse to $\mathscr{F}$. For a bicollar U of F_0, $U_+ = F_0 \cup$ (a component of $U - F_0$) is called a *collar* of F_0. A leaf $F \in \mathscr{F}$ is said to be *asymptotic* to F_0 in U_+ if $F \cap V \neq \phi$ for any neighborhood V of F_0 in U_+. Let F_V be a leaf asymptotic to F_0 of the restricted foliation $\mathscr{F} | V$, where V is a neighborhood of F_0 in U_+. A *plaque* of F is a leaf of $F | N$ diffeomorphic to an open $(n-1)$-ball, where N is a sufficiently small open n-ball in the n-manifold M. A C^r-covering $\tilde{\nu}: \tilde{F} \to F_0$ is said to be *associated* with F_V if there is an injection $i: F_V \to \tilde{F}$ such that $\tilde{\nu} i = \nu | F_V$ and that i maps any plaque of F_V C^r-diffeomorphically into $\tilde{F}$. The *one sided holonomy group* $\Phi_+(F_0)$ of F_0 is the holonomy group of F_0 defined by the restricted foliation $\mathscr{F} | U_+$.

The main purpose of this paper is to prove Theorem 2, which is an existence theorem of associated regular coverings. Theorem 1 is used in the proofs of Theorem 2 and Theorem 5. Theorem 3 and Theorem 4 are the properties of associated regular coverings. As an application we show Theorem 5, which is an unstability theorem of foliations.

THEOREM 2. *Let $\mathscr{F}$ be a transversely orientable C^r-foliation of codimension one, $r \geqq 1$, F_0 be an orientable closed leaf of $\mathscr{F}$, and let U_+ be a collar of F_0. Suppose that the one sided holonomy group $\Phi_+(F_0)$ is abelian. Then there is a neighborhood V_0 of F_0 in U_+ such that any*

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neighborhood V *of* F_0 *in* V_0 *satisfies the followings.*

For each asymptotic leaf F to F_0 in U_+ let F_V be an asymptotic leaf of $\mathscr{F}|V$ to F_0 contained in F. Then, an unique (in the sense of the equivalence of coverings) C^r-regular covering $\tilde{\nu}: \tilde{F} \to F_0$ is associated with F_V and $\nu_*(\pi_1(F_V)) = \tilde{\nu}_*(\pi_1(\tilde{F}))$ in $\pi_1(F_0)$. Furthermore, the equivalence class of $\tilde{\nu}$ does not depend on V, and so an unique normal subgroup $G(F) = \nu_*(\pi_1(F_V))$ of $\pi_1(F_0)$ is associated with F.

$\tilde{\nu}$ and $G(F)$ are considered as invariants on the behavior of F in a neighborhood of F_0 in U_+. There is an example of $\mathscr{F}$, F_0, and an asymptotic leaf F to F_0 such that, for any one sided neighborhood V of F_0, no regular covering is associated with F_V.

THEOREM 1. *Suppose that* $\mathscr{F}$, F_0, *and* U_+ *satisfy the same conditions as Theorem* 2. *Then, there are connected orientable codimension one submanifolds* $N_1, \cdots, N_\ell$ *of* F_0 *satisfying the followings.*

(i) $F_0 - N_1 \cup \cdots \cup N_\ell$ *is connected.*

(ii) *Let* F_* *be the manifold obtained by cutting open* F_0 *along* $N_1, \cdots, N_\ell$, *and let* $g: F_* \to F_0$ *be the map pasting* F_* *on* F_0 *naturally. (There are definitions of* F_* *and* g *in* § 3.) *Thus* $\partial F_* = \bigcup_{i=1}^{\ell} N_i' \cup N_i''$, $g^{-1}(N_i) = N_i' \cup N_i''$, *and* $g(N_i') = N_i = g(N_i'')$. *Then, there are injective diffeomorphisms* $f_i: [0, \varepsilon] \to [0, \varepsilon]$, $i = 1, \cdots, \ell$ *with the following properties.*

(a) $f_i(0) = 0$ *and* $f_i f_j(t) = f_j f_i(t)$ *for any* $i, j = 1, \cdots, \ell$ *and* t *such that* $f_i f_j(t)$ *and* $f_j f_i(t)$ *are defined.* (b) *Denote by* X_f *the quotient manifold obtained from* $F_* \times [0, \varepsilon]$ *by identifying* $(x, t) \in N_i' \times [0, \varepsilon]$ *and* $(x, f_i(t)) \in N_i'' \times [0, \varepsilon]$ *for all* $i = 1, \cdots, \ell$ *and* $t \in [0, \varepsilon]$. *By the commutativity of* f_i *and* f_j, X_f *is well defined. The product foliation of* $F_* \times [0, \varepsilon]$ *induces a foliation* $\mathscr{F}_f$ *on* X_f. *Then, there is a neighborhood* V *of* F_0 *in* U_+ *such that there is a leaf preserving* C^r*-diffeomorphism from* V *onto* X_f. (c) *The germs of* $f_1, \cdots, f_\ell$ *at* 0 *generate* $\Phi_+(F_0)$. *Moreover, if* $\dim F_0 > 2$, *they are chosen so that the germs of* $f_1, \cdots, f_\ell$ *are a basis of* $\Phi_+(F_0)$.

The following results are consequence of Theorem 1 and Theorem 2.

THEOREM 3. *Let* $\mathscr{F}$ *be a transversely orientable* C^1*-foliation of codimension one, and let* F_0 *be an orientable closed leaf of* $\mathscr{F}$. *Suppose that* $\pi_1(F_0) = \boldsymbol{Z}^m \times G$ *for a finite group* G *and that* $\{\log h'_{\alpha_1}, \cdots, \log h'_{\alpha_m}\}$

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is rationally independent for a basis $\alpha_1, \cdots, \alpha_m$ *of* Z^m, *where* h'_{α_i} *is the derivative of the holonomy of* α_i.

Then there are collars U_+ *and* U_- *in the both sides of* F_0 *such that any leaf meeting* U_σ *is asymptotic to* F_0 *in* U_σ *and that, for any neighborhood* V *of* F_0 *in* U_σ *and for any* $F \in \mathscr{F}$ *meeting* U_σ, *an unique regular covering* $\tilde{F}$ *with* $\pi_1(\tilde{F}) \cong G$ *is associated with* F_V. *Here* σ *denotes* $+$ *or* $-$.

THEOREM 4. *Let* $\mathscr{F}$ *be a transversely orientable codimension one foliation of class* C^r, *for* $r \geqq 2$, *and let* F_0 *be an orientable closed leaf of* $\mathscr{F}$. *Suppose that the holonomy group* $\Phi(F_0)$ *of* F_0 *is abelian and that there is* $\tilde{f} \in \Phi(F_0)$ *such that the derivative* $\tilde{f}'$ *of* $\tilde{f}$ *at* 0 *satisfies* $\tilde{f}' \neq 1$.

Then, there is a bicollar $U = U_+ \cup U_-$ *of* F_0 *satisfying the followings. Let* σ *denote* $+$ *or* $-$. (i) *Any leaf meeting* U_σ *is asymptotic to* F_0 *in* U_σ. (ii) *For any neighborhood* V *of* F_0 *in* U_σ *and for any leaf* F *meeting* U_σ, *an unique regular covering* $\tilde{\nu}$ *of* F_0 *is associated with* F_V *and the normal subgroup* $G(F)$ *of* $\pi_1(F_0)$ *is well defined. Moreover,* (iii) $\tilde{\nu}$ *and* $G(F)$ *do not depend on* U_+, U_-, *and* F.

This theorem shows that, under the above assumptions, all leaves near F_0 in a collar are in the same situation and $\mathscr{F}|U_+, \mathscr{F}|U_-$ have the same structure.

Let F be a closed submanifold of M, and let $\mathscr{F}, \mathscr{F}'$ be foliations on a neighborhood of F in M having F as a leaf. We say that $\mathscr{F}$ and $\mathscr{F}'$ are *locally equivalent at* F, if there are neighborhoods U and U' of F such that there is a homomorphism from U onto U' mapping any leaf of $\mathscr{F}|U$ onto a leaf of $\mathscr{F}'|U'$.

Let $\mathscr{F}^1_F$ be the set of germs at F of codimension k C^1-foliations $\mathscr{F}$ defined on neighborhoods $U_{\mathscr{F}}$ of F in M such that $\mathscr{F}$ has F as a leaf, and let $\mathscr{F}^1_F$ have a suitable topology defined by the germ of the section into the Grassmannian which defines the foliation. H. Levine and M. Shub show an unstability theorem [2] as follows: If $\pi_1(F)$ has the form $Z^m \times G$ for $m > 1$ and an arbitrary group G, there are no stable elements in $\mathscr{F}^1_F$ with respect to local equivalence at F.

Here, we show an unstability theorem for foliations defined on a fixed neighborhood U of F in M. Let $\mathrm{Fol}^r_F(U)$ be the space of C^r-foliations $\mathscr{F}$ of codimension one defined on a neighborhood U of F in M such that $\mathscr{F}$ has F as a leaf. Let $\mathrm{Fol}^r_F(U)$ have the C^r-topology defined in [1] using the charts $\{\varphi: I^{n-1} \times I \to M^n\}$.

THEOREM 5. *Let F be an orientable closed submanifold of M of codimension one such that $\pi_1(F) = \boldsymbol{Z}^m \times G$ for $m > 1$ and a finite group G. Let $\mathscr{F}$ be a transversely orientable codimension one foliation of class C^r on a neighborhood of F in M with F as a leaf. Then,*
(i) if $r = 2$, there is a neighborhood U of F such that for any neighborhood N of $\mathscr{F}|U$ in $\mathrm{Fol}^1_F(U)$ there is $\mathscr{F}'$ in N which is not locally equivalent at F to $\mathscr{F}$. Moreover,
(ii) if $r > 2$, assume that there is α in $\pi_1(F)$ such that $|h'_\alpha| \neq 1$, where h'_α is the derivative of the holonomy of α. Then, the same result as (i) *holds for $\mathrm{Fol}^{r-1}_F(U)$.*

In the preparation for this research the papers, [4] of Nishimori and [3] of Nakatsuka, were very helpful to the author.

§ 2. Preparation for Theorem 1

This section will be in the version of class C^∞. Let M be an oriented n-manifold, $n \geqq 3$, and let N be an oriented closed smooth submanifold of M with codimension one. Let $F: B^{n-1} \times I \to M$ be an orientation preserving embedding such that $F(B^{n-1} \times I) \cap N = F(B^{n-1} \times \partial I)$, where B^{n-1} denotes an $(n-1)$-ball in $\boldsymbol{R}^{n-1}$ with origin 0, $I = [0,1]$, and ∂ denotes the boundary. We obtain an $(n-1)$-submanifold

$$N_* = \{N - \operatorname{int} F(B^{n-1} \times \partial I)\} \cup F(\partial B^{n-1} \times I) .$$

By smoothing the corners, N_* can be regarded as a smooth manifold. Define a simple arc $f: I \to M$ by $f(t) = F(0, t)$, $t \in I$. We shall say that N_* is obtained from N *by attaching a* 1-*handle along* a simple arc f. If the intersection number of N and f is zero, N_* is orientable. In this case we assume that N_* has the orientation compatible with that of N. Then, $[N_*] = [N]$ in $H_{n-1}(M; \boldsymbol{Z})$, where [ ] denotes the homology class.

LEMMA 1. *Let M be an oriented manifold of dimension $n \geqq 3$, and let N' be a connected oriented closed $(n-1)$ submanifold of M. Then, for a simple closed path c in M which intersects N' at finite points, there is a connected oriented closed $(n-1)$ submanifold N of M satisfying the following conditions.*

(i) $[N] = [N']$ *in* $H_1(M; \boldsymbol{Z})$.

(ii) *N intersects c at only $|[c]\cdot[N]|$ points.*

(iii) *For a small neighborhood U of c in M, N is included in $N' \cup U$.*

Proof. We may assume $[c]\cdot[N'] \geqq 0$ and that N' intersects with c transversely at more than $[c]\cdot[N']$ points, $x_1 = c(t_1), \cdots, x_r = c(t_r)$, $0 < t_1 < \cdots < t_r < 1$. We construct by induction on r the desired manifold N. There is i such that $1 \leqq i \leqq r-1$ and that the intersection number of N' and $c|[t_i, t_{i+1}]$ is zero. By attaching a 1-handle to N' along the simple subarc $c|[t_i, t_{i+1}]$, we obtain N'_* which intersects at $(r-2)$ points and with $[N'_*] = [N']$. Then N'_* has the inductive property.

LEMMA 2. *Let $N \subset M$ be a pair of oriented connected manifolds of codimension one. If there is γ in $H_1(M; \boldsymbol{Z})$ such that the intersection number $\gamma\cdot[N]$ is 1, then $M - N$ is connected.*

Proof. First, we show that there is a closed path $u: I \to M$, $u(0) = u(1)$, such that u intersects with N at a single point. Let c be any closed path with $[c] = \gamma$. We may assume that c meets N transversely, and hence c meets N at finitely many points, $x_1 = c(t_1), \cdots, x_r = c(t_r)$, $0 < t_1 < \cdots < t_r < 1$. We shall construct by induction on r a closed path u as above. We may assume $r \geqq 3$. There is i with $1 \leqq i \leqq r-1$ such that the intersection number of N and $c|[t_i, t_{i+1}]$ is zero. Since N is connected there is a path d from x_i to x_{i+1} in N. Let ε be a sufficiently small positive real number. Then, we can take a path d' from $c(t_i - \varepsilon)$ to $c(t_{i+1} + \varepsilon)$ along d so that d' does not intersect with N. $c([0, t_i - \varepsilon]) \cup (\text{image } d') \cup c([t_{i+1} + \varepsilon, 1])$ is an image of a path $c': I \to M$ which meets N at $(r-2)$ points. Moreover, we have $[c']\cdot[N] = \gamma\cdot[N] = 1$, where $[c']$ denotes the homology class of c'. Then c' has the inductive property, and therefore u is constructed.

For any two points p_0 and p_1 in $M - N$ there is a path c from p_0 to p_1. We may assume as above that c intersects N transversely, and hence c meets N at finite points, $y_1 = c(s_1), \cdots, y_r = c(s_r)$, $0 < s_1 \cdots < s_r < 1$. We shall construct by induction on r a path v from p_0 to p_1 such that v does not intersect N. Let u be a closed path such that u intersects N at a single point $y_0 = u(t_0)$ for $t_0 \in (0, 1)$. There is a path d from y_1 to y_0 in N. Let $\varepsilon > 0$ be sufficiently small. Then, there is a path d_- in $M - N$ from $c(s_1 - \varepsilon)$ to $u(t_0 - \delta)$ along d, where δ is a positive or negative real number with a sufficiently small absolute value. Similarly, there is d_+ from $c(s_1 + \varepsilon)$ to $u(t_0 + \delta)$. $c([0, s_1 - \varepsilon]) \cup (\text{image } d_-)$

$\cup\, u(I-(t_0-\delta, t_0+\delta)) \cup (\text{image } d_+) \cup c([s_1+\varepsilon, 1])$ is an image of a path c' from p_0 to p_1 which intersects N at $(r-1)$ points. Then c' has the inductive property. This proves Lemma 2.

Let $H: H_1(M; \boldsymbol{Z}) \to \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)}$ be an epimorphism onto a free abelian group of rank m, $\boldsymbol{Z}_{(i)} \cong \boldsymbol{Z}$ $(i=1,\cdots,m)$. Let $p_i: \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)} \to \boldsymbol{Z}_{(i)}$ be the projection onto the i-th factor. By Künneth's theorem the map $\kappa: H^1(M; \boldsymbol{Z}) \to \operatorname{Hom}(H_1(M; \boldsymbol{Z}), \boldsymbol{Z})$ induced from slant operation is an isomorphism since $H_0(M; \boldsymbol{Z})$ is free abelian. Assume $\partial M = \phi$. Let $\delta: H^1(M; \boldsymbol{Z}) \to H_{n-1}(M; \boldsymbol{Z})$ be the Poincaré duality isomorphism, and let $\theta_i = \delta\kappa^{-1}(p_i H)$. For $\gamma \in H_1(M; \boldsymbol{Z})$,

$$\begin{aligned}\gamma\cdot\theta_i &= \gamma \cap \delta^{-1}(\theta_i)\\ &= \gamma \cap (\kappa^{-1}(p_i H))\\ &= p_i H(\gamma)\,,\end{aligned}$$

where $\cap$ denotes cup product. Thus we have

$$(1) \qquad \gamma\cdot\theta_i = p_i H(\gamma) \qquad \text{for } \gamma \in H_1(M; \boldsymbol{Z})\,.$$

Now, we set the following result of Nakatsuka.

LEMMA 3 ([3]). *Let M be a compact connected orientable manifold of dimension $n \geqq 3$ and $\theta \in H_{n-1}(M; \boldsymbol{Z})$. Then, there is a connected orientable $(n-1)$-submanifold N in M such that $\theta = [N]$ if and only if there is a homology class $\gamma \in H_1(M; \boldsymbol{Z})$ such that the intersection number $\gamma\cdot\theta = 1$.*

PROPOSITION 1. *Let M be a connected orientable closed manifold of dimension $n \geqq 3$, and let $H: H_1(M; \boldsymbol{Z}) \to \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)}$ be an epimorphism. Then, there are connected closed codimension one submanifolds $N_1, \cdots, N_m$ of M satisfying the followings.*

(i) *$N_1, \cdots, N_m$ are in general position in M.*

(ii) *$\gamma\cdot[N_i] = p_i H(\gamma)$ for any $\gamma \in H_1(M; \boldsymbol{Z})$, $i=1,\cdots,m$.*

(iii) *$N_i - N_1 \cup \cdots \cup N_{i-1}$ is connected for $i=2,\cdots,m$.*

(iv) *$M - N_1 \cup \cdots \cup N_m$ is connected.*

(v) *$Hj_*(H_1(M - N_1 \cup \cdots \cup N_i; \boldsymbol{Z})) = \boldsymbol{Z}_{(i+1)} + \cdots + \boldsymbol{Z}_{(m)}$ for $i=1, \cdots, m-1$, and $=0$ for $i=m$. Here, j is the inclusion $M - N_1 \cup \cdots \cup N_i \to M$.*

Proof. Since $p_i H: H_1(M; \boldsymbol{Z}) \to \boldsymbol{Z}_{(i)}$ is an epimorphism, there is $\gamma_i \in H_1(M; \boldsymbol{Z})$ such that $H(\gamma_i)$ is the generator of $\boldsymbol{Z}_{(i)}$, $i=1,\cdots,m$. Then, by Lemma 3, $\gamma_i \cdot \theta_i = p_i H(\gamma_i) = 1$ implies that there are connected orientable closed $(n-1)$-submanifolds $N'_1, \cdots, N'_m$ in M such that $[N'_i] = \theta_i$, $i = 1, \cdots, m$. $N'_1, \cdots, N'_m$ may be assumed to be in general position.

We vary N'_i to N_i, $i = 1, \cdots, m$, by induction on i so that $N_1, \cdots, N_i$ satisfy the following condition C(i). Denote $M_i = M - N_1 \cup \cdots \cup N_i$.

C(i) (i) $N_1, \cdots, N_i$ are in general position in M.

(ii) $[N_k] = \theta_k$, $k = 1, \cdots, i$.

(iii) $N_k - N_1 \cup \cdots \cup N_{k-1}$ is connected for $k = 2, \cdots, i$ if $i \geqq 2$.

(iv) M_i is connected.

(v) $H \circ j_*(H_1(M_k; \boldsymbol{Z}) = \boldsymbol{Z}_{(k+1)} + \cdots + \boldsymbol{Z}_{(m)}$ for $k = 1, \cdots, i$.

First, we construct N_1 as follows. Since $n \geqq 3$, there are simple closed paths $c_2, \cdots, c_m$ such that $[c_2] = \gamma_2, \cdots, [c_m] = \gamma_m$ and that they are mutually disjoint. By Lemma 1, there is a manifold N_1 such that $[N_1] = [N'_1]$ and that N_1 does not intersects $c_2, \cdots, c_m$. By Lemma 2, the existence of γ_1 implies that $M - N_1$ is connected. Since $c_2, \cdots, c_m$ are contained in M_1 and $0 = \gamma \cdot [N_1] = p_1 H(\gamma)$ for $\gamma \in H_1(M_1; \boldsymbol{Z})$, it is not difficult to see that $Hj_*(H_1(M_1; \boldsymbol{Z})) = \boldsymbol{Z}_{(2)} + \cdots + \boldsymbol{Z}_{(m)}$. Then, N_1 satisfies the condition $C(1)$.

Next, suppose that $N_1, \cdots, N_i$ are constructed so that the condition C(i) is satisfied. Now, we construct N_{i+1} so that $N_1, \cdots, N_i, N_{i+1}$ satisfy $C(i+1)$. By (v) of C(i) there is a simple closed path c_{i+1} in M_i realizing $\gamma_{i+1} \in H_1(M; \boldsymbol{Z})$, and hence the intersection number $c_{i+1} \cdot (N'_{i+1} - N_1 \cup \cdots \cup N_i) = c_{i+1} \cdot N'_{i+1} = [c_{i+1}] \cdot [N'_{i+1}] = \gamma_{i+1} \cdot \theta_{i+1}$ is 1. We can take c_{i+1} so that it intersects N'_{i+1} transversely. Then, by the method of the proof of Lemma 3 in [3], there is a closed manifold N''_{i+1} such that (i) $N''_{i+1} \cap M_i$, so N''_{i+1}, is connected and (ii) $[N''_{i+1}] = [N'_{i+1}]$ in $H_{n-1}(M; \boldsymbol{Z})$ and $[N'_{i+1} \cap M_i] = [N'_{i+1} \cap M_i]$ in $H_{n-1}(M_i; \boldsymbol{Z})$. Here, N''_{i+1} is obtained by attaching slender 1-handles to N'_{i+1} along simple arcs in M_i. Next, we vary N''_{i+1} to construct N_{i+1} so that $N_1, \cdots, N_i$ and N_{i+1} satisfy the condition $C(i+1)$. By (v) of C(i), there are simple closed paths $c_{i+2}, \cdots, c_m$ in M_i realizing $\gamma_{i+2}, \cdots, \gamma_m$, respectively. We may assume that they intersect N''_{i+1} transversely and that they are mutually disjoint. Similarly as the construction of N_1, we obtain N_{i+1} from N''_{i+1} by attaching slender 1-handles along simple arcs contained in $c_{i+2}, \cdots, c_m$ so that N_{i+1} does not intersects $c_{i+2}, \cdots, c_m$, that $[N_{i+1}] = [N'_{i+1}]$, and that $H \circ j_*(H_1(M - N_1 \cup \cdots \cup N_i \cup N_{i+1}; \boldsymbol{Z})) = \boldsymbol{Z}_{(i+2)} + \cdots + \boldsymbol{Z}_{(m)}$. Since c_{i+1} is a path in M_i and $c_{i+1} \cdot (N_{i+1} \cap M_i) = c_{i+1} \cdot N_{i+1} = 1$, Lemma 2 implies that $M_{i+1} = M_i$

$-N_{i+1}$ is connected. From the above, we can see that $N_1, \cdots, N_{i+1}$ satisfy the condition $C(i+1)$. This proves Proposition 1.

§ 3. Proof of Theorem 1

Let $\mathscr{F}$ be a codimension one foliation of class C^r of an orientable $(n+1)$-manifold M, and suppose that an orientable n-manifold F_0 is a closed leaf of $\mathscr{F}$. Let $\nu: U \to F_0$ is an $\boldsymbol{R}$-bundle of a bicollar U of F_0, and let $\nu_+: U_+ \to F_0$ is an $\boldsymbol{R}_+$-bundle of a collar U_+ of F_0, $\boldsymbol{R} = (-\infty, \infty)$ and $R_+ = [0, \infty)$. F_0 is identified with the zero section of ν or ν_+, and the fibres of ν and ν_+ are identified with $\boldsymbol{R}$ and $\boldsymbol{R}_+$ respectively.

A curve $u: [0,1] \to U$ is called a *leaf curve* from $u(0)$ to $u(1)$ if the image of u is contained in a leaf. Let $y \in \nu^{-1}u(0)$ and let $u_y: [0,1] \to U$ be a leaf curve such that $u_y(0) = y$ and $\nu u_y(t) = u(t)$ for any $t \in [0,1]$. We call u_y the *y-lift* of u. There exists at most one y-lift of u. If there is the y-lift of u for any y in $[y_1, y_2] \subset \boldsymbol{R} = \nu^{-1}u(0)$ the *holonomy map* h_u from $[y_1, y_2]$ into $\boldsymbol{R} = \nu^{-1}u(b)$ is defined by $h_u(y) = u_y(b)$.

Let $x_* \in F_0$ and u be a closed leaf curve with base point x_*. The germ of h_u at 0 is called the *holonomy* of u. The holonomy of u is determined by the homotopy class $[u]$ of u in $\pi_1(F_0, x_*)$ and is independent of the choice of ν up to conjugations by origin preserving diffeomorphism of $\boldsymbol{R}$. Let G^r be the group of the germs at 0 of all orientation-preserving local C^r-diffeomorphisms of $\boldsymbol{R}$ which leave the origin fixed. A homomorphism $h: \pi_1(F_0, x_*) \to G^r$ is defined by corresponding the holonomy of u to $[u] \in \pi_1(F_0, x_*)$. The image of the homomorphism h is called the *holonomy group* of F_0 and denoted by $\Phi(F_0)$. The *one-sided holonomy group* $\Phi_+(F_0)$ of F_0 is defined similarly by replacing ν and $\boldsymbol{R}$ by ν_+ and $\boldsymbol{R}_+$.

A proof of the following Lemma 4 is found in the proof of Lemma 2 in [4].

LEMMA 4. *If $\Phi_+(F_0)$ is the trivial group there is a neighborhood U_0 of F_0 in U_+ such that the restricted foliation $\mathscr{F} | U_0$ is trivial; i.e. for each leaf F of $\mathscr{F} | U_0$, $\nu: F \to F_0$ is a diffeomorphism.*

In this paper, we assume that $\Phi_+(F_0)$ is abelian, then $\Phi_+(F_0)$ is free abelian since G^r has no torsion element. Let $\iota: \Phi_+(F_0) \to \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)}$ be an isomorphism and let $\eta: \pi_1(F_0, x_*) \to H_1(F_0 : \boldsymbol{Z})$ be the Hurewicz homomorphism. Then, there is an epimorphism $H: H_1(F_0; \boldsymbol{Z}) \to \boldsymbol{Z}_{(1)} + \cdots$

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$+ Z_{(m)}$ such that $H\eta = \iota h$. Let p_i be the projection from $Z_{(1)} + \cdots + Z_{(m)}$ onto the i-th factor. Thus we have the following diagram.

$$\begin{array}{ccccccc} \pi_1(F_0, x_*) & \xrightarrow{h} & \Phi(F_0) & \xrightarrow{\iota} & Z_{(1)} + \cdots + Z_{(m)} & \xrightarrow{p_i} & Z_{(i)} \\ \downarrow{\scriptstyle \eta} & & & \nearrow{\scriptstyle H} & & & \\ H_1(F_0; Z) & & & & & & \end{array}$$

Let $N_1, \cdots, N_m$ be codimension one smooth submanifolds in F_0 such that they are in the general position and that $F_0 - N_1 \cup \cdots \cup N_m$ is connected. Denote by F_1 the compact manifold with boundary obtained by attaching two copies N_1' and N_1'' of N_1 to $F_0 - N_1$, so that $\partial F_1 = N_1' \cup N_1''$. Then, a local diffeomorphism $g_1 : F_1 \to F_0$ is defined by $g_1(x) = x$ for $x \notin \partial F_1$ and $g_1(y') = g_1(y'') = y$ for $y \in N_1$, where $y' \in N_1'$ and $y'' \in N_1''$ are the copies of $y \in N_1$. $g_1^{-1}(N_i) \subset F_1$ is denoted also by N_i, $i = 2, \cdots, m$. Inductively we define $F_2, \cdots, F_m$ and $g_i : F_i \to F_{i-1}$, $i = 2, \cdots, m$, similarly as above. The boundaries of $F_2, \cdots, F_m$ have possibly corners. Let $g : F_m \to F_0$ be the composition $g_j \cdots g_1$. F_m is said to be the manifold which is *obtained by cutting open* F_0 *along* $N_1, \cdots, N_m$. g is said to be the map *pasting* F_m *on* F_0.

Proof of Theorem 1. If $n = 1$, this theorem is well known in the theory of dynamical system. If $n > 2$, let $N_1, \cdots, N_m$ be the manifolds obtained by Proposition 1 for the epimorphism $H : H_1(F_0; Z) \to Z_{(1)} + \cdots + Z_{(m)} \cong \Phi_+(F_0)$ defined above. If $n = 2$, let p be the genus of F_0. Then we can take simple closed curves $N_1, \cdots, N_{2p}$ in F_0 such that $N_i \cap N_j$ is at most one point for any different i, j and that $F_0 - N_1 \cup \cdots \cup N_{2p}$ is an open 2-ball. We define $N_1, \cdots, N_\ell$ in the theorem as above.

Since $F_0 - N_1 \cup \cdots \cup N_\ell$ is a 2-ball for $n = 2$, $\Phi_+(F_0 - N_1 \cup \cdots \cup N_\ell) = 0$ in $\mathscr{F} | \nu_+^{-1}(F_0 - N_1 \cup \cdots \cup N_\ell)$. When $n > 2$, let c be a simple closed path in $F_0 - N_1 \cup \cdots \cup N_m$. Let γ be the homotopy class of c in $\pi_1(F_0, x_*)$, $x_* \in F_0 - N_1 \cup \cdots \cup N_m$, and $[N_i]$ be the homology class of N_i in $H_1(F_0; Z)$. Then, by Proposition 1,

$$\begin{aligned} p_i \iota h(\gamma) &= p_i H_\eta(\gamma) , \\ &= p_i H([c]) , \qquad [c] \in H_1(F_0; Z) \\ &= [c] \cdot [N_i] = 0 \end{aligned}$$

since $c \cap N_i = \phi$, for $i = 1, \cdots, m$. This implies $\Phi_+(F_0 - N_1 \cup \cdots \cup N_\ell) = 0$, if $n > 2$. By using Lemma 4, wee see that there is a injective

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C^r-diffeomorphism $\xi:(F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta] \to U_+$ such that (i) ξ maps each $(F_0 - N_1 \cup \cdots \cup N_\ell) \times \{t\}$ into a leaf of $\mathscr{F} | U_+$ and that (ii) $\nu_+\xi(x, t) = x$ for $x \in F_0 - N_1 \cup \cdots \cup N_\ell$ and $t \in [0, \delta]$. Put $\xi((F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta]) = \tilde{F}_* \subset \nu_+^{-1}(F_0 - N_1 \cup \cdots \cup N_\ell)$. By identifying $\xi(x, t)$ with (x, t), $(x, t) \in (F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta]$ is a coordinates of $\tilde{F}_*$. Putting $V' = \operatorname{cl} F_*$, V' is a closed neighborhood of F_0 in U_+. We are dealing with the holonomy maps and the holonomies for closed paths in F_0 with the fixed base point $x_* \in \operatorname{int} \tilde{F}_*$. From now on in this section a holonomy maps are considered as local diffeomorphisms of $[0, \delta]$ by identifying $[0, \delta]$ with $x_* \times [0, \delta]$, where $x_* \times [0, \delta]$ is the expression of the above coordinates.

The number of the connected components of $N_i - N_1 \cup \cdots \cup N_{i-1} \cup N_{i+1} \cup \cdots \cup N_\ell$ is only one if $n = 2$. For $n > 3$, let N_{ij} be one of these components. For any x in N_{ij} there is a closed path v_x in F_0 with base point x_* such that v_x intersects $N_1 \cup \cdots \cup N_\ell$ at only one point x, since $F_0 - N_1 \cup \cdots \cup N_\ell$ is connected. There is ε_x with $0 < \varepsilon_x \leqq \delta$ such that there is a leaf curve of $\mathscr{F} | V'$ which is the lift of v_x starting from $(x_*, \varepsilon_x) \in v_+^{-1}(x_*)$. So, the holonomy map f_x of v_x is defined on $[0, \varepsilon_x]$. Let $\tilde{v}_x$ be a lift of v_x and let $\tilde{v}_x(0) = s'$, $v_x(1) = s''$ in $\{x_*\} \times [0, s_x] \subset v_+^{-1}(x_*)$. Let $\tilde{v}_x(t_0) \in v_+^{-1}(x)$. For any t', t'' with $0 \leqq t' < t_0 < t'' \leqq 1$, we have $\tilde{v}_x(t') = (v_x(t'), s')$ and $\tilde{v}_x(t'') = (v_x(t''), s'')$ in the coordinates $\tilde{F}_* = (F_0 - N_1 \cup \cdots \cup N_\ell) \times [0, \delta]$, since $\mathscr{F} | \tilde{F}_*$ is trivial. Hence, we have $f_x(s') = s''$. Let N_{ij} have the orientation which is compatible with the inclusion $N_{ij} \subset N_i$ and the given orientation of N_i. For another point y in N_{ij} let v_y be a closed curve as above such that $[v_y] \cdot [N_{ij}] = [v_x] \cdot [N_{ij}]$. From the triviality of $\mathscr{F} | \tilde{F}_*$ it is easy to see that the source of the holonomy map f_y of v_y is same as f_x and that $f_y = f_x$ on it, i.e. $f_y(s) = f_x(s)$ for any $s \in [0, \varepsilon_x]$. Therefore, there are ε_{ij} with $0 < \varepsilon_{ij} < \delta$ and an injective diffeomorphism $f_{ij}: [0, \varepsilon_{ij}] \to [0, \delta]$ satisfying the following property; for any x in N_{ij} and any closed path v_x in F_0 with base point x_* such that v_x intersects $N_1 \cup \cdots \cup N_\ell$ at only one point x and that $[v_x] \cdot [N_i] = 1$, the holonomy map of v_x is defined on $[0, \varepsilon_{ij}]$ and is equal to f_{ij}. For two components N_{ij} and N_{ik} of $N_i - N_1 \cup \cdots \cup N_{i-1} \cup N_{i+1} \cup \cdots \cup N_\ell$ the holonomy maps f_{ij} and f_{ik} are coincide on a small neighborhood of 0, since $[v_x] \cdot [N_i] = [v_y] \cdot [N_i] = 1$ so the holonomies of v_x and v_y are coincide. Hence, there are ε_i with $0 < \varepsilon_i < \delta$ and an injective diffeomorphism $f_i: [0, \varepsilon_i] \to [0, \delta]$ satisfying the same property as above. Therefore, there are $0 < \varepsilon < \delta$ and injective diffeomorphisms $f_1, \cdots, f_\ell$ for $N_1, \cdots, N_\ell$ satisfying the following property; for any x in N_i and any closed path v_x in F_0 with base point x_* such that v_x intersects $N_1 \cup \cdots \cup N_\ell$ at only one point x and that $[v_x]\cdot[N_i] = 1$, the holonomy map of v_x is defined on $[0, \varepsilon]$ and is equal to f_i. Since $\Phi(F_0)$ is abelian, we may assume that $f_1, \cdots, f_\ell$ are mutually commutative by choosing ε sufficiently small.

Since f_i and f_i^{-1} are monotonously increasing, $f_i(\varepsilon) > \varepsilon$ implies $\varepsilon > f_i^{-1}(\varepsilon)$. So, replacing f_i by f_i^{-1} (i.e. replacing the orientation of N_i) if necessary, we can suppose that $\varepsilon \geqq f_i(\varepsilon)$ for all i. Notice that $N_i = \bigcup_j \mathrm{cl}\, N_{ij}$ and $g^{-1}(N_{ij}) = N'_{ij} \cup N''_{ij}$. Here, $g: F_* \to F_0$ is the diffeomorphism pasting F_* on F_0, F_* is the manifold obtained by cutting open F_0 along $N_1, \cdots, N_\ell$, and N'_{ij}, N''_{ij} are diffeomorphic manifolds such that $g(N'_{ij}) = N_{ij} = g(N''_{ij})$. Then, $g^{-1}(N_i) = N'_i \cup N''_i$, where N'_i and N''_i are diffeomorphic manifolds such that $N'_i = \bigcup_j \mathrm{cl}\, N'_{ij}$, $N''_i = \bigcup_j \mathrm{cl}\, N''_{ij}$ and $g(N'_i) = N_i = g(N''_i)$. Since $N_1, \cdots, N_\ell$ are in general position and $f_1, \cdots, f_\ell$ are mutually commutative, it is not difficult to show that a quotient manifold X_f is well defined from $F_* \times [0, \varepsilon]$ by identifying $(x, s) \in N'_i \times [0, \varepsilon]$ and $(x, f_i(s)) \in N''_i \times [0, \varepsilon]$. Let $\mathscr{F}_f$ be the foliation on X_f induced from the trivial foliation of $F_* \times [0, \varepsilon]$. Since $\mathrm{int}\, F_*$ is diffeomorphic to $F_0 - N_1 \cup \cdots \cup N_\ell$, we can see from the above facts that there is a C^r-diffeomorphism from a neighborhood V of F_0 in U_+ onto X_f mapping each leaf of $\mathscr{F} | V$ onto a leaf of $\mathscr{F}_f$.

By the constructions of $f_1, \cdots, f_\ell$, these maps satisfies the property (ii)–(c) in the theorem. This completes the proof of Theorem 1.

§ 4. Proof of Theorem 2

LEMMA 5. *Let $f_1, \cdots, f_\ell$ be injective homeomorphisms from $[0, \varepsilon]$ into $[0, \varepsilon]$ such that $f_i(0) = 0$ for $i = 1, \cdots, \ell$. Suppose*

$$f_i f_j(t) = f_j f_i(t)\ , \qquad i, j = 1, \cdots, \ell\ .$$

Put

$$(1) \qquad h_1(t) = f_{i_\alpha}^{\sigma_\alpha} \cdots f_{i_1}^{\sigma_1}(t)\ , \qquad \sigma_a = \pm 1\ ,$$

$$(2) \qquad h_2(t) = f_{j_\beta}^{\tau_\beta} \cdots f_{j_1}^{\tau_1}(t)\ , \qquad \tau_b = \pm 1\ .$$

Then $h_1(t) = h_2(t)$ for any t such that $h_1(t)$ and $h_2(t)$ are defined if

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(3) $\sum_{i_a=i}\sigma_{i_a}=\sum_{j_b=i}\tau_{j_b}$, $i=1,\cdots,\ell$, $a=1,\cdots,\alpha$, $b=1,\cdots,\beta$.

Here, f_i^{-1} is considered to be defined on $[0, f_i(\varepsilon)]$.

Proof. By the assumption we have

$$f_i^\sigma f_j^\tau(t)=f_j^\tau f_i^\sigma(t)\,,\qquad \sigma,\tau=\pm 1\,,\ i,j=1,\cdots,\ell$$

for any t such that both sides of the expression are defined. We define a linear order $<$ in the set $\{f_1,\cdots,f_\ell,f_1^{-1},\cdots,f_\ell^{-1}\}$ as follows; for f_i, f_j and f_i^{-1},f_j^{-1}, we define $f_i<f_j$ and $f_i^{-1}<f_j^{-1}$ respectively if $i<j$, and we define $f_i<f_j^{-1}$ for any f_i and f_j^{-1}. It is not difficult to see that $<$ is a linear order.

Next, we show that if $f_i^{\sigma_i}f_j^{\sigma_j}(t)$ is defined and $f_i^{\sigma_i}<f_j^{\sigma_j}$, $f_j^{\sigma_j}f_i^{\sigma_i}(t)$ is also defined and $f_i^{\sigma_i}f_j^{\sigma_j}(t)=f_j^{\sigma_j}f_i^{\sigma_i}(t)$. This property is trivial for f_i and f_j. For f_i^{-1} and f_j^{-1} it is shown as follows. Suppose $f_i^{-1}<f_j^{-1}$. If $f_i^{-1}f_j^{-1}(t)$ is defined, $f_j^{-1}(t)\leqq f_i(\varepsilon)$, so $t\leqq f_jf_i(\varepsilon)$. Since $f_jf_i(\varepsilon)=f_if_j(\varepsilon)$, $t\leqq f_if_j(\varepsilon)$. Hence, $f_i^{-1}(t)\leqq f_j(\varepsilon)$, and so $f_j^{-1}f_i^{-1}(t)$ is defined. Then $f_i^{-1}f_j^{-1}(t)=f_j^{-1}f_i^{-1}(t)$. Finally, for f_i and f_j^{-1} it is shown as follows. Suppose $f_i<f_j^{-1}$. If $f_if_j^{-1}(t)$ is defined, $t\leqq f_j(\varepsilon)$, so $t_i(t)\leqq f_if_j(\varepsilon)$. Since $f_if_j(\varepsilon)=f_jf_i(\varepsilon)\leqq f_j(\varepsilon)$, $f_i(t)\leqq f_j(\varepsilon)$. Then, $f_j^{-1}f_i(t)$ is defined, and so $f_if_j^{-1}(t)=f_j^{-1}f_i(t)$.

If $f_j(t)$ or $f_j^{-1}(t)$ is defined, $f_j(f_i^{-1}f_i)(t)$ or $(f_i^{-1}f_i)f_j^{-1}(t)$ is defined and $f_j(t)=f_j(f_i^{-1}f_i)(t)$ or $f_j^{-1}(t)=(f_i^{-1}f_i)f_j^{-1}(t)$, respectively. Next, we interplate $f_i^{-1}f_i$ in the right hand of the expressions of (1) and (2) if necessary so that the same number of f_i and f_i^{-1} are contained in these expressions for each $i=1,\cdots,\ell$. Finally, we change the order in the rows of the terms in these expressions to the order induced from $<$. Then, the obtained expressions are identical. This proves $h_1(t)=h_2(t)$.

LEMMA 6. *Let $\mathscr{F}$ be a transversely orientable C^r-foliation of codimension one, $r\geqq 1$, and let F_0 be a compact leaf of $\mathscr{F}$. Let ν be a normal $\boldsymbol{R}_+$-bundle map from a collar U_+ onto F_0 such that ν is transverse to $\mathscr{F}$, and let $F\in\mathscr{F}$ be asymptotic to F_0 in U_+. Then, the following properties are satisfied.*

(i) *For a neighborhood V of F_0 in U_+, let F_V be an asymptotic leaf of $\mathscr{F}|V$ to F_0 such that $F_V\cap F\neq\phi$. Then, an unique regular covering $\tilde\nu:\tilde F\to F_0$ is associated with F_V and $\nu_*(\pi_1(F_V))=\tilde\nu_*(\pi_1(\tilde F))$ in $\pi_1(F_0)$ if and only if the following condition $(*)$ is satisfied.*

$(*)$ *For a point x_* in F_0 and any closed path u in F_0 with the base*

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points x_* *let* y *and* z *be any two points in* $\nu^{-1}(x_*) \cap F_V$ *such that* $h_u(y)$ *and* $h_u(z)$ *are defined, where* h_u *is the holonomy map of* u. *Then,* $h_u(y) = y$ *if and only if* $h_u(z) = z$.

(ii) *Suppose* F *and* V *satisfies* $(*)$. *Then, for any neighborhood* V' *of* F_0 *in* V, *the same regular covering as* $\tilde{\nu}$ *is associated with* $F_{V'}$.

Proof. Let $\tilde{\nu}: \tilde{F} \to F_0$ be a regular covering and let u be a closed curve in F_0 with base point x_*. For y and z in $\tilde{\nu}^{-1}(x_*)$ let u_y and u_z be the lifts of u starting from y and z respectively. Then, u_y is a closed curve if and only if u_z is so. Therefore, if there is an associated regular covering with F_V, condition $(*)$ is satisfied.

Next, we prove the converse. Define a subgroup $G(F_V)$ of $\pi_1(F_0, x_*)$ by

$$G(F_V) = \{\alpha \in \pi_1(F_0, x_*) \mid \text{there is a closed curve } \bar{u} \text{ in } F_V \text{ such that } [\nu\bar{u}] = \alpha\},$$

where [ ] denotes the homotopy class. We must show that $G(F_V)$ is a subgroup of $\pi_1(F_0, x_*)$. For α and β in $G(F_V)$ there are closed curves $\bar{u}$ and $\bar{v}$ in F_V such that $[\nu\bar{u}] = \alpha$ and $[\nu\bar{v}] = \beta^{-1}$. Let $y, z \in \nu^{-1}(x_*)$ be the base point of $\bar{u}, \bar{v}$. Assume $x_* < y < z$ in the line $\nu^{-1}(x_*)$. Put $\nu\bar{u} = u$ and $\nu\bar{v} = v$. By the existence of $\bar{v}, h_v(y)$ is defined. Condition $(*)$ implies $h_v(y) = y$. So, there is the lift $\tilde{v}$ of v starting from y. $\tilde{v}$ is a closed curve in F_V. Then, $\bar{u}\tilde{v}$ is a closed curve in F_V such that $[\nu(\bar{u}\tilde{v})] = \alpha\beta^{-1}$. Therefore, $\alpha\beta^{-1} \in G(F_V)$.

To the conjugacy class of a subgroup of $\pi_1(F_0, x_*)$ an unique covering of F_0 exists. Let $\tilde{\nu}: \tilde{F} \to F_0$ be the covering corresponding to the conjugacy class including $G(F_V)$. Then, for $\tilde{y} \in \tilde{\nu}^{-1}(x_*)$, $\tilde{\nu}_*(\pi_1(F, y))$ is a subgroup of $\pi_1(F_0, x_*)$ which is conjugate to $G(F_V)$.

Next, we define the map $i: F_V \to \tilde{F}$. Fix two points $y_* \in F_V$ and $\tilde{y}_* \in \tilde{F}$ so that $\nu(y_*) = \tilde{\nu}(\tilde{y}_*) = x_*$ and that $\tilde{\nu}_*(\pi_1(\tilde{F}, \tilde{y}_*)) = G(F_V)$. For any point y in F_V there is a curve $u: [0, 1] \to F_V$ such that $u(0) = y_*$ and $u(1) = y$. Let $\tilde{u}$ be the lift of νu starting from $\tilde{y}_*$ for the covering $\tilde{\nu}$. We define $i(y) \in \tilde{F}$ by $i(y) = \tilde{u}(1)$. $i(y)$ is well defined, i.e. for another curve v in F_V from y_* to y, $\tilde{v}(1) = \tilde{u}(1)$. In fact, since $[\nu(uv^{-1})] \in G(F_V)$ and $G(F_V) = \tilde{\nu}_*(\pi_1(\tilde{F}, \tilde{y}_*))$, the lift of $\nu(uv^{-1})$ starting from $\tilde{y}_* \in \tilde{F}$ is a closed curve. Hence, $\tilde{u}^{-1}\tilde{v}$ is a closed curve with the base point $\tilde{u}(1)$. This implies $\tilde{v}(1) = \tilde{u}(1)$. By the definition of i, $\tilde{\nu} \circ i = \nu$ is obvious.

If $\nu(y) \neq \nu(y')$, clearly $i(y) \neq i(y')$. Next, we show that $i(y) \neq i(y')$

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when $\nu(y) = \nu(y')$ and $y \neq y'$. Let u' and v' be the curves in F_V from y_* to y and y' respectively. Put $\nu u' = u$ and $\nu v' = v$. We can assume that $y < y'$ in $\nu^{-1}(y)$. Since $h_{v^{-1}}(y') = y_*$, $h_{v^{-1}}(y)$ is defined and $h_{v^{-1}}(y) < y_*$ in $\nu^{-1}(x_*)$. Since $h_{uv^{-1}}(y_*) = h_{v^{-1}}(y) < y_*$, $[uv^{-1}] \notin G(F_V)$. So that, the lift of uv^{-1} starting from $\tilde{y}_*$ in $\tilde{F}$ is never a closed curve. Hence, $i(y) = \tilde{u}(1) \neq \tilde{v}(1) = i(y')$. Therefore, i is an injection.

It is obvious that i maps any plaque of F_V C^r-diffeomorphically into $\tilde{F}$.

To show $\tilde{\nu}$ is a regular covering we are sufficient to show that $G(F_V)$ is a normal subgroup of $\pi_1(F_0, x_*)$. Let u and v be closed curves in F_0 with the base point x_*. Assume $[u] \in G(F_V)$. Since F_V is asymptotic to F_0 there is y in $\nu^{-1}(x_*) \cap F_\nu$ such that $h_{vuv^{-1}}(y)$ is defined. Since $[u] \in G(F_V)$, $h_u h_v(y) = h_v(y)$. So that, $h_{vuv^{-1}}(y) = h_{v^{-1}} h_u h_v(y) = y$. Hence, $[vuv^{-1}] \in G(F_V)$. This implies that $G(F_V)$ is a normal subgroup. Therefore, (i) is proved.

To prove (ii) it is sufficient, if $G(F_V) = G(F_{V'})$ is shown. But, this is obvious since F_V is asymptotic to F_0.

Proof of Theorem 2. By Theorem 1 we obtain $N_1, \cdots, N_\ell \subset F_0, V$, and the functions $f_1, \cdots, f_\ell$. Let $x_* \in F_0 - N_1 \cup \cdots \cup N_\ell$. For an asymptotic leaf F of $\mathscr{F}|V$ to F_0, let F_V be an asymptotic leaf of $\mathscr{F}|V$ to F_0 such that $F_V \subset F$.

First, we show that, if u, v are closed paths in F_0 with base point x_* in a same homology class of $H_1(F_0; Z)$, $h_u(y) = h_v(y)$ for any $y \in \nu^{-1}(x_*) \cap V$ such that $h_u(y), h_v(y)$ are defined. Let $\tilde{u}, \tilde{v}$ be the leaf curves of $\mathscr{F}|V$ which are lifts of u, v starting from y. We may assume that $\tilde{u}, \tilde{v}$ intersect $\nu^{-1}(N_1 \cup \cdots \cup N_\ell)$ transversely. So, since $F_0 - N_1 \cup \cdots \cup N_\ell$ is connected, $\tilde{u}$ and $\tilde{v}$ are homotopic to $\tilde{u}_1 \cdots \tilde{u}_\alpha$ and $\tilde{v}_1 \cdots \tilde{v}_\beta$ by homotopies such that the homotopies preserve the end points of the paths and that each homotopy level is a leaf curve of $\mathscr{F}|V$, where $\tilde{u}_1 \cdots \tilde{u}_\alpha$ and $\tilde{v}_1 \cdots \tilde{v}_\beta$ are the paths which are the compositions of the paths $\tilde{u}_a, \tilde{v}_b$ with end points in $\nu^{-1}(x_*)$ such that putting $\nu\tilde{u}_a = u_a$ and $\nu\tilde{v}_b = v_b$, u_a and v_b are closed paths in F_0 each of which intersect $N_1 \cup \cdots \cup N_\ell$ at one point. Here, the composition of paths is defined by

$$uv(x) = \begin{cases} u(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2} \\ v(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1 \,. \end{cases}$$

Define N_{i_a} and N_{j_b} by $u_a \cap (N_1 \cup \cdots \cup N_\ell) = u_a \cap N_{i_a}$ and $v_b \cap (N_1$

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$\cup \cdots \cup N_\ell) = v_b \cap N_{j_b}$. Let the intersection numbers be $[u_a]\cdot[N_{i_a}] = \sigma_a$ and $[v_b]\cdot[n_{j_b}] = \tau_b$, where $\sigma_a, \tau_b = \pm 1$. Here, $N_1, \cdots, N_\ell$ are imposed the orientations such that if a closed path u intersects $N_1 \cup \cdots \cup N_\ell$ at only one point in N_i with the intersection number $[u]\cdot[N_i] = \sigma$, as in the proof of Theorem 1, then the holonomy map h_u of u is coincide with f_i^σ. Thus, we have

$$h_u(y) = h_{u_1\cdots h_\alpha}(y) = h_{u_\alpha}\cdots h_{u_1}(y)$$
$$= f_{i_\alpha}^{\sigma_\alpha}\cdots f_{i_1}^{\sigma_1}(y) .$$

Similarly,

$$h_v(y) = f_{j_\beta}^{\tau_\beta}\cdots f_{j_1}^{\tau_1}(y) .$$

Since u and v are in the same homology class, Lemma 5 implies $h_u(y) = h_v(y)$.

If we can show that V and F_V satisfy the condition $(*)$ in Lemma 6, the proof of Theorem 2 is completed by Lemma 6. $(*)$ is shown as follows. Let y, z be two points in $\nu^{-1}(x_*) \cap F_V$ such that $h_u(y)$ and $h_u(z)$ are defined, where u is a closed path in F_0 with end points x_*. We can assume $y \geqq h_u(y)$; if $y < h_u(y)$, consider the curve u^{-1} with the inverse direction of u. Here, $<$ is considered in the coordinates $\nu^{-1}(x_*) \cap V = x_* \times [0, \varepsilon]$. Let $y > z$. Since h_u is a homomorphism, $h_u(y) > h_u(z)$. There is a path $\tilde{w}$ in F_V from y to z. Put $w = \nu\tilde{w}$. Since $h_u(y) \leqq y$ and $h_w h_u(y)$ is defined. $h_u(y) \leqq y$ implies $z = h_w(y) \geqq h_w h_u(y)$. Notice that $y = h_u(y)$ if and only if $z = h_w h_u(y)$. We have $h_{w^{-1}uw}(z) = h_w h_u h_{w^{-1}}(z) = h_w h_u(y)$. Since $w^{-1}uw$ and u are in the same homology class, $h_{w^{-1}uv}(z) = h_u(z)$ by the fact that we proved above. Thus, $h_u(z) = h_w h_u(y)$. Since $z = h_w(y)$, we have $y = h_u(y)$ if and only if $z = h_u(z)$. This proves Theorem 2.

§ 5. Proof of Theorem 3

Let $\nu_+ : U'_+ \to F_0$ be a collar. Since $\{\log h'_{\alpha_1}, \cdots, \log h'_{\alpha_m}\}$ is rationally independent, there is a closed curve u in F_0 such that $h'_u \neq 1$. We can assume that $0 < h'_u < 1$. Let x be the base point of u. There is an interval $[t, z)$ in $\nu_+^{-1}(x)$ and a positive number $r < 1$ such that for any y in $[x, z)$ $h_u(y)$ is defined and that $h'_u(y) < r$. Hence, $\lim_{i\to\infty} (h_u)^i(y) = x$ for any y in $[x, z)$. Therefore, by taking a sufficiently small collar U_+, any leaf meeting U_+ is asymptotic to F_0. We can U_- similarly.

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By the assumption of $\pi_1(F_0)$, the one sided holonomy group $\Phi_\sigma(F_0)$ is abelian for $\sigma = +$ or $-$. Let V be any neighborhood of F_0 in U_σ. Then, for any leaf F meeting U_σ a regular covering $\tilde{\nu}: \tilde{F} \to F_0$ is associated with F_V, by Theorem 2.

Since holonomy has no torsion element, $G(F) = \nu_* \pi_1(F_V) = \tilde{\nu}_* \pi_1(\tilde{F}) \supset G$. ν_* and $\tilde{\nu}_*$ are injections. Suppose that there is a leaf F such that, for the associated covering $\tilde{\nu}: \tilde{F} \to F_0$ with F_V, $G(F) \neq G$. Then, there is a closed curve $\tilde{u}$ in $\tilde{F}$ with base point in $\tilde{\nu}^{-1}(x)$ such that the homotopy class $\alpha = [\tilde{\nu}\tilde{u}]$ is not contained in G. By the definition of $\tilde{F}$, there is a closed curve u in F_V starting from a point y in $\nu_\sigma^{-1}(x)$ such that $[\nu_\sigma u] = \alpha$. Then, for any y' in the interval $[x, y]$ in $\nu_\sigma^{-1}(x)$, the holonomy map $h_\alpha(y')$ is defined. As above, there is a sequence of points $y_0 = y, y_1, y_2, \cdots$ in $[x, y] \cap F_V$ such that $\lim_{i\to\infty} y_i = x$. By condition $(*)$ of Lemma 6, $h_\alpha(y_i) = y_i$ for each y_i. Since $\pi_1(F_0, x) = \boldsymbol{Z}_{(1)} + \cdots + \boldsymbol{Z}_{(m)} + G$, and $\nu_* \pi_1(F_V) \supset G$, we can put

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$$

for the integers $a_1, \cdots, a_m$ with $(a_1, \cdots, a_m) \neq (0, \cdots, 0)$. Let $u_1, \cdots, u_m$ be the closed curves with base point x realizing the homotopy classes $\alpha_1, \cdots, \alpha_m$ respectively. Then, the multiple $v = u_1^{a_1} \cdots u_m^{a_m}$ realizes α, so that, $[v] = [\nu_\sigma u]$. Let v_t be a homotopy from u to v, $t \in [0, 1]$. Since $h_{v_t}(y')$ is defined for arbitrary $y' \in [x, y]$ which is sufficiently close to x, we have $h_u(y') = h_v(y')$. Hence, for such y'

$$h_\alpha(y') = (h_{\alpha_m})^{a_m} \cdots (h_{\alpha_1})^{a_1}(y') .$$

Since $\lim_{i\to\infty} y_i = x$ and $h_\alpha(y_i) = y_i$, we have $h'_\alpha = 1$. Hence,

$$(h'_{\alpha_m})^{a_m} \cdots (h'_{\alpha_1})^{a_1} = 1 .$$

Therefore,

$$a_1 \log h'_{\alpha_1} + \cdots + a_m \log h'_{\alpha_m} = 0$$

with $(a_1, \cdots, a_m) \neq (0, \cdots, 0)$. But, this contradicts to the assumption of the theorem. This proves Theorem 3.

§ 6. Proof of Theorem 4

The proof of (i) and (ii) of Theorem 4 is contained in the proof of Theorem 3.

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Next we prove (iii). Since f is a local diffeomorphism of class C^2 with $f(0)=0$ and $f'(0)>1$, by a theorem of Sterenberg [5], there is a C^1-diffeomorphism g from a neighborhood of 0 of $\boldsymbol{R}$ into $\boldsymbol{R}$ such that $f'(0)\cdot t = gfg^{-1}(t)$ for any t in the image of g. Hence, by a C^1-alternation of the coordinate of $\nu^{-1}(x)\cap U$, we may assume that $f(t)=dt$, where $d=f'(0)<1$. Hereafter we use the new coordinate of $\nu^{-1}(x)\cap U$ translated by g. Let $f_1,\cdots,f_\ell$ be local diffeomorphisms of $\boldsymbol{R}$ generating $\Phi(F_0)$. Since $\Phi(F_0)$ is abelian, we may assume $f_if=ff_i$ for $i=1,\cdots,m$ by choosing U sufficiently small. Hence, $f_i'(f(t))\cdot f'(t)=f'(f_i(t))\cdot f_i'(t)$, and so $f_i'(f(t))=f_i'(t)$, for $f'(t)=d$. Then, $f_i'(t)=f_i'(0)$, since $\lim_{n\to\infty} f^n(t)=0$ and f_i is of class C^1. Therefore, $f_i(t)=d_i\cdot t$, where $d_i=f_i'(0)$. To show (iii), it is sufficient if $G(F)=G(F')$ is shown for any asymptotic leaves F and F' to F_0. Let α be a closed curve realizing an element of $G(F)$ and let h_α be the holonomy map defined by $\alpha\in\pi_1(F_0,x)$. Then, h_α can be written as $h_\alpha=f_1^{k_1}\cdots f_\ell^{k_\ell}$. By the definition of $G(F)$, there is a closed curve β in $F\cap U$ with the end point t in $\nu^{-1}(x)$ such that $\alpha=\nu\circ\beta$. Hence, $t=h_\alpha(t)=f_1^{k_1}\cdots f_\ell^{k_\ell}(t)=d_1^{k_1}\cdots d_\ell^{k_\ell}\circ t$. Thus, $h_\alpha=id$. since $d_1^{k_1}\cdots d_\ell^{k_\ell}=1$. Therefore, a lift of α to F' is a closed curve, and so the holonomy class of α is contained in $G(F')$. This implies $G(F)=G(F')$. This completes the proof of Theorem 3.

Remark 1. For $\tilde f\in\Phi(F_0)$ let $\tilde f'\in\boldsymbol{R}$ be the derivative of $\tilde f$ at 0. Denoting $D\Phi(F_0)=\{\tilde f'\mid\tilde f\in\Phi(F_0)\}$, $D\Phi(F_0)$ is a multiplicative subgroup of $\boldsymbol{R}-\{0\}$. Let $D:\Phi(F_0)\to D\Phi(F_0)$ be the homomorphism defined by the derivation. Then, for any asymptotic leaf F to F_0, we see that $G(F)\subset\ker Dh$, where h is the homomorphism $\pi_1(F_0,x_*)\to\Phi(F_0)$ defined in §3.

Remark 2. If $\mathscr{F}$ is of class C^2, then, by the method used in the proof of Theorem 4, we see that the sequence

$$1\longrightarrow G(F)\overset{\subset}{\longrightarrow}\pi_1(F_0)\overset{h}{\longrightarrow}\Phi(F_0)\longrightarrow 1$$

is exact for any asymptotic leaf F to F_0.

§7. Proof of Theorem 5

Assuming that $\pi_1(F)=\boldsymbol{Z}_{(1)}+\cdots+\boldsymbol{Z}_{(m)}+G$ for a finite group G, let $N_1,\cdots,N_m$ be the manifolds of F obtained by Proposition 1 for the isomorphism $H:H_1(F:\boldsymbol{Z})\to\boldsymbol{Z}_{(1)}+\cdots+\boldsymbol{Z}_{(m)}$. Here, we may assume

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that $\dim F \geqq 2$, because if $\dim F = 2$, F is a torus. By observing the proof of Theorem 1, the same conclusion of Theorem 1 is satisfied for these $N_1, \cdots, N_m$. Then, if $\mathscr{F}$ is a foliation of class C^r, there are injective C^r-diffeomorphisms $f_i^+ : [0, \varepsilon] \to [0, \varepsilon]$ for $i = 1, \cdots, m$ with the properties (a) and (b) of Theorem 1. By the proof of Theorem 1, f_i can be identified with an one sided holonomy map $h_{\alpha_i}^+$ of a generator α_i of $Z_{(i)}$.

We divide the stage into Case 1 and Case 2. (i) of Theorem 5 is divided into the both cases and (ii) is contained in Case 1.

Case 1: The case that $\mathscr{F}$ is of class C^r, $r \geqq 2$, and that there is i such that $(f_i^+)'(0) \neq 1$. Let f_j be a (both sided) holonomy map of α_j. Then $f_j'(0) = (f_j^+)'(0)$. By Sternberg's theorem, $f_1, \cdots, f_m$ are C^{r-1}-conjugate to linear functions by a same conjugation map g in a small neighborhood of 0. (See the proof of Theorem 4.) Then, $g f_j g^{-1}(t) = f_j'(0) \cdot t$ if $|t|$ is sufficiently small. Let U_- be a collar of F such that U_- is in the another side of U_+. Using Theorem 1 we get $f_i^- : [-\varepsilon', 0] \to [-\varepsilon', 0]$ for $i = 1, \cdots, m$. f_i^- is the other sided holonomy map of a generator α_i' of $Z_{(i)}$. $|f_i(t)| \leqq |t|$ for sufficiently small $|t|$ if and only if $|f_i'(0)| \leqq 1$ since $\bar{f}_i = g f_i g^{-1}$ is linear and $\bar{f}_i(t) = f_i'(0) \cdot t$, $i = 1, \cdots, m$. Hence, by taking ε' small, $\alpha_i' = \alpha_i$, i.e. f_i^+ and f_i^- are the one sided holonomies of the same generator α_i of $Z_{(i)}$. Therefore, there are injective linear maps $\bar{f}_i : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$, $i = 1, \cdots, m$ with the following properties: Let N_i', N_i'', and F_* be the manifolds defined in Theorem 1. Denote by $X_{\bar{f}}$ the quotient manifold obtained from $F_* \times [-\varepsilon, \varepsilon]$ by identifying $(x, t) \in N_i' \times [-\varepsilon, \varepsilon]$ and $(x, \bar{f}_i(t)) \in N_i'' \times [-\varepsilon, \varepsilon]$ for all $i = 1, \cdots, m$ and $t \in [-\varepsilon, \varepsilon]$. The product foliation of $F_* \times [-\varepsilon, \varepsilon]$ induces a foliation $\mathscr{F}_{\bar{f}}$ on $X_{\bar{f}}$. Then, there is a neighborhood V of F such that there is a leaf preserving C^{r-1}-diffeomorphism φ from V onto $X_{\bar{f}}$ which maps F onto $F_* \times 0/\sim$.

By Theorem 4, for any leaf F' meeting V an unique regular covering $\tilde{F}$ is associated with F_V'. Since $\bar{f}_i$ is linear, by Theorem 3, $\nu_* \pi_1(F_V') = \tilde{\nu}_* \pi_1(\tilde{F}) \cong \pi_1(\tilde{F}) \cong G$ if and only if $\log \bar{f}_1', \cdots, \log \bar{f}_m'$ are rationally independent. By an arbitrarily small perturbations of $\bar{f}_1, \cdots, \bar{f}_m$, we can take linear maps $\bar{g}_1, \cdots, \bar{g}_m : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$ such that $\log \bar{g}_1', \cdots, \log \bar{g}_m'$ are rationally independent or dependent when $\log \bar{f}_1', \cdots, \log \bar{f}_m'$ are rationally dependent or independent, respectively.

Let U be an open neighborhood of F contained in V. Let N be a

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neighborhood of $\mathscr{F}|U$ in $\mathrm{Fol}_F^{r-1}(U)$. $\varphi(U)$ is a neighborhood of $F_* \times 0/\sim$ in $X_{\bar{f}}$. Since $\bar{g}_1, \cdots, \bar{g}_m$ are close to $\bar{f}_1, \cdots, \bar{f}_m$ we may assume that $\varphi(U) \subset X_{\bar{g}} \subset X_{\bar{f}}$ and that $\mathscr{F}_{\bar{g}}|\varphi(U)$ is close to $\mathscr{F}_{\bar{f}}|\varphi(U)$. $\mathscr{F}_{\bar{g}}|\varphi(U)$ induce a foliation $\mathscr{F}'$ of U. By taking $\bar{g}_1, \cdots, \bar{g}_m$ sufficiently close to $\bar{f}_1, \cdots, \bar{f}_m$ we can $\mathscr{F}' \in N$.

Case 2. The case that $\mathscr{F}$ is of class C^1 and that $(f_i^+)'(0) = 1$ for all $i = 1, \cdots, m$. f_i^+ is the one sided holonomy map of α_i defined on $[0, \varepsilon]$. First, assume that there is no neighborhood U of F such that $F|U$ is a product foliation. For small $\delta > 0$ we define a C^1-diffeomorphism $g_i^+ : [0, \varepsilon + \delta] \to \boldsymbol{R}_+$ by

$$g_i^+(t) = \begin{cases} t & \text{for } 0 \leqq t < \delta \\ f_i^+(t - \delta) + \delta & \text{for } t > \delta \,. \end{cases}$$

Since $(f_i^+)'(0) = 1$, g_i^+ is of class C^1. It is easy to see that $g_1^+, \cdots, g_m^+$ are mutually commutative since $f_1^+, \cdots, f_m^+$ are so. $g_i^+|[0, \varepsilon]$ is a C^1-perturbation of f_i^+. Let $\mathscr{F}_f$ and X_f be the ones defined in Theorem 1 from f_i^+ and $F_* \times [0, \varepsilon]$. Define $\mathscr{F}_g$ and X_g similarly from g_i^+ and $F_* \times [0, \varepsilon + \delta]$. We can consider that $X_f \subset X_g$ and that $\mathscr{F}_g|X_f$ is C^1-close to $\mathscr{F}_f$ is δ is small enough. There is a neighborhood V_+ of F in U_+ and a C^1-diffeomorphism $\varphi : V_+ \to X_f$ mapping $\mathscr{F}|V_+$ to $\mathscr{F}_f$. Let $\mathscr{F}'_+$ be the foliation induced by φ^{-1} from $\mathscr{F}_g|X_f$. $\mathscr{F}'_+$ is C^1-close to $\mathscr{F}|V_+$ if δ is small enough. We get $\mathscr{F}'_-$ on V_- similarly. On small neighborhoods of $F, \mathscr{F}'_+$ and $\mathscr{F}'_-$ are product foliations. Let $U = V_+ \cup V_-$. Then, we get $\mathscr{F}'$ on U by $\mathscr{F}'|V_\sigma = \mathscr{F}'_\sigma$, $\sigma = \pm$. We can take $\mathscr{F}'$ in any neighborhood N of $\mathscr{F}|U$ in $\mathrm{Fol}_F^1(U)$. By the assumption $\mathscr{F}'$ is not locally equivalent to $\mathscr{F}|U$.

Next, we assume that there is a neighborhood V of F such that $\mathscr{F}|V$ is a product foliation. Then, V is leaf preservingly diffeomorphic to $F \times [-\varepsilon, \varepsilon]$. Consider that $V = F \times [-\varepsilon, \varepsilon]$ and $F = F \times 0$. Let $U = F \times (-\varepsilon/2, \varepsilon/2)$. Let α_i be a generator of $\boldsymbol{Z}_{(i)}$. Then, the holonomy map $f_i : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon]$ of α_i is the identity map. Let g_i be the perturbation of f_i such that $g_i = f_i$ for $i > 1$ and that $|g_1(t)| < |t|$ and $|g_1(\pm\varepsilon)| > \varepsilon/2$. Let $\mathscr{F}_g$ and X_g be as above defined from g_i and $F_* \times [-\varepsilon, \varepsilon]$. Then, we can consider that $U \subset X_g \subset V$ and that $\mathscr{F}_g|U$ is close to $\mathscr{F}|U$ if g_1 is close enough to f_1. Any leaf of $\mathscr{F}_g$ is asymptotic to F, but any leaf of $\mathscr{F}|V$ is not asymptotic. Hence, $\mathscr{F}_g|U$ is not locally equivalent to $\mathscr{F}|V$. This completes the proof of Theorem 5.

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