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ON THE PROJECTIVE COVER OF AN ORBIT SPACE

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Abstract

In this paper, we obtain the projective cover of the orbit space X/G in terms of the orbit space of the projective space of X, when X is a Tychonoff G-space and G is a finite discrete group. An example shows that finiteness of G is needed.

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1. Introduction

An *action* of a topological group G on a topological space X is a continuous map θ from $G \times X$ to X satisfying $\theta(e, x) = x$ and $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$, where $g_1, g_2 \in G$ and e is the identity of G: a topological space together with a given action is called a *G-space*. A subspace Y of X is called *invariant* if $\theta(G \times Y) \subseteq Y$, that is, Y becomes a G-space with the action induced by θ. Denote $\theta(g, x)$ by $g \cdot x$. For $x \in X$, the set $G_x = \{g \cdot x | g \in G\}$ is called the *orbit of* x. The collection of orbits is denoted by X/G and the topology on it is coinduced by the map p from X to X/G taking x to its orbit G_x. The space X/G is called the *orbit space* of X (with respect to G). The map p will be called the *orbit map*. It is open and if G is compact, it is closed as well.

The complete Boolean algebra of regular closed sets of a space X is denoted by $R(X)$ and its Stone space by $S(R(X))$. Denoting the closure and the interior of a set A in X by $\operatorname{Cl} A$ and $\operatorname{Int} A$, respectively, we recall that a set F in X is regular closed if $F = \mathrm{Cl}\,\mathrm{Int}\,F$, the complement F^c of F in $R(X)$ is $\mathrm{Cl}(X - F)$ and the meet $F_1 \wedge F_2$ of regular closed sets F_1 and F_2 of X is $\mathrm{Cl}(\mathrm{Int}\,F_1 \cap \mathrm{Int}\,F_2)$. The Stone-Čech compactification of X is denoted by βX. The map g from $S(R(\beta X))$ to βX taking a maximal filter $\mathscr{F}$ of regular closed sets of βX to $\bigcap \mathscr{F}$ (the singleton $\bigcap \mathscr{F}$ is identified with the point in it) is known to be continuous and irreducible. The projective cover of X is the pair $(E(X), h)$, where $E(X) = g^{-1}(X)$ and $h\colon E(X) \to X$ is the restriction of g; $E(X)$ is called the *projective space* of X. Projective covers have been constructed using different methods by Gleason (1958), Rainwater (1959), Strauss (1967), Banaschewski (1968) and Hager (1971).

Recently, Srivastava (1987) has extended an action of a discrete group G on a space X to an action on βX, which keeps X invariant. In Section 2 of this paper, after introducing an action on $S(R(X))$ through the given action of a discrete group G on X, and passing to the extended action on βX, we find $E(X)$ to be an invariant subspace of $S(R(\beta X))$; thus $E(X)$ becomes a G-space. We study the projective cover of an orbit space in Section 3. It is obtained that, in case G is finite, $S(R(X/G))$ is homeomorphic to $S(R(X))/G$. Since $E(X)$ is $S(R(X))$ if X is compact, $E(X)/G$ is homeomorphic to $E(X/G)$ for a compact G-space X with G finite. Taking X to be an arbitrary Tychonoff G-space, and passing to its Stone-Čech compactification we establish $E(X)/G$ to be homeomorphic to $E(X/G)$ with the application of the same result obtained for a compact space, to βX. Finally, an example is provided to show that the orbit space of the Stone space of $R(X)$ need not be homeomorphic to the Stone space of $R(X/G)$.

For terms not explained here we refer to Willard (1970), Bredon (1972) and Walker (1974).

2. Action on $E(X)$

Let Z be a zero-set of a G-space X. Then for $a \in G, a \cdot Z = \{a \cdot z | z \in Z\}$ is also a zero-set of X. It is easily seen that for a z-ultrafilter $\mathscr{F}$ on X, the family $a \cdot \mathscr{F}$ consisting of $a \cdot Z, Z \in \mathscr{F}$, is a z-ultrafilter on X. Taking G to be a discrete group, the map $\psi\colon G \times \beta X \to \beta X$ given by $\psi(a, \mathscr{F}) = a \cdot \mathscr{F}$ defines an action of G on βX keeping X invariant [see Srivastava (1987)].

In a similar way, the action on a G-space X, where G is a discrete group, gives rise to an action ν of G on $S(R(X))$. In fact, $\nu\colon G \times S(R(X)) \to S(R(X))$ is a map which sends $(a, \mathscr{F})$ to $a \cdot \mathscr{F} = \{a \cdot F | F \in \mathscr{F}\}$. If $\mathscr{F}$ is a maximal filter of regular closed sets of βX, then $a \cdot \mathscr{F}$ is also a maximal filter of regular closed sets of βX such that $\bigcap a \cdot \mathscr{F}$ is the point $a \cdot p$ in βX, where p is $\bigcap \mathscr{F}$. Noting that X is an invariant subspace of βX, we have

2.1 LEMMA. *$E(X)$ is an invariant subspace of $S(R(\beta X))$.*

3. The orbit space $S(R(X))/G$

Throughout this section, unless stated otherwise, G will denote a compact group. Let X be a G-space and let $p\colon X \to X/G$ be the orbit map. Then, for $\mathscr{F} \in S(R(X))$, the collection $\{p(F) | F \in \mathscr{F}\}$ is denoted by $p(\mathscr{F})$. We state the following Lemma without proof.

3.1 LEMMA. *We have*
(i) $p(F) \in R(X/G)$, *whenever* $F \in R(X)$;
(ii) $p^{-1}(H) \in R(X)$, *whenever* $H \in R(X/G)$;
(iii) *for* $H \in R(X/G), p^{-1}(H^c) = (p^{-1}(H))^c$;
(iv) $p(\mathscr{F}) \in S(R(X/G))$, *whenever* $\mathscr{F} \in S(R(X))$.

3.2 PROPOSITION. *The map $S(p)\colon S(R(X)) \to S(R(X/G))$ defined by $S(p)$ $(\mathscr{F}) = p(\mathscr{F})$, $\mathscr{F} \in S(R(X))$, is onto and continuous.*

PROOF. Let $\mathscr{H} \in S(R(X/G))$ and let $\mathscr{F}$ be a maximal filter in $R(X)$ containing the filter generated by $p^{-1}(\mathscr{H})$. Then $p(\mathscr{F}) = \mathscr{H}$, which proves that $S(p)$ is onto. The continuity of $S(p)$ follows by noting that, for $\mathscr{G} \in S(R(X))$, $H \in p(\mathscr{G})$ if and only if $p^{-1}(H) \in \mathscr{G}$.

3.3 LEMMA. *Let X be a G-space, where G is a finite discrete group and let $\mathscr{F}, \mathscr{H} \in S(R(X))$. Then $p(\mathscr{F}) = p(\mathscr{H})$ if and only if $\mathscr{F} = a \cdot \mathscr{H}$ for some $a \in G$.*

PROOF. We prove the necessary part only. Suppose to the contrary that $\mathscr{F} \neq a \cdot \mathscr{H}$, for any $a \in G$. Then, for each $a \in G$, there exists an $F_a \in \mathscr{F}$ such that $F_a \notin a \cdot \mathscr{H}$. Put $F = \bigwedge_{a \in G} F_a$. Then $F \in \mathscr{F}$ and $F \notin a \cdot \mathscr{H}$, for any $a \in G$. Since, for each $a \in G$, $a \cdot \mathscr{H}$ is a maximal filter, there exists $H_a \in \mathscr{H}$ such that $F \wedge a \cdot H_a = \varnothing$. Let $H = \bigwedge_{a \in G} H_a$. Then, for each $a \in G$, $F \wedge a \cdot H = \varnothing$, that is $F \cap a \cdot \operatorname{Int} H = \varnothing$. For $h \in \operatorname{Int} H$, $G_h \neq G_x$, for any $x \in F$ and hence $p(F) \cap p(\operatorname{Int} H) = \varnothing$. This implies that $\operatorname{Int} p(F) \cap \operatorname{Cl} p(\operatorname{Int} H) = \operatorname{Int} p(F) \cap p(H) = \varnothing$ and therefore $p(F) \wedge p(H) = \varnothing$. Hence $p(\mathscr{F}) \neq p(\mathscr{H})$.

The above lemma gives rise to an injective map $p_G\colon S(R(X))/G \to S(R(X/G))$ defined by $p_G(G_{\mathscr{F}}) = p(\mathscr{F})$, $G_{\mathscr{F}} \in S(R(X))/G$. Since $S(p)$ is the composition of the orbit map $q: S(R(X)) \to S(R(X))/G$ with p_G, it follows that p_G is continuous and onto. From the compactness of $S(R(X))/G$, we obtain that p_G is a homeomorphism. Thus we have the following theorem.

3.4 THEOREM. *If G is a finite discrete group and X is a G-space, then $S(R(X))/G$ is homeomorphic to $S(R(X/G))$. In particular, if X is compact then $E(X)/G$ is homeomorphic to $E(X/G)$.*

Note that the above theorem determines the projective cover of the orbit space X/G in terms of the orbit space of the projective space of X, when X is compact and G is finite. Using this result, we generalize it below to an arbitrary Tychonoff G-space.

Let G be a finite discrete group and let X be a G-space. Since X/G is a dense subspace of $\beta X/G$, the projective space $E(X/G)$ of X/G is $h_\beta^{-1}(X/G)$, where $(S(R(\beta X/G)), h_\beta)$ is the projective cover of $\beta X/G$. In view of Lemma 2.1, $E(X)$ is a G-space. Now we have

3.5 THEOREM. *Let X be a G-space, where G is a finite discrete group. Then $E(X/G)$ is homeomorphic to $E(X)/G$.*

PROOF. Let $q: \beta X \to \beta X/G$ be the orbit map. Then the map

$$q_G: S(R(\beta X))/G \to S(R(\beta X/G))$$

defined by $q_G(G_{\mathscr{F}}) = q(\mathscr{F})$, where $\mathscr{F} \in S(R(\beta X))$, describes a homeomorphism [see, Theorem 3.4]. Since, for $\mathscr{H} \in S(R(\beta X))$, $\bigcap \mathscr{H} \in X$ if and only if $\bigcap q(\mathscr{H}) \in X/G$, it follows that $q_G(E(X)/G) = E(X/G)$ and we have the result.

3.6 REMARK. Let $(E(X/G), g_1)$ be the projective cover of X/G. Then, in view of Theorem 3.5, $(E(X)/G, h_1)$ can be regarded as the projective cover of X/G, where h_1 is the composition of the restriction of the homeomorphism q_G to $E(X)/G$ and g_1. It may also be noted that h_1 maps an orbit $G_{\mathscr{F}}$, $\mathscr{F} \in E(X)$, to the orbit in X/G determined by $\bigcap \mathscr{F}$.

3.7 EXAMPLE. Let D be the open interval $(0,1)$ of the real line. Consider the G-space D, where G is the discrete group consisting of all non-decreasing homeomorphisms from D to D (group operation being the composition of homeomorphisms) and the action θ of G on D is given by $T \cdot x = T(x)$, $x \in D$, $T \in G$. Let $\mathscr{F}$ be the filter in $R(D)$ generated by the collection consisting of regular closed sets containing $1/4$ in their interiors, and closed intervals $[s, 1/4]$, $0 < s < 1/4$; and let $\mathscr{H}$ be the filter in $R(D)$ generated by the collection consisting of regular closed sets containing $3/4$ in their interiors, and closed intervals $[3/4, t]$, $3/4 < t < 1$. It is easy to check that both $\mathscr{F}$ and $\mathscr{H}$ are in $S(R(D))$

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and that $\mathscr{F} \neq T \cdot \mathscr{H}$, for any $T \in G$. This shows that $G_{\mathscr{F}}$ and $G_{\mathscr{H}}$ are distinct and both belong to $S(R(D))/G$, whereas $S(R(D/G))$ is the singleton.

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