A GENERALIZATION OF THE COFINITENESS PROBLEM IN LOCAL COHOMOLOGY MODULES

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Abstract

Let R be a commutative Noetherian ring with nonzero identity and let M be a finitely generated R-module. In this paper, we prove that if an ideal I of R is generated by a u.s.d-sequence on M then the local cohomology module $H_I^j(M)$ is I-cofinite. Furthermore, for any system of ideals Φ of R, we study the cofiniteness problem in the context of general local cohomology modules.

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1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity and M is a finitely generated R-module. We shall only assume that R is local, when this is explicitly stated.

For a Noetherian local ring R with the maximal ideal $\mathfrak{m}$ and a finitely generated R-module M, it is well known that the local cohomology modules $H_{\mathfrak{m}}^j(M)$ are Artinian. It is an easy consequence of Matlis duality [21] and the basic work of Grothendieck [10] that this property is equivalent to say that $\mathrm{Supp}_R(H_{\mathfrak{m}}^j(M)) \subseteq V(\mathfrak{m})$ and $\mathrm{Hom}_R(k, H_{\mathfrak{m}}^j(M))$ is finitely generated. In view of these facts, the following conjecture was made by Grothendieck (see [11, Expose XIII, Conjecture 1.2]):

CONJECTURE 1.1. Let M be a finitely generated R-module and let I be an ideal of R. Then the module $\mathrm{Hom}_R(R/I, H_I^j(M))$ is finitely generated for all $j \geq 0$.

The conjecture is false in general, as was shown by Hartshorne [12] with a counterexample, which states that the conjecture is not true even if the ring R is regular. Hartshorne defined a module N to be I-cofinite if the support of N is contained in $V(I)$ and $\operatorname{Ext}_R^i(R/I, N)$ is finitely generated for all i and asked the following question:

QUESTION 1.2. If I is an ideal of R and M is a finitely generated R-module, when are $\operatorname{Ext}_R^i(R/I, H_I^j(M))$ finitely generated for all i and all j?

By working in the derived category, Hartshorne showed that if M is a finitely generated R-module, where R is a complete regular local ring, then $H_I^j(M)$ is I-cofinite in two cases:

(i) I is a non-zero principal ideal [12, Corollary 6.3].
(ii) I is a prime ideal with dimension 1 [12, Corollary 7.7].

There are several papers devoted to the extension of Hartshorne's second result to more general situations. We refer the reader to the papers of Huneke and Koh [14, Theorem 4.1], Delfino [8, Theorem 3], Delfino and Marley [9, Theorem 1], and Yoshida [29, Theorem 1.1].

On the other hand, there have been attempts to generalize result (i) for a ring without any restriction. The general case was proved by Kawasaki [17]. He showed that if an ideal I of a Noetherian ring is principal, up to radical, then the local cohomology modules with support in $V(I)$ are I-cofinite.

There are some generalizations of the theory of local cohomology modules. The following is given in [3].

DEFINITION 1.1. Let Φ be a non-empty set of ideals of R. We call Φ a *system of ideals* (of R) if, whenever $I, I' \in \Phi$, then there is an ideal $J \in \Phi$ such that $J \subseteq II'$. For such a system, for every R-module M, one can define

$$\Gamma_\Phi(M) = \{x \in M : Jx = 0 \text{ for some } J \in \Phi\}.$$

Then $\Gamma_\Phi(-)$ is a functor from $\mathscr{C}(\mathscr{R})$ to itself (where $\mathscr{C}(\mathscr{R})$ denotes the category of all R-modules and all R-homomorphisms).

The functor $\Gamma_\Phi(-)$ is additive, covariant, R-linear and left exact. In [3], $\Gamma_\Phi(-)$ is denoted by $L_\Phi(-)$ and is called the 'general local cohomology functor with respect to Φ'. For each $i \geq 0$, the i-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $H_\Phi^i(-)$. It is shown that the study of torsion theory over R is equivalent to study the general local cohomology theory (see [3]).

The purpose of this note is to make a suitable generalization of Conjecture 1.1 and Question 1.2 in the context of general local cohomology modules and then obtain some results as above, especially for Hartshorne's first result, but for general local cohomology modules.

To go into a detail about our main results, let us begin with a definition.

DEFINITION 1.2. Let Φ be a system of ideals of R. The general local cohomology module $H_\Phi^j(M)$ is defined to be Φ-*cofinite* if there exists an ideal $I \in \Phi$ such that $\mathrm{Ext}_R^i(R/I, H_\Phi^j(M))$ is finitely generated for all i.

Concerning this, we are led to the following analogue of Conjecture 1.1 and Question 1.2 in the general local cohomology context:

QUESTION 1.3. Let Φ be a system of ideals of R and $I \in \Phi$. Let M be a finitely generated R-module. When is $\mathrm{Hom}_R(R/I, H_\Phi^j(M))$ finitely generated?

QUESTION 1.4. Let Φ be a system of ideals of R. Let M be a finitely generated R-module. When are $H_\Phi^j(M)$ Φ-cofinite for all j ?

The origins of our idea go back to the following lemma of Huneke and Koh:

LEMMA 1.3 ([14, Lemma 4.2]). *Let I be an ideal of a Noetherian ring R. Then $H_I^j(M)$ is I-cofinite if and only if $H_I^j(M)$ is I'-cofinite, where I' is the radical of I.*

This result tells us that, in one special case at least, where $\Phi = \{I^i : i \geq 1\}$, our generalization is compatible with the original ones.

However, after some preliminary results in section one, we provide partial answers to the above questions. For instance, we prove that whenever Φ is a system of ideals of R, M is a finitely generated R-module and n is a positive integer such that $H_\Phi^j(M)$ is finitely generated for all $j < n$ and that $H_J^j(M) = 0$ for all $j > n$ and all $J \in \Phi$, then, $H_\Phi^i(M)$ is Φ-cofinite for all i. We also show that if n is a non-negative integer such that for an ideal I of a Noetherian ring R and a finitely generated R-module M, $H_I^i(M)$ is finitely generated for all $i < n$, then $\mathrm{Hom}_R(R/I, H_I^n(M))$ is finitely generated. By the same argument, we will be able to extend the result to the context of general local cohomology modules.

2. Preliminaries

In this section we review some definitions and results. We begin with the following definition

DEFINITION 2.1. A sequence $x_1, \ldots, x_n$ of elements of I is said to be an I-*filter regular M-sequence*, if $\mathrm{Supp}_R((x_1, \ldots, x_{i-1})M :_M x_i/(x_1, \ldots, x_{i-1})M) \subseteq V(I)$ for all $i = 1, \ldots, n$, where $V(I)$ denotes the set of prime ideals of R containing I.

This concept is a generalization of the one of a filter regular sequence which has been studied in [25, 27, 28] and has led to some interesting results. Both concepts coincide if I is the maximal ideal of a local ring. It is easy to see that the analogue of [27, Appendix 2 (ii)] holds true whenever M is finitely generated and $\mathfrak{m}$ replaced by I; so that, if $x_1, \ldots, x_n$ is an I-filter regular M-sequence, then there is an element $y \in I$ such that $x_1, \ldots, x_n, y$ is an I-filter regular M-sequence. Thus, for a positive integer n, there exists an I-filter regular M-sequence of length n.

Throughout the paper, we shall appeal to the following proposition without further comments.

PROPOSITION 2.2 (Nagel-Schenzel). *Let* $x_1, \ldots, x_n \in I$ *be an* I-*filter regular* M-*sequence. Then there are the following isomorphisms* $H_I^i(M) \cong H_{(x_1, \ldots, x_n)}^i(M)$ *for all* $i < n$, *and* $H_I^i(M) \cong H_I^{i-n}(H_{(x_1, \ldots, x_n)}^n(M))$ *for all* $i \geq n$.

PROOF. See [22, Lemma 3.4] and [18, Proposition 1.2].   □

DEFINITION 2.3. A sequence $x_1, \ldots, x_n$ of elements of R is said to be an I-*weak* M-*sequence* if $(x_1, \ldots, x_{i-1})M :_M x_i \subseteq (x_1, \ldots, x_{i-1})M :_M I$ for all $i = 0, \ldots, n$.

REMARK 2.4. Clearly every I-weak M-sequence in I is an I-filter regular M-sequence.

PROPOSITION 2.5. *For a system of ideals* Φ *of a Noetherian ring* R, *a finitely generated* R-*module* M, *and a positive integer* n, *the following conditions are equivalent*:

(1) $H_\Phi^j(M)$ *is finitely generated for all* $j < n$.
(2)   *There exists an ideal* I *in* Φ *such that* $I H_\Phi^j(M) = 0$ *for all* $j < n$.
(3)   *There exists an ideal* I *in* Φ *such that, for all* $J \in \Phi$, *every* J-*filter regular* M-*sequence, of length* n, *is an* I-*weak* M-*sequence*.

PROOF. See [1, Theorem 3].   □

For convenience we write $H_\Phi^j(M) = 0$, whenever j is a negative integer. Also, for an ideal I of R, we use $D_I(-)$ to denote the functor $\varprojlim_\alpha \text{Hom}_R(I^\alpha, -)$.

3. Cofiniteness of local cohomology modules for an ideal generated by a u.s.d-sequence

DEFINITION 3.1. The sequence $x_1, \ldots, x_n$ of elements of R is called a d-*sequence* on M if, for each $i = 0, 1, \ldots, n - 1$, the equality

$$(x_1, \ldots, x_i)M :_M x_{i+1}x_k = (x_1, \ldots, x_i)M :_M x_k$$

holds for all $k \geq i+1$ (this is actually slight weakening of Huneke's definition in [13]); it is an unconditioned strong d-sequence (a u.s.d-sequence) on M if $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ is a d-sequence in any order for all positive integers $\alpha_1, \ldots, \alpha_n$.

REMARK 3.2. If I is an ideal generated by a u.s.d-sequence $x_1, \ldots, x_n$ on M, then $0 :_M x_i^l = 0 :_M x_i = 0 :_M x_i x_j = 0 :_M x_j$ for all i, j with $1 \leq i, j \leq n$ and all $l \geq 1$ and so $\Gamma_I(M) = (0 :_M x_i)$ for all $i = 1, 2, \ldots, n$.

REMARK 3.3. If I is an ideal generated by a u.s.d-sequence $x_1, \ldots, x_n$ on M, then $x_1, \ldots, x_n$ is an I-filter regular M-sequence.

THEOREM 3.4. *If I is an ideal generated by a u.s.d-sequence $x_1, \ldots, x_n$ on M, then the local cohomology module $H_I^i(M)$ is I-cofinite for all i.*

REMARK 3.5. Note that, by [29, Proposition 3.1], in the above theorem $H_I^n(M)$ is not finitely generated, unless $H_I^i(M) = 0$ for all $i > 0$.

PROOF OF THEOREM 3.4. It is well known that $H_I^i(M) = 0$ for all $i > n$. Also, by Remark 3.3 and [19, Theorem A], $H_I^i(M)$ is finitely generated for all $i < n$. Hence we need only to show that $\text{Ext}_R^j(R/I, H_I^n(M))$ is finitely generated for all $j \geq 0$. To do this, we use induction on n, the length of the sequence. When $n = 0$, there is nothing to prove. Assume, inductively, that $n \geq 1$ and that the result has been proved for positive integer smaller than n. Since $x_1, \ldots, x_n$ is a u.s.d-sequence on M, by Remark 3.2, $H_I^0(M) = (0 :_M x_1)$. Now, in view of the natural isomorphism $H_I^n(M/(0 :_M x_1)) \cong H_I^n(M)$, we may replace M by $M/(0 :_M x_1)$ and assume that x_1 is a non-zero divisor on M. Set $\overline{M} := M/x_1 M$ and $J := (x_2, \ldots, x_n)$. It is clear that $x_2, \ldots, x_n$ is a u.s.d-sequence on $\overline{M}$ and so, by the inductive hypothesis, $H_J^{n-1}(\overline{M})$ is J-cofinite. Since, by the Independence Theorem, $H_I^{n-1}(\overline{M}) \cong H_J^{n-1}(\overline{M})$, it follows from [14, Lemma 4.2] that $H_I^{n-1}(\overline{M})$ is I-cofinite. Consider the exact sequence

$$0 \to M \xrightarrow{x_1} M \to \overline{M} \to 0$$

to obtain the long exact sequence

$$H_I^{n-1}(M) \xrightarrow{f} H_I^{n-1}(\overline{M}) \xrightarrow{g} H_I^n(M) \xrightarrow{x_1} H_I^n(M) \to 0.$$

By [19, Theorem A], $H_I^{n-1}(M)$ is finitely generated and so is $\text{Im} f$. Therefore we conclude from the exact sequence $0 \to \text{Im} f \to H_I^{n-1}(\overline{M}) \to \text{Im} g \to 0$ that $\text{Im} g$ is I-cofinite. Now, using the exact sequence

$$0 \to \text{Im} g \to H_I^n(M) \xrightarrow{x_1} H_I^n(M) \to 0$$

and the long exact sequence of 'Ext' modules, by noting that $x \in I$, we obtain that $\text{Ext}_R^j(R/I, H_I^n(M))$ is finitely generated as required.   □

The next theorem is another main result of this paper.

THEOREM 3.6. *Let M be a finitely generated R-module and I be an ideal of R. Suppose that $n \geq 0$ is such that $H_I^j(M)$ is finitely generated for all $j < n$ and that $H_I^j(M) = 0$ for all $j > n$. Then $H_I^i(M)$ is I-cofinite for all i.*

PROOF. We only need to prove the result for $i = n$. By [19, Theorem A], there exists a u.s.d-sequence $x_1, \ldots, x_n$ on M which is an I-filter regular M-sequence. Put $N := H_{(x_1,\ldots,x_n)}^n(M)$. Hence $H_I^0(N) \cong H_I^n(M)$ and $H_I^1(N) \cong H_I^{n+1}(M) = 0$. Also, by [7, Theorem 2.2.4], we have the exact sequence

$$0 \to H_I^n(M) \to N \to D_I(N) \to 0.$$

Now, using the following claim in conjunction with [14, Lemma 4.2], we may assume that I can be generated by a u.s.d-sequence on M, of length n. Now the result follows from Theorem 3.4.   □

CLAIM 3.7. *Let M and I be as above and let $x_1, \ldots, x_n$ be a u.s.d-sequence on M which is an I-filter regular M-sequence. Put $N := H_{(x_1,\ldots,x_n)}^n(M)$. Then*

$$\operatorname{Ext}_R^i(R/I, D_I(N)) = 0$$

for all $i \geq 0$.

PROOF. Let E be an injective R-module. Then by [7, Theorem 2.2.4 (ii)], the sequence

$$0 \to \Gamma_I(E) \to E \to D_I(E) \to 0$$

is exact. Since, by [7, Proposition 2.1.4], $\Gamma_I(E)$ is injective, the above sequence is split and so $D_I(E)$ is injective. Now, let $L^\bullet$ be an injective resolution of N. Thus $D_I(L^\bullet)$ is an injective complex, whose terms, by [7, Corollary 2.2.8 (iv)], are all I-torsion-free. Note that, by [7, Theorem 2.2.4 (ii)], $\mathcal{R}^i D_I(N) \cong H_I^{i+1}(N) = 0$ for all $i \geq 1$ and $D_I(-)$ is a left exact functor. Hence $D_I(L^\bullet)$ is an injective resolution of $D_I(N)$ and so $\operatorname{Ext}_R^i(R/I, D_I(N)) = 0$ for all $i \geq 0$.   □

4. Results on cofiniteness of general local cohomology modules

The following theorem is one of the main results of this paper. With the following theorem, we not only offer a partial answer to the Question 1.3, but we also provide an improved form of Proposition 6 of [1] (see [11, Expose XIII, Conjecture 1.1]).

THEOREM 4.1. *Let I be an ideal of a Noetherian ring R and M be a finitely generated R-module. Let n be a non-negative integer such that $H_I^i(M)$ is finitely generated for all $i < n$. Then $\mathrm{Hom}_R(R/I, H_I^n(M))$ is finitely generated.*

PROOF. We use induction on n. When $n = 0$, the module $H_I^0(M) \cong \Gamma_I(M)$ is finitely generated. Assume, inductively, that $n > 0$ and we have established the result for positive integers smaller than n. Consider the exact sequence

$$0 \to \Gamma_I(M) \to M \to \frac{M}{\Gamma_I(M)} \to 0.$$

By [7, Corollary 2.1.7 (ii)], $H_I^i(\Gamma_I(M)) = 0$ for all $i > 0$. This means that $H_I^i(M) \cong H_I^i(M/\Gamma_I(M))$, for all $i > 0$. Hence we can make the additional assumption that M is an I-torsion-free R-module. By [7, Lemma 2.1.1 (ii)], the ideal I contains an element r which is a non-zero divisor on M. The exact sequence $0 \to M \xrightarrow{r} M \to (M/rM) \to 0$ induces an exact sequence

$$H_I^{n-1}(M) \to H_I^{n-1}\left(\frac{M}{rM}\right) \xrightarrow{f} H_I^n(M) \xrightarrow{r} H_I^n(M).$$

Now we conclude from the exact sequence $0 \to \mathrm{Im}\,f \to H_I^n(M) \xrightarrow{r} H_I^n(M)$ and the fact that $r \in I$ that it suffices for us to show that $\mathrm{Hom}_R(R/I, \mathrm{Im}\,f)$ is finitely generated. Since $H_I^{n-1}(M)$ is finitely generated, so is $\mathrm{Ker}\,f$. Hence consider the exact sequence

$$0 \to \mathrm{Ker}\,f \to H_I^{n-1}\left(\frac{M}{rM}\right) \to \mathrm{Im}\,f \to 0$$

to deduce that $\mathrm{Hom}_R(R/I, \mathrm{Im}\,f)$ is finitely generated, because

$$\mathrm{Hom}_R(R/I, H_I^{n-1}(M/rM))$$

is finitely generated by induction hypothesis. This completes the inductive step and so the proof. □

REMARK 4.2. Let Φ be a system of ideals of R and M be a finitely generated R-module. Let n be a non-negative integer such that $H_\Phi^i(M)$ is finitely generated for all $i < n$. Then $\mathrm{Hom}_R(R/I, H_\Phi^n(M))$ is finitely generated for all $I \in \Phi$.

LEMMA 4.3. *Let the notation be as in Remark 4.2. Then $H_\Phi^i(M) \cong \lim_{\overrightarrow{a \in \Phi}} H_a^i(M)$ for all $i \geq 0$.*

PROOF. See [4, Lemma 2.1]. □

LEMMA 4.4. *Let the notation be as in Remark 4.2. If E is an injective R-module, then so is $\Gamma_\Phi(E)$.*

PROOF. By Lemma 4.3, $\Gamma_\Phi(M) \cong \varinjlim_{\mathfrak{a} \in \Phi} \Gamma_\mathfrak{a}(M)$. On the other hand, for an ideal $\mathfrak{a}$ of R, by [7, Proposition 2.1.4], $\Gamma_\mathfrak{a}(E)$ is injective since R is Noetherian. Therefore, by [24, Theorem 4.10], $\Gamma_\Phi(E)$ is injective. ☐

DEFINITION 4.5. We say that an R-module M is Φ-*torsion* if $\Gamma_\Phi(M) = M$. We say that M is Φ-*torsion-free* if $\Gamma_\Phi(M) = 0$.

LEMMA 4.6. *If N is an Φ-torsion R-module, then $H_\Phi^i(N) = 0$ for all $i > 0$.*

PROOF. By using Lemma 4.4, we can deduce, as in the proof of [7, Corollary 2.1.6], that for every Φ-torsion R-module N, there exists an injective resolution in which each term is an Φ-torsion R-module. Hence $H_\Phi^i(N) = 0$ for all $i > 0$. ☐

COROLLARY 4.7 (See [19, Theorem B (β)] and [6, Theorem 2.2]). *Let I be an ideal of R and n be a non-negative integer such that $H_I^i(M)$ is finitely generated for all $i < n$. Then the set $\mathrm{Ass}_R(H_I^n(M))$ is finite.*

PROOF. It follows from Theorem 4.1 and the fact that

$$\mathrm{Ass}_R(H_I^n(M)) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H_I^n(M))).$$ ☐

The following theorem is, in fact, a general version of Theorem 3.6 in the context of general local cohomology modules. With it, we are able to prove I-cofiniteness in several cases.

THEOREM 4.8. *Let Φ be a system of ideals of R and M be a finitely generated R-module. Suppose that $n \geq 0$ is such that $H_\Phi^j(M)$ is finitely generated for all $j < n$ and that $H_J^j(M) = 0$ for all $j > n$ and all $J \in \Phi$. Then, $H_\Phi^i(M)$ is Φ-cofinite for all i.*

REMARK 4.9. Let I be in Φ, and put $\Phi_I = \{J \in \Phi \mid J \subseteq \mathfrak{a}I \text{ for some } \mathfrak{a} \in \Phi\}$. Then Φ_I is cofinal in Φ. Note that $J \subseteq I$ for all $J \in \Phi_I$.

PROOF OF THEOREM 4.8. We have only to prove the theorem in the case $i = n$. By Proposition 2.5 and Remark 2.4, there exists an ideal I in Φ such that, for all $J \in \Phi$, every J-filter regular M-sequence, of length n, is I-filter regular M-sequence. Now, by Remark 4.9, we may assume that $J \subseteq I$, for all $J \in \Phi$. Hence every J-filter regular M-sequence, of length n, is also I-filter regular M-sequence. Fix $J \in \Phi$. Let

$x_1, \ldots, x_n$ be an J-filter regular M-sequence. Hence by Proposition 2.2, we conclude that

$$H_I^n(M) \cong H_I^0(H_{(x_1,\ldots,x_n)}^n(M)) \cong H_I^0(H_J^0(H_{(x_1,\ldots,x_n)}^n(M))) \cong H_I^0(H_J^n(M)).$$

Since the local cohomology functor $H_I^0(-)$ commutes with direct limits, it follows that $H_I^n(M) \cong H_I^0(H_\Phi^n(M))$. Now, by using Claim 4.10 and [7, Theorem 2.2.4], we have the exact sequence

$$0 \to H_I^n(M) \to H_\Phi^n(M) \to D_I(H_\Phi^n(M)) \to 0.$$

Therefore we have the following long exact sequence

$$\cdots \to \operatorname{Ext}_R^i(R/I, H_I^n(M)) \to \operatorname{Ext}_R^i(R/I, H_\Phi^n(M))$$
$$\to \operatorname{Ext}_R^i(R/I, D_I(H_\Phi^n(M))) \to \cdots.$$

Now, by Claim 4.11, in order to conclude the result it suffices for us to prove that $\operatorname{Ext}_R^i(R/I, H_I^n(M))$ is finitely generated for all i. Note that, by Proposition 2.5 (3) and [18, Theorem (3)], $H_I^i(M)$ is finitely generated for all $i < n$. Hence the conclusion follows immediately from Theorem 3.6.   □

CLAIM 4.10. *Let Φ, M, I and n be as in Theorem 4.8. Then $H_I^i(H_\Phi^n(M)) = 0$ for all $i > 0$.*

PROOF. Fix $J \in \Phi$. Let $x_1, \ldots, x_n$ be an J-filter and so I-filter regular M-sequence. Set $N := H_{(x_1,\ldots,x_n)}^n(M)$. Let $E^\bullet$ be an injective resolution of N. Thus, by Lemma 4.4, $H_J^0(E^\bullet)$ is an injective complex. Since $H_J^j(N) \cong H_J^{j+n}(M) = 0$ for all $j \geq 1$ and $H_J^0(-)$ is a left exact functor, $H_J^0(E^\bullet)$ is an injective resolution of $H_J^0(N)$. Now, it follows from Proposition 2.2 that

$$H_I^i(H_J^n(M)) \cong H_I^i(H_J^0(N)) \cong H^i(H_I^0(H_J^0(E^\bullet))).$$

Since $J \subseteq I$, $H_I^0(H_J^0(E^\bullet)) \cong H_I^0(E^\bullet)$ and hence

$$H_I^i(H_J^n(M)) \cong H^i(H_I^0(E^\bullet)) \cong H_I^i(N) \cong H_I^{i+n}(M),$$

where, by our assumption, the latter module is zero for all $i > 0$. Now the claim follows, since $H_I^0(-)$ commutes with direct limits.   □

CLAIM 4.11. *Let Φ, M, I and n be as in Theorem 4.8. Then*

$$\operatorname{Ext}_R^i(R/I, D_I(H_\Phi^n(M))) = 0$$

for all $i \geq 0$.

PROOF. Let $L^\bullet$ be an injective resolution of $H^n_\Phi(M)$. Application of the functor $D_I(-)$ yields an injective complex, whose terms are all I-torsion-free. Observe that, as we mentioned in the proof of Claim 3.7, for an injective R-module E, $D_I(E)$ is injective and I-torsion-free. By [7, Theorem 2.2.4 (ii)], for all $i \geq 1$, $\mathscr{R}^i D_I(H^n_\Phi(M)) \cong H^{i+1}_I(H^n_\Phi(M))$. Since, by the above claim, $H^{i+1}_I(H^n_\Phi(M)) = 0$ and $D_I(-)$ is a left exact functor, $D_I(L^\bullet)$ is, in fact, an injective resolution of $D_I(H^n_\Phi(M))$. Therefore $\operatorname{Ext}^i_R(R/I, D_I(H^n_\Phi(M))) = 0$ for all $i \geq 0$. □

The following definitions are recoded for another result of this paper. These definitions are generalizations of the familiar concepts, 'arithmetic rank of I', denoted by $\operatorname{ara}(I)$, and 'finiteness dimension of M relative to I', denoted by $f_I(M)$ (see [7, Definition 3.3.2 and Definition 9.1.3]).

DEFINITION 4.12. Let Φ be a system of ideals of R. The *arithmetic rank of* Φ, denoted by $\operatorname{ara}(\Phi)$, is defined as follows $\operatorname{ara}(\Phi) = \operatorname{Max}\{\operatorname{ara}(I) : I \in \Phi\}$.

Note that $\operatorname{ara}(\Phi)$ is either a positive integer, whenever Φ is non-empty, or ∞. In particular, if $\Phi = \{I^i : i > 0\}$, then $\operatorname{ara}(\Phi) = \operatorname{ara}(I)$.

DEFINITION 4.13 (See [1, Definition 5]). Let M be a finitely generated R-module. We define the *finiteness dimension* $f_\Phi(M)$ *of M relative to* Φ by

$$f_\Phi(M) = \inf\{i \geq 0 : H^i_\Phi(M) \text{ is not finitely generated}\}$$
$$= \inf\{i \geq 0 : I H^i_\Phi(M) \neq 0 \text{ for all } I \in \Phi\}.$$

COROLLARY 4.14. *Let Φ be a system of ideals of R. Let M be a finitely generated R-module and I be an arbitrary ideal of R.*

(i)   *If $\operatorname{ara}(\Phi) \leq f_\Phi(M)$, then $H^i_\Phi(M)$ is Φ-cofinite for all i.*
(ii) *If $H^i_I(M) = 0$ for all $i > f_I(M)$, then $H^i_I(M)$ is I-cofinite for all i.*
(iii) *If $\operatorname{ara}(I) \leq f_I(M)$, then $H^i_I(M)$ is I-cofinite for all i.*

Note that (iii) provides an improved form of the main result of [17].

PROOF. (i) If $f_\Phi(M)$ is infinite then there is nothing to do any more. Hence we may assume that $f_\Phi(M)$ is finite. So $f_\Phi(M) = \operatorname{ara}(\Phi)$, by [7, Corollary 3.3.3]. Now the result follows easily from Theorem 4.8.
The statements (ii) and (iii) are immediate consequence of Theorem 3.6. □

COROLLARY 4.15. *Let n be a positive integer and let*

$$\Phi = \left\{ \sum_{i=1}^n Rx_i : x_1, \dots, x_n \text{ is an } M\text{-regular sequence} \right\}.$$

Then $H^j_\Phi(M)$ is Φ-cofinite for all j.

PROOF. Note that, by [26, Example 3.10] and [5, Theorem], Φ is a system of ideals of R. Now use [7, Theorem 6.2.7 and Corollary 3.3.3] and then Theorem 4.8.   □

There are some results on the finiteness of the Bass numbers of local cohomology modules (compare [15, 16, 20, 17]). For a prime ideal $\mathfrak{p}$ of R and an R-module W, the i-th Bass number $\mu^i(\mathfrak{p}, W)$, is defined to be $\dim_{k(\mathfrak{p})} \text{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), W_\mathfrak{p})$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ (compare [2, Lemma 2.7]). If W is I-cofinite, then, by [14, Lemma 4.2], the Bass numbers of W are finite. Hence one can put all our various results together to obtain some corollaries about the finiteness of the Bass numbers of local cohomology modules.

Acknowledgments

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