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A NOTE ON INTEGRAL REPRESENTATIONS IN STOKES FLOW

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Abstract

An alternative method via generalised functions is used to obtain the surface integral representation for a finite body in an infinite fluid in Stokes flow. The problem is further generalised to a finite number of intersecting finite bodies in an infinite and semi-infinite fluid. Possible applications to line distributions for axi-symmetric bodies are discussed.

1. Introduction

Volume and surface integral representations for finite bodies in an infinite fluid for Stokes flow have been known for many years (see for example Oseen [8], Ladyzhenskaya [7] and Happel and Brenner [5]). The derivation of the integral representation has been via Green's theorem techniques and the subsequent substitution of the fundamental velocity and pressure singularity into the integral equations. In this paper a simpler and more straightforward approach is developed using generalised functions for a finite body in an infinite fluid. This can be extended to a finite number of finite bodies. The equivalent problem for a semi-infinite fluid has been obtained in terms of previously known solutions (Blake [1] and Blake and Chwang [2]) for singular point forces and mass sources.

2. Finite body in an infinite fluid

Let us consider a finite body, volume V^*, surface S, with outward normal $\boldsymbol{n}$ defined by a scalar function $g(\boldsymbol{x})$ such that $g(\boldsymbol{x})$ is less than zero inside the body and greater than zero outside (i.e. $g(\boldsymbol{x}) = 0$ defines the surface S)*. The Heaviside step function $H(g(\boldsymbol{x}))$ is defined such that it takes a zero value inside S and a unit value outside. The volume V consists of the remaining

* We suppose $g(x)$ is infinitely differentiable such that $\nabla g \neq 0$ anywhere on $g = 0$; thus the hypersurface has no singular points.

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volume outside of V^* (i.e. $V \cup V^*$ is the complete Euclidean three-dimensional space).

The Stokes flow equations of motion for an incompressible fluid are defined by,

$$\nabla p = \mu \nabla^2 \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where p is the pressure and $\boldsymbol{u}$ is the velocity vector. These equations are valid in V and on its boundary S. The validity of these equations can be extended to the complete space $V \cup V^*$ by introducing the terms $pH(g(\boldsymbol{x}))$ and $\boldsymbol{u}H(g(\boldsymbol{x}))$; thus on reverting to tensor notation,

$$\frac{\partial (pH)}{\partial x_i} = \mu \frac{\partial^2 (Hu_i)}{\partial x_j^2} + F_i,$$
$$\frac{\partial (Hu_i)}{\partial x_i} = M, \tag{2}$$

where

$$F_i = pn_i\delta(g)|\nabla g| - \mu \frac{\partial}{\partial n}(u_i|\nabla g|\delta(g)) - \mu\left(\frac{\partial u_i}{\partial n}|\nabla g|\delta(g)\right)$$

and

$$M = u_n|\nabla g|\delta(g). \tag{3}$$

The normal velocity is denoted by u_n and likewise $\partial/\partial n$ is the normal derivative. The formulae in (3) were derived by using the relation

$$\frac{\partial H}{\partial x_j} = \frac{\partial g}{\partial x_j}\delta(g) = n_j|\nabla g|\delta(g), \tag{4}$$

where $\delta(g)$ is the Dirac delta function and ∇g is the gradient. The expressions for $F(y)$ and $M(y)$ can now be interpreted as a generalised volume force distribution and a mass per unit volume source distribution respectively.

Equations (2) can be solved in a straightforward manner by Fourier transform methods. The solution for a force $\boldsymbol{F}(\boldsymbol{y})$ and mass/unit volume source $M(\boldsymbol{y})$ is

$$u_i = \int_{V \cup V^*} \left\{\frac{F_j(\boldsymbol{y})}{8\pi\mu}\left[\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right] + \frac{M(\boldsymbol{y})}{4\pi}\frac{(x_i - y_i)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right\} dV \tag{5a}$$

and

$$p = \int_{V \cup V^*} \frac{1}{4\pi}\frac{F_i(\boldsymbol{y})(x_i - y_i)}{|\boldsymbol{x}-\boldsymbol{y}|^3} dV + \mu M(\boldsymbol{x}), \tag{5b}$$

where the integral dV is over all space $V \cup V^*$. On substitution of the expressions in (3) into (5a), using the integral properties of the delta function (Jones [6], p. 263) and integrating by parts where necessary, we obtain for $x \in V$

$$u_i(x) = \int_S \left[-(u_n q_i + v_{ni} p) + \mu \left(v_{ji} \frac{\partial u_j}{\partial n} - u_j \frac{\partial v_{ji}}{\partial n} \right) \right] dS$$

and

$$p = \int_S \left[-p n_i q_i + \mu \left(\frac{\partial u_i}{\partial n} q_i - u_i \frac{\partial q_i}{\partial n} \right) \right] dS, \tag{6}$$

where we have defined

$$q_i = \frac{1}{4\pi} \frac{(x_i - y_i)}{|x - y|^3}$$

and

$$v_{ji} = \frac{1}{8\pi\mu} \left(\frac{\delta_{ij}}{|x - y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^3} \right). \tag{7}$$

This is the surface integral representation derived by Oseen using a Green's theorem for Stokes flow (see Happel and Brenner [5]).

The problem can quite easily be extended to a finite number of non-singular, non-intersecting hypersurfaces S_i defined by

$$S_i = \{x : g_i(x) = 0\}, \qquad i = 1, 2, \cdots, n. \tag{8}$$

If we use the expressions (Jones [6])

$$\delta\left(\prod_{i=1}^{n} g_i(x) \right) = \sum_{j=1}^{n} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} |g_i| \right)^{-1} \delta(g_j) \tag{9a}$$

$$\frac{\partial}{\partial x_k} H\left(\prod_{i=1}^{n} g_i(x) \right) = \delta\left(\prod_{i=1}^{n} g_i \right) \sum_{j=1}^{n} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} g_i \right) \frac{\partial g_j}{\partial x_k} \tag{9b}$$

then it follows in an analogous way that

$$u_i(x) = \sum_{k=1}^{n} \int_{S_k} \left[-(u_n q_i + v_{ni} p) + \mu \left(v_{ij} \frac{\partial u_j}{\partial n} - u_j \frac{\partial v_{ji}}{\partial n} \right) \right] dS \tag{10a}$$

and

$$p = \sum_{k=1}^{n} \int_{S_k} \left[-p n_i q_i + \mu \left(\frac{\partial u_i}{\partial n} q_i - u_n \frac{\partial q_i}{\partial n} \right) \right] dS \tag{10b}$$

where $x \in V$, the volume exterior to S_k, $k = 1, \cdots, n$. Thus the integral representation derived by Oseen [8] can be extended to a finite number of bodies in Stokes flow.

We can further extend this derivation to a semi-infinite fluid; all we need in this case are the Green's functions for a force and a mass source subject to a

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no-slip condition on a plane boundary, say $x_3 = 0$. We require that the surfaces $g_i(\boldsymbol{x}) = 0$ be wholly contained in the half space $x_3 > 0$, and that they are, as before, non-singular and non-intersecting. Explicit formulae and interpretation for the Green's functions can be found in Blake [1] for the Green's function due to a point force, and Blake and Chwang [2] for the Green's functions due to a mass source. We can then substitute these Green's functions into (10a) and (10b) to obtain the integral representations for the semi-infinite case.

Recently Chwang and Wu [3], [4] have obtained numerous exact solutions for flow around axi-symmetric bodies, by using a line distribution of singularities along the axis of symmetry. Their method consists of empirical rules to obtain the correct distribution of singularities for a spheroid in different flow fields. Although their method is refreshingly simple and elegant, the method of extension to general axi-symmetric bodies is not entirely clear. It should be possible by using the above surface integral representation to calculate the exact strengths and singularities needed on the axis of symmetry for any flow field. This could be obtained by a Taylor series expansion in the direction normal to the surface.

In conclusion an alternative method of deriving the Oseen surface integral representations via generalised functions has been used for the cases of a single body and a finite number in either an infinite or semi-infinite fluid. The method is straightforward and elegant, in comparison to the more complicated derivation of Oseen [8].

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