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RESEARCH ARTICLE

k-leaky double Hurwitz descendants

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Abstract
We define a new class of enumerative invariants called k-leaky double Hurwitz descendants, generalizing both descendant integrals of double ramification cycles and the k-leaky double Hurwitz numbers introduced in [CMR25]. These numbers are defined as intersection numbers of the logarithmic DR cycle against ψ-classes and logarithmic classes coming from piecewise polynomials encoding fixed branch point conditions. We give a tropical graph sum formula for these new invariants, allowing us to show their piecewise polynomiality in any genus. Investigating the piecewise polynomial structure further (and restricting to genus zero for this purpose), we also show a wall-crossing formula. We also prove that in genus zero the invariants are always nonnegative and give a complete classification of the cases where they vanish.

Contents

1. Introduction

The goal of this paper is to study enumerative invariants of the (logarithmic) double ramification cycles, giving an overview of the existing landscape of results and defining new invariants using techniques from logarithmic and tropical geometry.

1.1. The double ramification cycle and its intersection numbers

Inside the moduli space $\mathcal{M}_{g,n}$ of smooth curves, the *double ramification locus* is a closed substack cut out by an equality of line bundles

$$\mathrm{DRL}_g(\mathbf{x}) = \left\{(C, p_1, \dots, p_n) : (\omega_C^{\log})^{\otimes k} \cong \mathcal{O}_C\left(\sum_{i=1}^n x_i p_i\right)\right\} \subseteq \mathcal{M}_{g,n}\,, \tag{1.1}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ is an integer vector with $|\mathbf{x}| = \sum_i x_i = k(2g-2+n)$. The (virtual) fundamental class of $\mathrm{DRL}_g(\mathbf{x})$ admits a natural extension

$$\mathrm{DR}_g(\mathbf{x}) \in \mathrm{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}), \tag{1.2}$$

in the Chow group of dimension $2g-3+n$ cycles on $\overline{\mathcal{M}}_{g,n}$, called the *double-ramification cycle (class)*.[1] For $k=0$, this class can be defined as the pushforward of the virtual fundamental class of the space $\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}$ of stable maps to *rubber* [GV05, Li01, Li02]. This is a compactification of the space of maps

$$f : (C, p_1, \dots, p_n) \to \mathbb{P}^1 \text{ with } f^*([0] - [\infty]) = \sum_{i=1}^n x_i p_i, \tag{1.3}$$

with ramification profile over $0, \infty$ given by the positive and negative parts of $\mathbf{x}$ (and taken modulo the $\mathbb{C}^*$-action on $\mathbb{P}^1$). For the history and general properties of the double ramification cycle, we refer the reader to [JPPZ17].

An established approach to extract intersection numbers from the cycle $\mathrm{DR}_g(\mathbf{x})$ is to calculate its *descendant invariants*

$$\int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(\mathbf{x}) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n}, \tag{1.4}$$

for a vector $\mathbf{e} = (e_1, \dots, e_n) \in (\mathbb{Z}_{\geq 0})^n$ of exponents with $|\mathbf{e}| = 2g-3+n$. An explicit formula for these numbers was calculated in [BSSZ15] for $k=0$ and in [CSS21] for arbitrary k, but with $\mathbf{e}$ restricted to vectors with only a single nonzero entry. A separate direction of study is the *double ramification hierarchy*, which studies intersection numbers of $\mathrm{DR}_g(\mathbf{x})$ against Hodge classes, a (partial) cohomological field theory and ψ-classes as above (see [Bur15, BR21]).

There are natural enumerative questions concerning the double ramification geometry which have no known expression via intersection numbers of $\mathrm{DR}_g(\mathbf{x})$ on $\overline{\mathcal{M}}_{g,n}$. A prominent example is the *double Hurwitz numbers* $\mathrm{H}_g(\mathbf{x})$ [GJV05]. These count covers f as in (1.3) with $b = 2g-3+n$ simple branch points at fixed positions in $\mathbb{P}^1$. The double Hurwitz numbers *can* be defined by intersection theory: the space $\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}$ of rubber maps has a natural branch morphism

$$\mathrm{br} : \overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}} \to [\mathrm{LM}(b)/S_b] \tag{1.5}$$

to a (stack quotient of a) *Losev-Manin space*, which remembers the position of the simple branch points and allows them to coincide away from $0, \infty$ [CM14]. Then we have

[1] Since k can be uniquely reconstructed from g, n and $\mathbf{x}$, we omit it from the notation.

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$$\mathrm{H}_g(\mathbf{x}) = \int_{[\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}]^{\mathrm{vir}}} \mathrm{br}^*[\mathrm{pt}]. \tag{1.6}$$

Since the branch morphism (1.5) does not factor through the forgetful morphism $F : \overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}} \to \overline{\mathcal{M}}_{g,n}$, there is no obvious way to express the intersection number (1.6) as a product of $\mathrm{DR}_g(\mathbf{x}) = F_*[\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}]^{\mathrm{vir}}$ with a class on $\overline{\mathcal{M}}_{g,n}$.

1.2. Logarithmic double ramification cycles and double Hurwitz numbers

The failure of $\overline{\mathcal{M}}_{g,n}$ in supporting a class that restricts to $\mathrm{br}^*[\mathrm{pt}]$ is part of a broader picture that emerged in recent years, and which can be summarized as follows:

$\overline{\mathcal{M}}_{g,n}$ is not the right ambient space for the double ramification cycle.

The first hint of this appeared in a construction of $\mathrm{DR}_g(\mathbf{x})$ [Hol19, MW20], which proceeds by constructing a cycle $\widehat{\mathrm{DR}}_g(\mathbf{x})$ on a *log blowup*[2] $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ and then obtains $\mathrm{DR}_g(\mathbf{x})$ as the pushforward of $\widehat{\mathrm{DR}}_g(\mathbf{x})$ under the map $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. This construction works for arbitrary $k \in \mathbb{Z}$ (compared to the one via stable maps to rubber when $k = 0$) and gives a natural lift of $\mathrm{DR}_g(\mathbf{x})$ to a log blowup of $\overline{\mathcal{M}}_{g,n}$. While the blowup $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ is not unique, the resulting cycle stabilizes on sufficiently fine blowups: given another (sufficiently fine) choice of blowup $\widetilde{\mathcal{M}}^{\mathbf{x}}_{g,n}$, the cycle $\widetilde{\mathrm{DR}}_g(\mathbf{x})$ constructed there pulls back to the same cycle as $\widehat{\mathrm{DR}}_g(\mathbf{x})$ on any blowup dominating $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ and $\widetilde{\mathcal{M}}^{\mathbf{x}}_{g,n}$. Thus, one obtains a well-defined element in the *logarithmic Chow ring*

$$\mathrm{logCH}^*(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}} \mathrm{CH}^*(\widehat{\mathcal{M}}), \tag{1.7}$$

defined as the direct limit of Chow rings of smooth log-blowups $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$, with maps given by pullback. The constructed lift

$$\mathrm{logDR}_g(\mathbf{x}) = [\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}, \widehat{\mathrm{DR}}_g(\mathbf{x})] \in \mathrm{logCH}^*(\overline{\mathcal{M}}_{g,n}) \tag{1.8}$$

is called the *logarithmic double-ramification cycle*.

One immediate advantage of $\mathrm{logDR}_g(\mathbf{x})$ compared to $\mathrm{DR}_g(\mathbf{x})$ is that we can recover the double Hurwitz numbers from $\mathrm{logDR}_g(\mathbf{x})$. Indeed, as shown in [CMR25, Theorem A], there exists a cycle $\mathrm{Br}_g(\mathbf{x}) \in \mathrm{logCH}^{2g-3+n}(\overline{\mathcal{M}}_{g.n})$ such that

$$\mathrm{H}_g(\mathbf{x}) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{logDR}_g(\mathbf{x}) \cdot \mathrm{Br}_g(\mathbf{x}). \tag{1.9}$$

To see where the cycle $\mathrm{Br}_g(\mathbf{x})$ comes from, recall from equation (1.6) that we want to pair the virtual fundamental class of $\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}$ with $\mathrm{br}^*[\mathrm{pt}]$. For a suitable choice of blowup $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, one can ensure that the map $\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}} \to \overline{\mathcal{M}}_{g,n}$ factors via an embedding ι into $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ as illustrated on the left-hand side of Figure 1 (see Proposition 2.3). Moreover, the pushforward of the virtual class of $\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}$ under ι gives the lifted double ramification cycle $\widehat{\mathrm{DR}}_g(\mathbf{x})$ (see, for example, [Hol19, Proposition 7.1]).

While the map br does not immediately factor through $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$, we can nevertheless get a proxy for the cycle $\mathrm{br}^*[\mathrm{pt}]$ using an auxiliary space: the stack Ex of *expansions* parameterizes chains of rational

[2] See e.g. [Bar19, Section 2] for a definition of log blowups. In the context of our paper, the reader can think of log blowups of $\overline{\mathcal{M}}_{g,n}$ as iterated blowups at smooth boundary strata and their strict transforms.

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$$
\begin{tikzcd}
 & \mathrm{br}^*[\mathrm{pt}] & {[\mathrm{pt}]} \arrow[l, mapsto, "\mathrm{br}^*"'] & {[T_b]} \arrow[l, mapsto, "F_b^*"'] \\
{[\widetilde{\overline{\mathcal{M}}}_{g,\mathbf{x}}]^{\mathrm{vir}}} \arrow[d, mapsto, "\iota_*"'] & \widetilde{\overline{\mathcal{M}}}_{g,\mathbf{x}} \arrow[r, "\mathrm{br}"] \arrow[d, "\iota"'] & {[\mathrm{LM}(b)/S_b]} \arrow[r, "F_b"] & \mathrm{Ex} \\
\widehat{\mathrm{DR}}_g(\mathbf{x}) \arrow[d, mapsto, "q_*"'] & \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \arrow[urr, bend right, "\mathrm{t}"] \arrow[d, "q"'] & & \\
\mathrm{DR}_g(\mathbf{x}) & \overline{\mathcal{M}}_{g,n} & &
\end{tikzcd}
$$

Figure 1. *Factoring the branch morphism to the stack* Ex *of expansions through the log blowup* $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$.

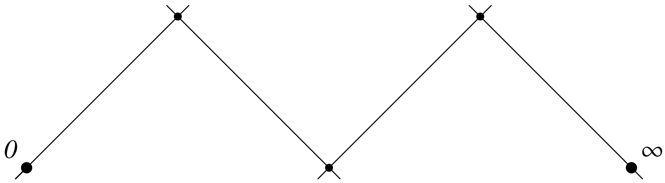

Figure 2. *A point in the stack of expansions* Ex.

curves with marked points $0, \infty$ at the opposite ends of the chain (see Figure 2 and [CMR25, Section 2.8] for a discussion).

For any $c \geq 0$, the stack Ex has a closed codimension c stratum T_c parameterizing chains of length at least c. Then for the map $F_b : [\mathrm{LM}(b)/S_b] \longrightarrow \mathrm{Ex}$ forgetting the $b = 2g - 3 + n$ unordered points, it is easy to see that the class $[T_b]$ of the codimension b stratum pulls back to the class of a point in the Losev-Manin space. However, by Proposition 2.3, the composition $F_b \circ \mathrm{br}$ factors through a map t as illustrated in Figure 1. Then defining $\mathrm{Br}_g(\mathbf{x}) = \mathrm{t}^*[T_b]$, we conclude

$$
\begin{aligned}
\mathrm{H}_g(\mathbf{x}) &= \int_{[\widetilde{\overline{\mathcal{M}}}_{g,\mathbf{x}}]^{\mathrm{vir}}} \mathrm{br}^*[\mathrm{pt}] = \int_{[\widetilde{\overline{\mathcal{M}}}_{g,\mathbf{x}}]^{\mathrm{vir}}} \mathrm{br}^* F_b^*[T_b] = \int_{\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}} (\iota_*[\widetilde{\overline{\mathcal{M}}}_{g,\mathbf{x}}]^{\mathrm{vir}}) \cdot \mathrm{t}^*[T_b] \\
&= \int_{\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}} \widehat{\mathrm{DR}}_g(\mathbf{x}) \cdot \mathrm{Br}_g(\mathbf{x}) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{logDR}_g(\mathbf{x}) \cdot \mathrm{Br}_g(\mathbf{x}) \,.
\end{aligned}
$$

1.3. Generalized double Hurwitz numbers

In the previous section, we saw a formula for the double Hurwitz numbers in terms of intersection numbers of $\mathrm{logDR}_g(\mathbf{x})$. Now, turning the story around, one may *define* a generalization of these Hurwitz numbers by pairing $\mathrm{logDR}_g(\mathbf{x})$ with more general cycle classes.

The first step in this direction was already discussed in [CMR25]: while the double Hurwitz numbers require the entries of the ramification profile $\mathbf{x}$ to sum to zero (corresponding to setting $k = 0$), the formula (1.9) can be extended to arbitrary values of $k \in \mathbb{Z}$. More precisely, by Proposition 2.3, we have the commutative diagram depicted in Figure 3.

Thus, the definition $\mathrm{Br}_g(\mathbf{x}) = \mathrm{t}^*[T_b]$ still makes sense, and the paper [CMR25] defines the *k-leaky double Hurwitz numbers*[3]

$$
\mathrm{H}_g(\mathbf{x}) = \mathrm{H}_g(\mathbf{x}, k) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{logDR}_g(\mathbf{x}) \cdot \mathrm{Br}_g(\mathbf{x}) \,. \tag{1.10}
$$

[3]Again, since k is determined by g and $\mathbf{x}$, we will in general omit the parameter k in the notation for generalized double Hurwitz numbers.

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$$\begin{tikzcd}
{[\overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}}]^{\mathrm{vir}}} \arrow[d, mapsto, "\iota_*"'] & \overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}} \arrow[r, "\mathrm{b}"] \arrow[d, "\iota"'] & \mathrm{Ex} \\
\widehat{\mathrm{DR}}_g(\mathbf{x}) \arrow[d, mapsto, "q_*"'] & \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \arrow[d, "q"'] \arrow[ur, bend right, "\mathrm{t}"'] & \\
\mathrm{DR}_g(\mathbf{x}) & \overline{\mathcal{M}}_{g,n} &
\end{tikzcd}$$

Figure 3. *Maps to the stack of expansions.*

While the enumerative meaning of the k-leaky Hurwitz numbers is less clear, they share many properties of classical double-Hurwitz numbers ([CMR25, Theorem B and Section 5]):

- they are nonnegative rational numbers, piecewise polynomial in $\mathbf{x}$ and given by polynomials of degree $4g - 3 + n$ on the chambers of polynomiality,
- there is a tropical graph sum formula calculating them as a weighted count of tropical covers of the real line,
- they arise as matrix elements for powers of the *k-leaky cut and join operator* on the Fock space.

1.4. Results

The main object of study of the present paper generalizes k-leaky Hurwitz numbers by introducing descendant insertions.

Definition 1.1. Given a vector $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ with $0 \leq |\mathbf{e}| \leq 2g - 3 + n$, let $c = 2g - 3 + n - |e|$ and consider the *branch class*

$$\mathrm{Br}^c_g(\mathbf{x}) = \mathrm{t}^*[T_c] \in \mathrm{logCH}^c(\overline{\mathcal{M}}_{g,n}). \tag{1.11}$$

Then we define the *k-leaky double Hurwitz descendants*

$$\mathrm{H}_g(\mathbf{x}, \mathbf{e}) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{logDR}_g(\mathbf{x}) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n} \cdot \mathrm{Br}^c_g(\mathbf{x}). \tag{1.12}$$

For $\mathbf{e} = 0$, we recover the k-leaky double Hurwitz numbers, whereas for $|e| = 2g - 3 + n$ by the projection formula, we obtain the descendant invariants (1.4) of the double ramification cycle. Thus, these numbers form a natural interpolation between two previously studied enumerative invariants.

Our first result establishes a correspondence theorem between k-leaky double Hurwitz descendants numbers and certain tropical counts.

Theorem 1.2. *The k-leaky double Hurwitz descendants equal the count of tropical k-leaky covers satisfying Psi-conditions (Definition 3.5):*

$$\mathrm{H}^{\mathrm{trop}}_g(\mathbf{x}, \mathbf{e}) = \mathrm{H}_g(\mathbf{x}, \mathbf{e}).$$

The graphs that are being counted are leaky covers as in [CMR25] (i.e., piecewise linear maps from tropical curves to the real line satifying the leaky condition (2.2)). The valence, and genus at a given vertex, depends on the degree of the descendant insertions at its incident legs (3.1). Each graph is counted with a multiplicity which is a product of various local factors: besides automorphism, and edge factors, there are now multiplicities assigned to each vertex, which consist of double-ramification descendants (1.4) with no appearence of the branch class (Definition 3.3). The key ingredient in proving this result is expressing the pushforward to $\overline{\mathcal{M}}_{g,n}$ of the product $\mathrm{logDR}_g(\mathbf{x}) \cdot \mathrm{Br}^c_g(\mathbf{x})$ as a linear combination of boundary classes that are described by dual graphs decorated with vertex terms given by smaller-dimensional double ramification cycles.

The correspondence theorem provides a combinatorial approach to the computation of k-leaky double Hurwitz descendants that allows to observe and characterize their structural properties.

Theorem 1.3 (Piecewise polynomiality, see Theorem 5.1). *The k-leaky double Hurwitz descendant numbers $\mathrm{H}_g(\mathbf{x}, \mathbf{e})$ are piecewise polynomial in k and $\mathbf{x}$ of degree $4g-3+n-|\mathbf{e}|$.*

As in the case for double Hurwitz numbers [CJM11], the wall-crossing formulas are modular, in the sense that they are expressed as sums of products of descendant leaky numbers with smaller discrete invariants. We do not work out general wall-crossing formulas in this paper, as the combinatorial complexity seems to outweight the benefits, but we do present the result in genus zero, where the formulas are succint and elegant.

Theorem 1.4 (Wall-crossing in genus zero). *Fix a wall $\delta := \sum_{i\in I} x_i - k\cdot(\sharp I - 1) = 0$ and denote by P_1^δ the polynomial expression for $\mathrm{H}_0^{\mathrm{trop}}(\mathbf{x}, \mathbf{e})$ we have on one side of the wall and by P_2^δ the expression on the other side of the wall.*

Define $\mathbf{e}_I = (e_i)_{i\in I}$. Let $r = n-2-|\mathbf{e}|$, $r_1 = \sharp I - 1 - |\mathbf{e}_I|$ and $r_2 = \sharp I^c - 1 - |\mathbf{e}_{I^c}|$.

Then the wall-crossing (i.e., the difference between the two polynomial expressions on both sides of the wall) equals

$$P_1^\delta - P_2^\delta = \binom{r}{r_1, r_2} \cdot \delta \cdot \mathrm{H}_0(\mathbf{x}_I \cup \{\delta\}, \mathbf{e}_I) \cdot \mathrm{H}_0(\mathbf{x}_{I^c} \cup \{-\delta\}, \mathbf{e}_{I^c}).$$

The tropical correspondence formula in Theorem 1.2 reduces the computation of all descendant leaky double Hurwitz numbers to the case (1.4) where the branch insertion is trivial. We seek to further improve the situation by shrinking the class of initial conditions needed. We observe that any descendant insertion is equivalent to a linear combination of boundary divisors together with the class κ_1, thanks to a second splitting formula for the double ramification cycle (essentially proven in [CSS21]). It shows that the cycles

$$x_i\psi_i \cdot \mathrm{DR}_g(\mathbf{x}) \text{ and } \frac{k}{2g-2+n}\kappa_1 \cdot \mathrm{DR}_g(\mathbf{x})$$

differ by a sum over graphs with two vertices carrying DR-cycles (see Proposition 4.2). Using this relation, we can calculate the intersection number (1.4) by exchanging one ψ-class after the other for either a multiple of κ_1 or a term supported in the boundary. After performing this procedure $2g-3+n$ times, we are left with a graph sum with vertex terms only involving powers of the class κ_1.

To state a formal recursion, we introduce the notation

$$\mathrm{H}_g(\mathbf{x}, \mathbf{e}, f) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(\mathbf{x}) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n} \cdot \kappa_1^f \tag{1.13}$$

mixing ψ-insertions with a power of κ_1.

Theorem 1.5. *The numbers $\mathrm{H}_g(\mathbf{x}, \mathbf{e}, f)$ can be recursively computed (see (4.6)) from the intersection numbers*

$$\mathrm{H}_{g'}(\mathbf{x}', \mathbf{0}, 2g'-3+n') = \int_{\overline{\mathcal{M}}_{g',n'}} (k\kappa_1)^{2g'-3+n'} \cdot \mathrm{DR}_{g'}(\mathbf{x}') \,. \tag{1.14}$$

Combined with the tropical graph sum formula from Theorem 1.2, this uniquely determines the k-leaky double Hurwitz descendants from the initial data of the numbers (1.14). These intersection numbers with powers of κ_1 in turn have been characterized explicitly in upcoming work [Sau] by Sauvaget. In fact, Sauvaget's paper will give a full formula for the integrals $H_g(\mathbf{x}, \mathbf{e})$ with $|e| = 2g-3+n$

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Table 1. *The landscape of known results on double Hurwitz descendants* $\mathrm{H}_g(\mathbf{x}, \mathbf{e})$. *Here, [CMS24] denotes the present paper.*

			$\mathrm{Br}_g(\mathbf{x})$	$\mathrm{Br}_g^c(\mathbf{x}) \cdot \psi^{\mathbf{e}}$		$\psi^{\mathbf{e}}$
				$\mathbf{e} = e_1\delta_1$	$\mathbf{e}$ arbitrary	
$g = 0$	tot. ram.	$k = 0$	formula [GJ92]	[CMS]		classical formula $\binom{n-2}{\mathbf{e}}$
		$k > 0$	formula [CMS]	[CMS]		
	arb. ram.	$k = 0$	wall-crossing [SSV08]			
		$k > 0$			wall-crossing [CMS24]	
$g > 0$	tot. ram.	$k = 0$	one-part double Hurwitz numbers [GJV05]	[CMS]		see below
		$k > 0$				see below
	arb. ram.	$k = 0$	double Hurwitz numbers wall-crossing [CJM11]			generating function [BSSZ15]
		$k > 0$	k-leaky double Hurwitz numbers piecew. polynomiality [CMR25]		piecewise polynomiality [CMS24]	generating function [CSS21] for $\mathbf{e} = e_1\delta_1$, [Sau] for all $\mathbf{e}$

appearing as the vertex multiplicities of our tropical graph sum formula. After the initial publication of the paper, a second recursive procedure for calculating the numbers (1.4), reducing them to a formula given in [CSS21, Theorem 1.1], was pointed out to us by the referee (see Remark 4.5). We thank them for sharing the suggestion!

Lastly, we turn our attention to unexpected vanishings of leaky descendant double Hurwitz numbers. Thinking of such enumerative invariants as generalization of Hurwitz numbers, this problem is analogous to the yet unsolved question in Hurwitz theory of characterizing what discrete data satisfying the Riemann-Hurwitz condition does not yield any corresponding covering. In genus 0, we compute leaky double Hurwitz descendants as weighted graph sums where each graph carries a nonnegative contribution. (This is not true in higher genus; see Example 6.1.) This gives a blunt but powerful tool to show nonvanishing, and in fact positivity: it suffices to exhibit a single graph with a nonzero contribution.

If $k = 0$, an easy argument shows that $H_0(\mathbf{x}, \mathbf{e}) > 0$ unless $\mathbf{x} = 0$ and $n > |\mathbf{e}| + 3$; see Remark 6.2. For the more interesting case $k \neq 0$, we are able to witness and classify some exotic vanishing behavior, as summarized in the following theorem.

Theorem 1.6. *Let* $g = 0$ *and* $k \neq 0$. *Let* $|\mathbf{x}| = (n-2)k$ *and* $0 \leq |\mathbf{e}| \leq n-3$.

The k-leaky descendant $H_0(\mathbf{x}, \mathbf{e})$ *vanishes if and only if k is even,* $x_i = m_i \cdot \frac{k}{2}$ *for* $m_i \in \mathbb{N}_{>0}$, *and for every subset* $I \subset \{1, \ldots, n\}$, *we have*

$$\sum_{i \in I} e_i < \sum_{i \in I} m_i - |I| + 1.$$

In all other cases, we have $H_0(\mathbf{x}, \mathbf{e}) > 0$.

1.5. Current landscape and future directions

The k-leaky descendant double Hurwitz numbers are a large family of enumerative invariants rich in combinatorial structure and closely tied to the geometry of double ramification cycles, and have been studied and progressively generalized over the last few decades. In Table 1, we list the known results on the numbers $\mathrm{H}_g(\mathbf{x}, \mathbf{e})$.

Depending on the parameters $g, \mathbf{x}, \mathbf{e}$ they have been characterized to different extents, ranging from structural properties to explicit formulas:

- *explicit formulas* for (non-leaky) double Hurwitz numbers have been found mostly in the special case where all but one entry of $\mathbf{e}$ have the same sign (the case of total ramification; see [GJ92, GJV05]). In the forthcoming paper [CMS], we extend these formulas to k-leaky double Hurwitz numbers and even to the case of ψ-insertions at the marking of total ramification;

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- for k-leaky double ramification descendants (without branch conditions), explicit *generating functions* have been found in [BSSZ15] (for $k = 0$) and [CSS21] (for $k > 0$ and insertions given by a power of a single ψ-class). The forthcoming paper [Sau] proves a full formula for the generating function;
- the *piecewise polynomiality* of $\mathrm{H}_g(\mathbf{x}, \mathbf{0})$ and *wall-crossing formulas* for different values of $\mathbf{x}$ were found in [SSV08, CJM11] for $k = 0$ and [CMR25] for $k > 0$. In our paper, we show the piecewise polynomiality in full generality and work out the wall-crossing structure for $g = 0$.

As illustrated in Table 1, there is lots of room for progress. In particular, in the total ramification case, it seems feasible to hope for an explicit formula in genus 0, and an approach via generating functions in arbitrary genus.

A second direction of study is the enumerative interpretation of k-leaky double Hurwitz numbers. As explained before, for $k = 0$, the number $\mathrm{H}_g(\mathbf{x}, \mathbf{0})$ counts covers of the projective line with fixed ramification profiles over $0, \infty$ and simple ramification over $2g - 2 + n$ other points. It is natural to expect that for arbitrary k, the number $\mathrm{H}_g(\mathbf{x}, \mathbf{0})$ could be a count of k-differentials with given zero and pole-orders together with $2g - 2 + n$ further conditions reducing the dimension to zero.

One approach in this direction is studied by [GT22, BR24, CP23] in the case of $g = 0$, $k = 1$ and total ramification. More precisely, consider a vector

$$\mathbf{x} = (d, -a_1, \ldots, -a_n) \in \mathbb{Z}^{n+1} \text{ (with } d, a_1, \ldots, a_n > 0)$$

such that $|\mathbf{x}| = n-1$. In the notation of our paper, the authors study the moduli space $\mathcal{H}_0(\mathbf{x})$ parameterizing tuples

$$(C, q, p_1, \ldots, p_n, \eta)$$

of a smooth genus 0 curve C with $n + 1$ distinct marked points and a $k = 1$ differentials η on C with zeros and poles of orders $x_i - 1$ at the marked points. Given a fixed vector $\vec{r} = (r_1, \ldots, r_n) \in \mathbb{C}^n \setminus \{0\}$ satisfying $\sum r_i = 0$, the subset $\mathcal{H}_0(\mathbf{x})^{\vec{r}} \subseteq \mathcal{H}_0(\mathbf{x})$ where the differential η has residue r_i at p_i is a finite set. The papers above count the number of points in dependence of $\vec{r}$. As an example, for a general vector $\vec{r}$, the count is given by the formula

$$|\mathcal{H}_0(\mathbf{x})^{\vec{r}}| = (d - 1) \cdot (d - 2) \cdots (d - (n - 2)).$$

However, in [CMS], we show that the k-leaky double Hurwitz number is given by

$$\mathrm{H}_0(\mathbf{x}) = (n - 1)!(d - \frac{1}{2}) \cdot (d - \frac{2}{2}) \cdots (d - \frac{n-2}{2}).$$

While the formulas do not agree, they bear a strong resemblance and share structural properties (such as being a polynomial of degree $n - 2$ in the entries of $\mathbf{x}$). It seems interesting to explore whether the setup in [GT22, BR24, CP23] can be modified to give an enumerative interpretation for the number $\mathrm{H}_0(\mathbf{x})$. This could also show a path to extending their counting problem to higher genus, where fixing the residues at poles no longer cuts the dimension to zero.

2. Background

2.1. Tropical curve counts

An *abstract tropical curve* is a connected metric graph Γ with unbounded edges (called *ends*) which have infinite length, together with a genus function $g : \Gamma \to \mathbb{Z}_{\geq 0}$ with finite support. Locally around a point p, Γ is homeomorphic to a star with r halfrays. The number r is called the *valence* of the point p and denoted by $\mathrm{val}(p)$. The *minimal vertex set* of Γ is defined to be the points where the genus function is nonzero, together with points of valence different from 2. The vertices of valence greater than 1 are called *inner vertices*. Besides *edges*, we introduce the notion of *flags* of Γ. A flag is a pair (v, e) of a vertex v and an edge e incident to it ($v \in \partial e$). Edges that are not ends are required to have finite length and are referred to as *bounded* or *internal* edges.

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A *marked tropical curve* is a tropical curve whose leaves are labeled. An isomorphism of a tropical curve is an isometry respecting the leaf markings and the genus function. The *genus* of a tropical curve Γ is given by

$$g(\Gamma) = h_1(\Gamma) + \sum_{p\in\Gamma} g(p).$$

The *combinatorial type* is the equivalence class of tropical curves obtained by identifying any two tropical curves which differ only by edge lengths.

We want to examine covers of $\mathbb{R}$ by graphs up to additive translation, and equip $\mathbb{R}$ with a polyhedral subdivision to ensure the result is a map of metric graphs (see, for example, Section 5.4 and Figure 3 in [MW20]). A *metric line graph* is any metric graph obtained from a polyhedral subdivision of $\mathbb{R}$. The metric line graph determines the polyhedral subdivision up to translation. We fix an orientation of a metric line graph going from left to right (i.e., from negative values in $\mathbb{R}$ to positive values).

Definition 2.1 (Leaky cover, [CMR25]). Let $\pi : \Gamma \to T$ be a surjective map of metric graphs where T is a metric line graph. We require that π is piecewise integer affine linear; the slope of π on a flag or edge e is a positive integer called the *expansion factor* $\omega(e) \in \mathbb{N}_{\geq 0}$.

For a vertex $v \in \Gamma$, the *left (resp. right) degree of π at v* is defined as follows. Let f_l be the flag of $\pi(v)$ in T pointing to the left and f_r the flag pointing to the right. Add the expansion factors of all flags f adjacent to v that map to f_l (resp. f_r):

$$d_v^l = \sum_{f\mapsto f_l} \omega(f), \quad d_v^r = \sum_{f\mapsto f_r} \omega(f). \tag{2.1}$$

We say that the k-leaky condition is satisfied at $v \in \Gamma$ if

$$d_v^l - d_v^r = k(2g(v) - 2 + \mathrm{val}(v)). \tag{2.2}$$

We impose a stability condition: $\Gamma \to T$ is called stable if the preimage of every vertex of T contains a vertex of Γ in its preimage which is of genus greater than 0 or valence greater than 2.

Furthermore, we stabilize the source tropical curve further by passing to its minimal vertex set (containing only the points where the genus function is nonzero, together with points of valence different from 2). The outcome $\pi : \Gamma \to T$ is called a *k-leaky cover*.

By our stabilization procedure, we lose the property that the cover is a map of graphs; however, this vertex structure is relevant to determine valencies correctly for the purpose of Psi-conditions.

Definition 2.2 (Left and right degree). The *left (resp. right) degree* of a leaky cover is the tuple of expansion factors of its ends mapping asymptotically to $-\infty$ (resp. $+\infty$). The tuple is indexed by the labels of the ends mapping to $-\infty$ (resp. $+\infty$). When the order imposed by the labels of the ends plays no role, we drop the information and treat the left and right degree only as a multiset.

By convention, we denote the left degree by $\mathbf{x}^+$ and the right degree by $\mathbf{x}^-$. In the right degree, we use negative signs for the expansion factors, in the left degree positive signs. We also merge the two to one vector which we denote $\mathbf{x} = (x_1, \ldots, x_n)$ called the *degree*. The labeling of the ends plays a role: the expansion factor of the end with the label i is x_i. In $\mathbf{x}$, we distinguish the expansion factors of the left ends from those of the right ends by their sign. A Euler characteristic calculation, combined with the leaky cover condition, shows that

$$\sum_{i=1}^{n} x_i = k \cdot (2g - 2 + n),$$

where g denotes the genus of Γ.

An automorphism of a leaky cover is an automorphism of Γ compatible with π.

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2.2. Insertions from the stack of expansions

The stack Ex of expansions has been studied extensively in the context of Gromov-Witten theory (see [Li01, Li02, ACFW13]). Recall from Section 1.2 that it parameterizes chains of rational curves with markings $0, \infty$ at the opposite ends of the chain. It has a locally closed decomposition

$$\mathrm{Ex} = \mathrm{pt} \sqcup B\mathbb{G}_m \sqcup B\mathbb{G}_m^2 \sqcup \dots , \tag{2.3}$$

where the locally closed stratum $B\mathbb{G}_m^r \subseteq \mathrm{Ex}$ represents the locus where the chain has $r+1$ components.[4]

One way to define Ex is to see it as the Artin fan associated to the cone stack tEx of *tropical expansions*. For the original definition and background on Artin fans and cone stacks, we refer the reader to Sections 2.7 and 2.8 of [CMR25] and the references therein. Roughly, a (combinatorial) cone stack is a collection of rational polyhedral cones with a system of face morphisms between them (see [CCUW20, Definition 2.15]). The cone stack tEx has one object $\mathbb{R}^r_{\geq 0}$ for each $r \geq 0$, and the face morphisms $\mathbb{R}^r_{\geq 0} \to \mathbb{R}^s_{\geq 0}$ are all coordinate inclusions obtained by inserting $s - r$ zero coordinates (and leaving the orders of the other coordinates fixed). For example, for $r = 2, s = 4$, these are the six morphisms:

$$(x, y) \mapsto (x, y, 0, 0), (x, 0, y, 0), (x, 0, 0, y), (0, x, y, 0), (0, x, 0, y), (0, 0, x, y) .$$

The resulting cone stack can be interpreted as parameterizing subdivisions of the real line $\mathbb{R}$ at $r+1$ points, up to translation. The coordinates of the cone $\mathbb{R}^r_{\geq 0}$ parameterize the r distances between neighboring vertices in the subdivision.

The stack Ex can then be defined as the unique Artin fan $\mathrm{Ex} = \mathcal{A}_{\mathrm{tEx}}$ corresponding to the cone stack tEx above. By [MPS23, Theorem 14], the Chow ring of Ex can be seen to be isomorphic to the ring $\mathrm{sPP}^*(\mathrm{tEx})$ via an algebra isomorphism

$$\Phi : \mathrm{sPP}^*(\mathrm{tEx}) \to \mathrm{CH}^*(\mathrm{Ex}) . \tag{2.4}$$

Note that this Chow ring has previously been computed independently by Oesinghaus [Oes19] – the above gives a new way to obtain this result. Using [HMP+22, Lemma 54], one can verify that the fundamental class of the closure $T_c = \overline{B\mathbb{G}_m^c}$ of the codimension c stratum is precisely given as

$$[T_c] = \Phi(\varphi_c) \text{ with } \varphi_c \in \mathrm{sPP}^c(\mathrm{tEx}) \text{ given by } \varphi_c = \sum_{I \subseteq \{1,\dots,r\}, |I|=c} \prod_{i \in I} x_i \text{ on } \mathbb{R}^r_{\geq 0} .$$

Indeed, the formula φ_c is easily seen to define a strict piecewise polynomial. On the dimension c cone $\mathbb{R}^c_{\geq 0}$ in tEx, it is given by the product $\prod_{i=1}^c x_i$ of the coordinate functions. Thus, it maps to the fundamental class of the associated strata closure T_c under Φ.

In Definition 1.1, we defined the branch class $\mathrm{Br}^c_g(\mathbf{x})$ as a pull-back of a codimension c class on Ex under a map $\mathrm{t} : \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \mathrm{Ex}$. We start by giving a concrete description of this map t and its claimed properties from the introduction.

To define it, let $\widetilde{\mathcal{M}}^{\mathbf{x}}_{g,n}$ be the blowup of $\overline{\mathcal{M}}_{g,n}$ associated to the vector $\mathbf{x}$ and a small nondegenerate stability condition θ (see [HMP+22, Section 4]). Its tropicalization $\widetilde{\Sigma}^{\mathbf{x}}_{g,n}$ has been described in [HMP+22, Section 4.2.2]. Its cones are indexed by tuples $(\widehat{\Gamma}, D, I)$, where $\widehat{\Gamma}$ is a quasi-stable graph, D is a θ-stable divisor on $\widehat{\Gamma}$ and I is an acyclic flow on $\widehat{\Gamma}$ with

$$\mathrm{div}(I) = \underline{\deg}((\omega^{\log})^k(-\sum a_i x_i)) - D .$$

[4]A priori one would expect this object to have stabilizer group $\mathbb{G}_m^{r+1}$, but we rigidify by the simultaneous standard action of $\mathbb{G}_m$ on all components simultaneously. On the tropical side (see below), this corresponds to considering subdivisions of the line $\mathbb{R}$ up to translation.

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The cone $\sigma_{(\widehat{\Gamma},D,I)}$ associated to this tuple parameterizes tropical covers $\pi : \widehat{\Gamma} \to \mathbb{R}$ from (a tropical curve with underlying graph) $\widehat{\Gamma}$ to $\mathbb{R}$ having slope $I(e)$ on each edge e of $\widehat{\Gamma}$. Inside this cone, there are codimension 1 walls defined by the condition that two vertices of $\widehat{\Gamma}$ have the same image in $\mathbb{R}$. Let $\widehat{\Sigma}^{\mathbf{x}}_{g,n} \to \widetilde{\Sigma}^{\mathbf{x}}_{g,n}$ be the subdivision obtained by introducing all of these walls (it is straightforward to check that the walls are compatible under face maps and thus do define a subdivision). We denote by $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \widetilde{\mathcal{M}}^{\mathbf{x}}_{g,n}$ the log blowup associated to this subdivision.[5]

Proposition 2.3. *The space $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ admits a map $\mathrm{t} : \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \mathrm{Ex}$ to the stack of expansions such that for the inclusion $\iota : \overline{\mathcal{M}}^{\sim}_{g,\mathbf{x}} \to \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ of the log double-ramification locus, we have*

- *the diagram in Figure 3 commutes,*
- *the tropicalization of the composition $\mathrm{t} \circ \iota$ is the map $\mathrm{DR}^{\mathrm{trop}}_g(\mathbf{x}) \to \mathrm{tEx}$ sending a tropical cover $\Gamma \to \mathbb{R}$ to the induced subdivision of its target $\mathbb{R}$ at the images of vertices of Γ.*

Proof. Given $\pi \in \sigma_{(\widehat{\Gamma},D,I)}$, taking the image of the stable vertices of $\widehat{\Gamma}$ in $\mathbb{R}$ defines a subdivision of the real line (i.e. an element of the tropicalization tEx). By taking the subdivision $\widehat{\Sigma}^{\mathbf{x}}_{g,n} \to \widetilde{\Sigma}^{\mathbf{x}}_{g,n}$, one ensures that this operation defines a morphism

$$\widehat{\Sigma}^{\mathbf{x}}_{g,n} \to \mathrm{tEx} \tag{2.5}$$

of cone stacks. Since $\mathrm{Ex} = \mathcal{A}_{\mathrm{tEx}}$ is its own Artin fan, we can define the map

$$\mathrm{t} : \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \mathcal{A}_{\widehat{\Sigma}^{\mathbf{x}}_{g,n}} \to \mathcal{A}_{\mathrm{tEx}} = \mathrm{Ex}$$

as the composition of the map from $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ to its Artin fan, with the morphism of Artin fans induced from the cone stack map (2.5). The tropicalization $\mathrm{DR}^{\mathrm{trop}}_g(\mathbf{x})$ is the sub-complex of $\widehat{\Sigma}^{\mathbf{x}}_{g,n}$ where the curve $\widehat{\Gamma}$ is stable, and on there, the map to tEx continues to record the edge lengths of the subdivided target $\mathbb{R}$. This shows the second bullet point, and for the first, we simply observe that two maps to Ex are equal if and only if their tropicalizations coincide (again, since Ex is its own Artin fan). □

In our paper, the only type of insertion from $\mathrm{CH}_*(\mathrm{Ex})$ that we consider is the fundamental class $[T_c] \in \mathrm{CH}^c(\mathrm{Ex})$ of the codimension c boundary stratum of Ex. Using a variant of Ex where we do not rigidify by the simultaneous action of $\mathbb{G}_m$ on all components (as in Footnote 4), further natural insertions would be the cotangent line classes Ψ_0, Ψ_∞ of the expanded target at $0, \infty$. For $\pi : \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, the blowup on which $\mathrm{logDR}_g(\mathbf{x})$ is supported, a description for the class

$$\pi_*\Big(\mathrm{logDR}_g(\mathbf{x}) \cdot \mathrm{t}^*\Psi_\infty^u\Big) \in \mathrm{CH}^{g+u}(\overline{\mathcal{M}}_{g,n})$$

was proposed in [CGH+22, Conjecture 1.4], and recently proven in [CH24]. This formula would allow to calculate the intersection numbers of $\mathrm{logDR}_g(\mathbf{x})$ against both powers of Ψ_∞ and further ψ-classes in examples. However, for now, we restrict our attention to the insertions $[T_c]$ mentioned above.

3. Tropical leaky descendants

Definition 3.1 (Psi-conditions for leaky covers). Let $g, n \geq 0$ such that $2g - 2 + n > 0$ and consider vectors $\mathbf{x} \in \mathbb{Z}^n$ such that $|\mathbf{x}| = k(2g - 2 + n)$ for some $k \in \mathbb{Z}$. Let $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ such that $0 \leq |\mathbf{e}| \leq 2g - 3 + n$.

[5] This modification is related to the distinction between the spaces $\mathbf{Div}_{g,x}$ and $\mathbf{Rub}_{g,x}$ in [MW20]: we have that $\widetilde{\mathcal{M}}^{\mathbf{x}}_{g,n} \subseteq \mathbf{Div}_{g,x}$ is an open substack, whereas $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ is *almost* an open substack of $\mathbf{Rub}_{g,x}$. The discrepancy between the latter two stems from the fact that we do not modify the integral structure inherited from the moduli space of tropical curves since this will make our lives easier in the proof of Theorem 3.7 below.

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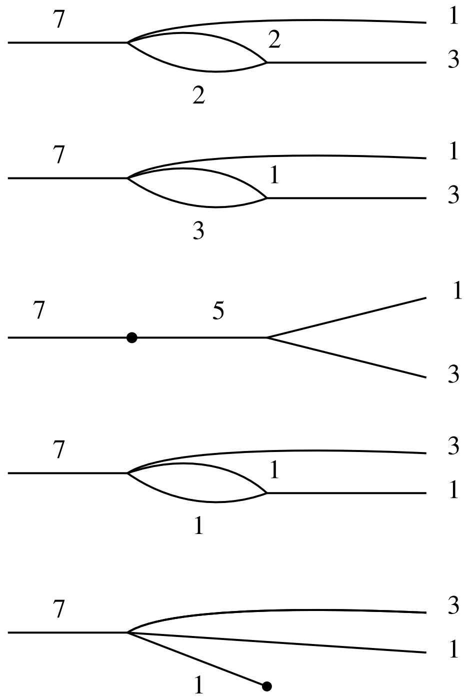

Figure 4. *Five* 1*-leaky tropical covers of genus* 1 *and degree* $\mathbf{x} = (7, -3, -1)$ *satisfying the Psi-conditions* $\mathbf{e} = (1, 0, 0)$*. The vertices marked with a dot are vertices of genus* 1*. All other vertices are of genus* 0*. All five pictures cover a line graph T with* 2 *vertices. We did not specify lengths in the picture, as the lengths in the source graph* Γ *are determined by the expansion factors and the lengths in T.*

Let $\pi : \Gamma \longrightarrow T$ be a k-leaky cover. For a vertex v, let $I_v \subset \{1, \ldots, n\}$ be the subset of ends adjacent to v after passing to the minimal vertex set of Γ.

We say that $\pi : \Gamma \longrightarrow T$ *satisfies the Psi-conditions* $\mathbf{e}$ if for all vertices v of Γ,

$$\mathrm{val}(v) = \sum_{i \in I_v} e_i + 3 - 2g(v). \tag{3.1}$$

Example 3.2. For an example of three 1-leaky tropical covers of genus 1 and degree $\mathbf{x} = (7, -3, -1)$ satisfying the Psi-conditions $\mathbf{e} = (1, 0, 0)$, see Figure 4.

Definition 3.3 (Vertex multiplicities). Let $\pi : \Gamma \longrightarrow T$ be a k-leaky cover satisfying the Psi-conditions $\mathbf{e}$. For a vertex v, let $I_v \subset \{1, \ldots, n\}$ be the subset of ends adjacent to v after passing to the minimal vertex set of Γ. Let $\mathbf{x}(v)$ denote the vector containing the (left and right) local degree of v, and let $g(v)$ denote the genus of v.

We define the *vertex multiplicity* to be

$$\mathrm{mult}_v := \int_{\overline{M}_{g(v),\mathrm{val}(v)}} \mathrm{DR}_{g(v)}(\mathbf{x}(v)) \cdot \prod_{i \in I_v} \psi_i^{e_i}$$

Example 3.4. For the 3-valent genus 0 vertices in the covers of Figure 4, the vertex multiplicity is 1. This is true because in genus 0, DR equals $\overline{\mathcal{M}}_{0,3}$ which is just a point. For the 4-valent vertices of genus 0, it is still true that DR equals $\overline{\mathcal{M}}_{0,4}$, but now we take the integral over ψ_1. This is also just a point, so again, these vertex multiplicities are 1. The genus 1 vertex may be evaluated using the software `admcycles` [DSvZ21] to obtain $\int_{\overline{\mathcal{M}}_{1,2}} \mathrm{DR}_1(7, -5) \cdot \psi_1 = 35/24$.

Definition 3.5 (Count of k-leaky covers satisfying Psi-conditions). Let $g, n \geq 0$ such that $2g-2+n > 0$ and consider vectors $\mathbf{x} \in \mathbb{Z}^n$ such that $|\mathbf{x}| = k(2g-2+n)$ for some $k \in \mathbb{Z}$. Let $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ such that $0 \leq |\mathbf{e}| \leq 2g-3+n$, and $c = 2g-3+n-|\mathbf{e}| \geq 0$.

We define

$$\mathrm{H}_g^{\mathrm{trop}}(\mathbf{x}, \mathbf{e}) = \sum_{\pi} \mathrm{mult}(\pi) \cdot \prod_{v} \mathrm{mult}_v, \tag{3.2}$$

where

- $\pi : \Gamma \to T$ ranges among all leaky covers of degree $\mathbf{x}$ and genus g (Definition 2.1) and satisfying the Psi-conditions $\mathbf{e}$ (Definition 3.1); we require that every vertex of T has precisely one vertex in its preimage[6];
- the multiplicity

$$\mathrm{mult}(\pi) = \frac{1}{|\mathrm{Aut}(\pi)|} \cdot \prod_{e} \omega(e) \in \mathbb{Q} \tag{3.3}$$

  is the product of the expansion factors at the bounded edges of Γ (according to its minimal vertex set), weighted by the number of automorphisms of π;
- the product goes over the set of vertices of Γ and mult_v is as in Definition 3.3.

Example 3.6. Fix the leaking 1, $n = 3$ ends, genus 1, degree $\mathbf{x} = (7, -3, -1)$ and the Psi-conditions $\mathbf{e} = (1, 0, 0)$. Then the covers we have to consider are depicted in Figure 4. For these three 1-leaky covers, we obtain the following multiplicities, using Example 3.4 discussing vertex multiplicities:

π_i	$1/\lvert\mathrm{Aut}(\pi_i)$	$\prod_e \omega(e)$	mult_v	$\mathrm{mult}(\pi_i)$
π_1	$1/2$	4	$(1, 1)$	2
π_2	1	3	$(1, 1)$	3
π_3	1	5	$(35/24, 1)$	$175/24$
π_4	$1/2$	1	$(1, 1)$	$1/2$
π_5	1	1	$(1, -1/24)$	$-1/24$

In total, we obtain

$$H_1^{\mathrm{trop}}(\mathbf{x}, \mathbf{e}) = 2 + 3 + \frac{175}{24} + \frac{1}{2} - \frac{1}{24} = \frac{51}{4}.$$

Theorem 3.7 (Correspondence Theorem for leaky covers with Psi-conditions). *Let $g, n \geq 0$ such that $2g-2+n > 0$ and consider vectors $\mathbf{x} \in \mathbb{Z}^n$ such that $|\mathbf{x}| = k(2g-2+n)$ for some $k \in \mathbb{Z}$. Let $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ such that $0 \leq |\mathbf{e}| \leq 2g-3+n$.*

Then the k-leaky double Hurwitz descendant (defined in 1.1) equals the count of tropical k-leaky covers satisfying Psi-conditions (defined in 3.5):

$$\mathrm{H}_g^{\mathrm{trop}}(\mathbf{x}, \mathbf{e}) = \mathrm{H}_g(\mathbf{x}, \mathbf{e}).$$

Proof. We begin the proof by giving a geometric interpretation of the product

$$\mathrm{logDR}_g(\mathbf{x}) \cdot Br_g^c(\mathbf{x}) \in \mathrm{logCH}^{g+c}(\overline{\mathcal{M}}_{g,n}) \tag{3.4}$$

[6]As we will see in the proof of Theorem 3.7 below, the latter condition actually follows from a dimension counting argument and does not necessarily have to be imposed. For the sake of clarity, we impose it here already.

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of the logarithmic double ramification cycle with the *branch class* (i.e., the logarithmic Chow class $Br_g^c(\mathbf{x}) = \mathrm{t}^*[T_c]$ associated to a piecewise polynomial function pulled back from tEx).

Let $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ be the non-fs log blowup of $\overline{\mathcal{M}}_{g,n}$ (in the category LogSch$^{\mathrm{coh}}$ of log schemes that étale locally admit charts by finitely generated monoids), performed using the subdivision introduced in Proposition 2.3. First, we claim that by a purely tropical argument (similar to Definition 3.2.4 and the proof of Theorem B in [CMR25]), we can decompose

$$Br_g^c(\mathbf{x}) = \sum_{\pi:\Gamma\to T} \mathrm{mult}(\pi)\cdot(\iota_\pi)_*[\widehat{\mathcal{M}}_\pi]\,. \tag{3.5}$$

Indeed, below we explain the notation in this formula and how to prove it:

- The summation goes over all k-leaky tropical covers π of the real line T subdivided at $c+1$ vertices, not imposing any dimension constraints or the condition that any vertex downstairs should have at most one (stable) vertex upstairs.
- Associated to π, there is a codimension c stratum $\widehat{\mathcal{M}}^\pi$ in the log blowup $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$. In the language of [HMP+22, Section 4.2.1], its generic element satisfies that the quasi-stable model of the universal curve is in fact stable and the piecewise linear function α has slopes given by the $\omega(e)$ from π. Then the bundle

$$\mathcal{L} = (\omega_C^{\log})^{\otimes k}(-\sum_i x_i p_i)(\alpha) \tag{3.6}$$

  is stable since it has multidegree 0 and the stability condition is assumed to be small.
- In the Artin fan of $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$, described in detail in [HMP+22, Section 1.7.2], the automorphism group of the cone associated to $\widehat{\mathcal{M}}^\pi$ is equal to $\mathrm{Aut}(\pi)$. We denote by

$$\iota_\pi : \widehat{\mathcal{M}}_\pi \to \widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$$

  the corresponding monodromy torsor over the normalization of the closure of $\widehat{\mathcal{M}}^\pi$, as constructed in [PRSS24, Section 4.1.2].

The formula (3.5) then follows by decomposing the pullback of the piecewise linear function $\varphi \in \mathrm{sPP}^c(\mathrm{tEx})$ from Section 2.2 into piecewise polynomials on the cone stack of $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ associated to the strata $\widehat{\mathcal{M}}^\pi$. The factor $\mathrm{mult}(\pi)$ then appears from comparisons of the integral structure on the cone stack of $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$ and tEx.[7]

Substituting the decomposition (3.5) into the formula (3.4), we are left to calculate the pullback

$$\iota_\pi^* \operatorname{logDR}_g(\mathbf{x}) \in \mathrm{CH}^g(\widehat{\mathcal{M}}_\pi)\,,$$

and then to compute its intersection number with the ψ-insertions. To do this, we record some useful information:

- Writing as before $\mathbf{x}(v)$ for the insertion at the vertex v induced by π, we claim that the stability condition chosen before induces stability conditions at all vertices v such that the corresponding log blowup $\widehat{\mathcal{M}}^{\mathbf{x}(v)}_{g(v),n(v)}$ of $\overline{\mathcal{M}}_{g(v),n(v)}$ supports the log double ramification cycle $\operatorname{logDR}_{g(v)}(\mathbf{x}(v))$.

[7] Obtaining the factor $\mathrm{mult}(\pi)$ is easier by working with the non-fs log blowup $\widehat{\mathcal{M}}^{\mathbf{x}}_{g,n}$, since there it is immediately obtained from an index calculation of lattices. When working with an fs log modification of $\overline{\mathcal{M}}_{g,n}$, the final multiplicity with which this term contributes to the intersection number can still be recovered, but it splits into contributions from greatest common divisors of edge slopes and étale covers obtained in the saturation process. For the purpose of obtaining the formula, it is thus easier to work in the non fs-setting at this point. For a related discussion, see also [HS21, Remark 1.4].

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- These spaces satisfy the functoriality that we obtain a natural projection map

$$\widehat{p} : \widehat{\mathcal{M}}_\pi \to \prod_{v \in V(\Gamma)} \widehat{\mathcal{M}}^{\mathbf{x}(v)}_{g(v),n(v)}, \tag{3.7}$$

  sending the log curve C and piecewise linear function α to the normalization of C at the nodes associated to edges of Γ, and the corresponding restriction of α to these components.[8]

We claim that

$$\widehat{p}_* \iota_\pi^* \operatorname{logDR}_g(\mathbf{x}) = \begin{cases} 0 & \text{if } \exists v_1 \neq v_2 \in V(\Gamma) \text{ with } \pi(v_1) = \pi(v_2) \\ \bigotimes_{v \in V(\Gamma)} \operatorname{logDR}_{g(v)}(\mathbf{x}(v)) & \text{otherwise.} \end{cases} \tag{3.8}$$

Assuming this formula, the theorem follows immediately, using the projection formula to convert the intersection product with $\psi^{\mathbf{e}}$ into a product of intersection numbers of cycles $\operatorname{logDR}_{g(v)}(\mathbf{x}(v))$ with ψ-classes. By another application of the projection formula, these are equal to the vertex multiplicities mult_v from Definition 3.3.

Denote by $\widehat{\mathcal{P}ic}_\pi$ the Picard stack of the universal curve over $\widehat{\mathcal{M}}_\pi$. Via the restriction to the normalization at nodes associated to edges of Γ, it has a map q to the product of stacks $\widehat{\mathcal{P}ic}_{g(v),n(v)}$ for the spaces $\widehat{\mathcal{M}}^{\mathbf{x}(v)}_{g(v),n(v)}$. The corresponding map is a $(\mathbb{G}_m)^{b_1(\Gamma)}$-torsor, fitting into the following diagram:

$$\begin{tikzcd} \widehat{\mathcal{M}}_\pi \arrow[r, "\mathrm{aj}_{\mathbf{x}}"] \arrow[d, "\widehat{p}"] & \widehat{\mathcal{P}ic}_\pi \arrow[d, "q"] \\ \prod_{v \in V(\Gamma)} \widehat{\mathcal{M}}^{\mathbf{x}(v)}_{g(v),n(v)} \arrow[r, "\prod_v \mathrm{aj}_{\mathbf{x}(v)}"] & \prod_{v \in V(\Gamma)} \widehat{\mathcal{P}ic}_{g(v),n(v)} \end{tikzcd} \tag{3.9}$$

Let $e_\pi \subseteq \widehat{\mathcal{P}ic}_\pi$ and $e_v \subseteq \widehat{\mathcal{P}ic}_{g(v),n(v)}$ be the closures of the zero sections. Then, as shown in [HMP+22, Proof of Theorem A], we have

$$(\mathrm{aj}_{\mathbf{x}})^*[e_\pi] = \iota_\pi^* \operatorname{logDR}_g(\mathbf{x}) \text{ and } (\mathrm{aj}_{\mathbf{x}(v)})^*[e_v] = \operatorname{logDR}_{g(v)}(\mathbf{x}(v)). \tag{3.10}$$

Moreover, we have

$$[e_\pi] = E \cdot q^*\left(\prod_{v \in V(\Gamma)} [e_v]\right) \in \mathrm{CH}^g(\widehat{\mathcal{P}ic}_\pi), \tag{3.11}$$

where $E \in \mathrm{CH}^{b_1(\Gamma)}(\widehat{\mathcal{P}ic}_\pi)$ is a class whose restriction over the locus

$$\prod_{v \in V(\Gamma)} e_v \subseteq \prod_{v \in V(\Gamma)} \widehat{\mathcal{P}ic}_{g(v),n(v)}$$

is the zero section.[9] A quick dimension calculation shows that the fibers of $\widehat{p}$ are of dimension

$$\begin{aligned} 3g - 3 + n - c - \sum_{v \in V(\Gamma)} (3g(v) - 3 + n(v)) &= |E(\Gamma)| - c = b_1(\Gamma) + |V(\Gamma)| - (c+1) \\ &= b_1(\Gamma) + \sum_{w \in V(T)} (|\pi^{-1}(w)| - 1). \end{aligned}$$

[8]A priori, working with the modular interpretation in terms of stable bundles on quasi-stable curves would yield such maps between the space $\widetilde{\mathcal{M}}$ discussed in the paragraph before Proposition 2.3. But then all constructions are compatible with the subdivision introduced to obtain the spaces $\widehat{\mathcal{M}}$, and thus, we obtain the map as described above. Similar arguments apply in all further cases where we cite properties from [HMP+22].

[9]An expression for $(\mathrm{aj}_{\mathbf{x}})^*E$ in terms of piecewise polynomials on $\widehat{\mathcal{M}}_\pi$ will appear in [Spe25].

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Combining the equations above, we obtain

$$\widehat{p}_*\iota_\pi^* \operatorname{logDR}_g(\mathbf{x}) = (\widehat{p}_*(\mathrm{aj}_\mathbf{x})^*E) \cdot \prod_{v\in V(\Gamma)} \pi_v^* \operatorname{logDR}_{g(v)}(\mathbf{x}(v)) \,. \tag{3.12}$$

If any vertex w of T has at least two preimages, the term $\widehat{p}_*(\mathrm{aj}_\mathbf{x})^*E$ vanishes for dimension reasons. If this is not the case, then for dimension reasons, $\widehat{p}_*(\mathrm{aj}_\mathbf{x})^*E$ gives a multiple of the fundamental class of $\prod_{v\in V(\Gamma)} \widehat{\mathcal{M}}^{\mathbf{x}(v)}_{g(v),n(v)}$. By restricting to a fiber, we see that the corresponding degree is 1. In both cases, we prove the claim above and thus conclude our argument. □

4. Splitting formulas for (logarithmic) double ramification cycles

Below, we discuss how to convert products of double ramification cycles with certain linear combinations of κ- and ψ-classes into a sum of boundary terms described via further double ramification cycles. These so-called splitting formulas generalize analogous results that first appeared in [BSSZ15, CSS21], and allow us to recursively compute double ramification descendants in terms of intersection numbers of $\mathrm{DR}_g(\mathbf{x})$ against powers of κ_1 in Section 4.2.

4.1. Splittings of ψ-classes

Definition 4.1. Let $\pi : \Gamma \to T$ be a stable k-leaky cover with expansion factors $\omega(e)$ on its flags or edges e. For each vertex $v \in V(\Gamma)$, let $f_l(v), f_r(v)$ be the left and right flag of the vertex $\pi(v)$. Consider the gluing map

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n} \tag{4.1}$$

associated to the underlying stable graph of Γ. Then we define

$$\mathrm{DR}_\pi = (\xi_\Gamma)_* \prod_{v\in V(\Gamma)} \pi_v^* \, \mathrm{DR}_{g(v)}\big((\omega(f))_{f\mapsto f_l(v)}, (-\omega(f))_{f\mapsto f_r(v)}\big) \in \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n}) \,, \tag{4.2}$$

where $\pi_v : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g(v),n(v)}$ is the projection to the factor associated to $v \in V(\Gamma)$.

For $\ell \geq 0$, let $T_\ell = T_\ell(\vec{w})$ be the metric line graph obtained by subdividing $\mathbb{R}$ at the $\ell + 1$ vertices $w_0 < w_1 < \ldots < w_\ell$.

Proposition 4.2. *Let $g, n \geq 0$ with $2g - 2 + n > 0$ and $\mathbf{x} \in \mathbb{Z}^n$ with $|\mathbf{x}| = k(2g - 2 + n)$. Then for any $1 \leq s \leq n$, we have*

$$((2g - 2 + n)x_s\psi_s - k\kappa_1) \cdot \mathrm{DR}_g(\mathbf{x}) = \sum_{\pi:\Gamma\to T_1} \rho(\pi, s) \cdot \mathrm{mult}(\pi) \cdot \mathrm{DR}_\pi \,, \tag{4.3}$$

where the sum runs over stable k-leaky covers $\pi : \Gamma \to T_1$ with precisely two vertices v_0, v_1 of Γ which map to the two vertices w_0, w_1 of T_1 (in that order), and

$$\rho(\pi, s) = \begin{cases} 2g(v_1) - 2 + n(v_1) & \textit{if } s \textit{ adjacent to } v_0, \\ -(2g(v_0) - 2 + n(v_0)) & \textit{if } s \textit{ adjacent to } v_1. \end{cases}$$

Proof. For $k = 0$ and $x_s \neq 0$, this statement was proven in [BSSZ15, Theorem 4]. To prove the general case, consider the vector $\widehat{\mathbf{x}} = (x_1, \ldots, x_n, k)$ associated to a double ramification cycle with an additional free marking. For the forgetful map $F : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, we have

$$F^* \,\mathrm{DR}_g(\mathbf{x}) = \mathrm{DR}_g(\widehat{\mathbf{x}}) \in \mathrm{CH}^g(\overline{\mathcal{M}}_{g,n+1}) \,.$$

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Applying [CSS21, Proposition 3.1] to this extended double ramification cycle (with $s = s, t = n + 1$ in the notation of [CSS21]), we obtain

$$(x_s\psi_s - k\psi_{n+1}) \cdot F^* \mathrm{DR}_g(\mathbf{x}) = \sum_{\widehat{\pi}:\Gamma\to T_1} f_{s,n+1}(\widehat{\pi}) \cdot \mathrm{mult}(\widehat{\pi}) \cdot \mathrm{DR}_{\widehat{\pi}}\,, \tag{4.4}$$

where $\widehat{\pi}$ runs over $(n+1)$-pointed k-leaky covers with exactly one vertex v_0, v_1 over each of the vertices $w_0, w_1 \in T_1$ and

$$f_{s,t}(\widehat{\pi}) = \begin{cases} 0 & \text{if } s \text{ and } n+1 \text{ are adjacent to the same vertex,} \\ 1 & \text{if } s \text{ is adjacent to } v_0 \text{ and } n+1 \text{ to } v_1, \\ -1 & \text{otherwise.} \end{cases}$$

We claim that equation (4.3) follows by multiplying boths sides of (4.4) with ψ_{n+1} and pushing forward under the forgetful map F. Indeed, using that

$$F_*(\psi_s\psi_{n+1}) = (2g-2+n)\cdot\psi_s \text{ and } F_*(\psi_{n+1}^2) = \kappa_1\,,$$

one sees that applying $F_*(\psi_{n+1}\cdot -)$ to the left-hand side of (4.4) gives the left-hand side of (4.3).

For the comparison of the right-hand sides, we first note that for any stable k-leaky cover $\widehat{\pi}$ such that marking $n+1$ lies on a vertex v with $g(v) = 0, n(v) = 3$ that becomes unstable under forgetting $n+1$, we have $\psi_{n+1}\cdot \mathrm{DR}_{\widehat{\pi}} = 0$ for dimension reasons. The remaining covers $\widehat{\pi}$ appearing in the summation are in bijective correspondence to the tuples (π, v_{n+1}) recording the cover π obtained by forgetting marking $n+1$ and the choice $v_{n+1} \in \{v_0, v_1\}$ of the vertex where this marking was attached. Indeed, the fact that marking $n+1$ carries weight k precisely means that its position on the graph Γ does not influence the balancing condition on the two vertices. Note also that $\mathrm{mult}(\widehat{\pi}) = \mathrm{mult}(\pi)$ is preserved under this correspondence.

To conclude the proof, observe that for any cover π appearing on the right-hand side of (4.3), there is precisely one choice of v_{n+1} such that the corresponding lift $\widehat{\pi} = (\pi, v_{n+1})$ satisfies $f_{s,n+1}(\widehat{\pi}) \neq 0$ (namely, $v_{n+1} = v_1$ for s adjacent to v_0 in π, and $v_{n+1} = v_0$ otherwise). Then indeed, the map $F_*(\psi_{n+1}\cdot -)$ sends the right-hand side of (4.4) to the right-hand side of (4.3). Here, the sign of the factor $\rho(\pi, s)$ comes from $f_{s,n+1}(\widehat{\pi})$, and its absolute value $2g(v_{n+1}) - 2 + n(v_{n+1})$ comes from the forgetful pushforward of the class ψ_{n+1} on the vertex v_{n+1}. □

Remark 4.3. It is an interesting question how to lift equation (4.3) to a splitting formula for the logarithmic double ramification cycle $\mathrm{logDR}_g(\mathbf{x})$. We expect that the right-hand side of (4.3) generalizes by allowing arbitrary stable k-leaky covers π, with the associated contribution logDR_π given as a suitable log-boundary pushforward of logarithmic double ramification cycles on the vertices of Γ. The associated language of log-boundary pushforwards is currently being developed in [PRSS24].

4.2. Recursions for double ramification descendants

Given a vector $\mathbf{x} \in \mathbb{Z}^n$ with $|\mathbf{x}| = k(2g-2+n)$ and $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ with $|\mathbf{e}| = 2g-3+n$, we want to give a recursion determining all intersection numbers

$$\mathrm{H}_g(\mathbf{x}, \mathbf{e}) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(\mathbf{x}) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n}\,. \tag{4.5}$$

Note that a priori, the invariant $\mathrm{H}_g(\mathbf{x}, \mathbf{e})$ is defined as an intersection number of the logarithmic double ramification cycle $\mathrm{logDR}_g(\mathbf{x})$ supported on a log blowup of $\overline{\mathcal{M}}_{g,n}$. However, in the absence of branch cycles, all the insertions ψ_i above are pulled back from $\overline{\mathcal{M}}_{g,n}$ and so by applying the projection formula, we can replace $\mathrm{logDR}_g(\mathbf{x})$ by its pushforward $\mathrm{DR}_g(\mathbf{x})$ in the above intersection number.

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The splitting formula for ψ-classes can be used to recursively calculate the numbers (4.5). However, when $k \neq 0$, the recursion naturally features a generalization of these numbers defined in equation (1.13) as

$$\mathrm{H}_g(\mathbf{x}, \mathbf{e}, f) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(\mathbf{x}) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n} \cdot \kappa_1^f ,$$

where now $|e| + f = 2g - 3 + n$.

Proposition 4.4. *Assume that for $1 \leq s \leq n$, the component e_s of $\mathbf{e}$ is positive and denote $\mathbf{e}_s = \mathbf{e} - \delta_s$. Then*

$$x_s(2g-2+n) \cdot \mathrm{H}_g(\mathbf{x}, \mathbf{e}, f) = k \cdot \mathrm{H}_g(\mathbf{x}, \mathbf{e}_s, f+1) + \sum_{\pi:\Gamma \to T_1} \rho(\pi, s) \cdot \mathrm{mult}(\pi) \cdot \mathrm{Cont}_{\pi,\mathbf{e},f} , \tag{4.6}$$

where the sum goes over covers $\pi : \Gamma \longrightarrow T_1$ with two vertices v_0, v_1 of Γ which map to the two vertices w_0, w_1 of T_1 (in that order). The contribution of this cover to the formula above is given by

$$\mathrm{Cont}_{\pi,\mathbf{e},f} = \binom{f}{f_0, f_1} \mathrm{H}_{g(v_0)}(\mathbf{x}[v_0], \mathbf{e}_s[v_0], f_0) \cdot \mathrm{H}_{g(v_1)}(\mathbf{x}[v_1], \mathbf{e}_s[v_1], f_1) , \tag{4.7}$$

where $\mathbf{x}[v_j]$, $\mathbf{e}_s[v_j]$ are the entries of the vectors $\mathbf{x}, \mathbf{e}_s$ associated to markings attached to vertex v_j and

$$f_j = 2g(v_j) - 3 + n(v_j) - |\mathbf{e}_s[v_j]| .$$

Proof. This immediately follows from multiplying (4.3) by $\psi_1^{e_1} \cdots \psi_s^{e_s-1} \cdots \psi_n^{e_n}$ and taking the integral over $\overline{\mathcal{M}}_{g,n}$. The factor $\binom{f}{f_0,f_1}$ arises when the factor κ_1^f in (1.13) splits as $(\kappa_{1,v_0} + \kappa_{1,v_1})^f$ when restricted to the boundary stratum associated to Γ. By dimension reasons, the only term which survives is $\kappa_{1,v_0}^{f_0} \kappa_{1,v_1}^{f_1}$, which appears with the above binomial factor. Note that when either f_0 or f_1 are negative, the integral vanishes for dimension reasons, and the binomial vanishes by definition. □

The numbers $\mathrm{H}_g(\mathbf{x}, \mathbf{e}, f)$ are polynomial in $\mathbf{x}$ by [PZ]. Seeing the entries x_t of $\mathbf{x}$ as formal variables and dividing equation (4.6) by $x_s(2g-2+n)$, this equation determines $\mathrm{H}_g(\mathbf{x}, \mathbf{e}, f)$ in terms of the polynomials $\mathrm{H}_{g'}(\mathbf{x}', \mathbf{e}', f')$ with $|\mathbf{e}'| < |\mathbf{e}|$. Iterating this procedure, the initial data of the recursion is given by the numbers

$$\mathrm{H}_g(\mathbf{x}, \mathbf{0}, 2g-3+n) = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(\mathbf{x}) \cdot \kappa_1^{2g-3+n} . \tag{4.8}$$

These numbers are determined explicitly in forthcoming work [Sau] by Sauvaget.

Remark 4.5. Another possibility to calculate the numbers (4.5) was pointed out to us by the referee of this paper: an explicit formula for the numbers in the case $e_1 = 2g - 3 + n$ and $e_i = 0$ for $i > 1$ was given in [CSS21]. Moreover, in [CSS21, Proposition 3.1], a formula is given for the class

$$(2g-2+n) \cdot (x_s\psi_s - x_t\psi_t) \cdot \mathrm{DR}_g(\mathbf{x})$$

in terms of boundary classes involving smaller DR-cycles. This formula is obtained by subtracting equation (4.3) for index t from the same equation for index s, eliminating the κ_1-term. Using this method, one can one-by-one shift all ψ-insertions to the first marked point, with all correction terms being controlled recursively. This second recursion does not need the initial data (4.8) and is thus entirely explicit. We once again thank the referee for pointing it out!

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5. Piecewise polynomiality in any genus and wall crossings for genus 0

Theorem 5.1. *Let $n \geq 3$ and let $\mathcal{X} = \{\mathbf{x} \in \mathbb{Z}^n \mid |\mathbf{x}| = k(2g-2+n)$ for some $k \in \mathbb{Z}\}$. Let $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ such that $0 \leq |\mathbf{e}| \leq 2g-3+n$.*

We can then view the k-leaky double Hurwitz descendant as a function

$$\mathcal{X} \to \mathbb{Q} : \mathbf{x} \mapsto \mathrm{H}_g(\mathbf{x}, \mathbf{e}).$$

This function is piecewise polynomial in the x_i and k, where the polynomials are of degree

$$n - 3 + 4g - |\mathbf{e}|.$$

Using the Correspondence Theorem 3.7, the function mapping $\mathbf{x}$ to the count of tropical k-leaky covers satisfying Psi-conditions is also piecewise polynomial of course. In fact, we will prove Theorem 5.1 on the tropical side.

Remark 5.2. In genus 0, the vertex multiplicity

$$\mathrm{mult}_V := \int_{\overline{M}_{0,\mathrm{val}(V)}} \log\mathrm{DR}_0(\mathbf{a}_\mathbf{V}) \cdot \prod_{i \in I_V} \psi_i^{e_i} = \int_{\overline{M}_{0,\mathrm{val}(V)}} \prod_{i \in I_V} \psi_i^{e_i} = \frac{(\mathrm{val}(V)-3)!}{\prod_{i \in I_V} e_i!}$$

of a leaky cover satisfying Psi-conditions does not depend on k or on the expansion factors of its adjacent edges; it equals a multinomial coefficient that only depends on its valency and the Psi-conditions.

More generally, the vertex multiplicities are given by the intersection numbers

$$\int_{\overline{M}_{g(v),\mathrm{val}(v)}} \mathrm{DR}_{g(v)}(\mathbf{x}(v)) \cdot \prod_{i \in I_v} \psi_i^{e_i}.$$

By [PZ, Spe24], the cycle $\mathrm{DR}_{g(v)}(\mathbf{x}(v))$ is a tautological class with coefficients which are polynomials in the entries of the vector $\mathbf{x}(v)$ of degree equal to $2g(v)$. Accordingly, the vertex multiplicities are polynomials in the entries of the vector $\mathbf{x}(v)$ of degree equal to $2g(v)$.

Proof of Theorem 5.1. By the Correspondence Theorem 3.7,

$$\mathrm{H}_g(\mathbf{x}, \mathbf{e}) = \mathrm{H}_g^{\mathrm{trop}}(\mathbf{x}, \mathbf{e}),$$

and the latter is a sum over all tropical k-leaky covers π of degree $\mathbf{x}$ and genus g satisfying the Psi-conditions $\mathbf{e}$ and mapping to a fixed metric line graph, where each cover π is counted with multiplicity $\mathrm{mult}(\pi)$ equal to the product of expansion vectors, vertex multiplicities and $\frac{1}{|\mathrm{Aut}(\pi)|}$.

Given the combinatorial type of an abstract tropical curve of genus g with n labeled ends (such that the valence of a vertex v of genus $g(v)$ adjacent to the ends with labels in I_v equals $\sum_{i \in I_V} e_i + 3 - 2g(v)$), we associate the expansion factor $|x_i|$ to the end with label i. We orient the ends pointing inward if $x_i > 0$ and outward if $x_i < 0$. We view this degree now as something varying with the vector $\mathbf{x}$.

Furthermore, we fix $g' := g - \sum_v g(v)$ edges whose removal produces a tree and view their expansion factors as variables $i_1, \ldots, i_{g'}$.

We pick an arbitrary orientation for the bounded edges, for which we ask ourselves whether there exists a map to the line graph respecting this orientation.

Using the k-leaky condition, the expansion factor of every other edge is then uniquely determined, and it is equal to a linear form in the x_j, k and i_s.

A map to the line graph exists if and only if each expansion factor is positive.

If a map to the line graph exists, there is a unique way to add a metric to the graph which is compatible with the metric of the target.

Consider the space with coordinates $i_1, \ldots, i_{g'}$. Each expansion factor defines a hyperplane equation in this space, such that the expansion factor is positive if and only if we are on the right side of the hyperplane.

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The sum over all leaky tropical covers can thus be viewed as a weighted sum over all integer points $(i_1, \ldots, i_{g'})$ in a bounded chamber of a hyperplane arrangement defined by an oriented labeled graph (with orientations of the ends matching the x_i), where each summand contributes $\mathrm{mult}(\pi)$ for the associated leaky cover π. The fact that the chamber is bounded follows from the arguments used in the proof of Corollary 2.13 in [CJM11].

By Definition 3.5, the multiplicity $\mathrm{mult}(\pi)$ with which π contributes to the count of k-leaky covers satisfying Psi-conditions is a product of expansion vectors, vertex multiplicities and $\frac{1}{|\mathrm{Aut}(\pi)|}$. The last factor is a number, independent of the expansion factors of the ends. The first factor is a product of linear forms in the x_j, k and the i_s of degree equal to the number of bounded edges, which is $n-3+3g' - \sum_v (\mathrm{val}(v)-3) = n-3+3g' - |\mathbf{e}| + \sum_v 2g(v)$.

By Remark 5.2, the vertex multiplicities are polynomial of degree $g(v)$ in the expansion factors of the adjacent edges, which are themselves affine-linear forms in the x_j and the i_k.

Thus, viewed as polynomial in the x_j, k and i_s, the multiplicity $\mathrm{mult}(\pi)$ is of degree $n-3+3g' - |\mathbf{e}| + \sum_v 4g(v)$.

Summing over the points $(i_1, \ldots, i_{g'})$ in the bounded chamber increases the degree by g'. Here, we use that the matrix defining the walls of the hyperplane arrangement is a network matrix (calculated from the domain graph Γ of the tropical cover π), and thus totally unimodular [Sch03, Chapter 13]. Then the fact that this summation is a piecewise polynomial follows, for example, from [Mou00].

Thus, the multiplicity $\mathrm{mult}(\pi)$ is a polynomial in the x_j and k of degree $n-3+4g' - |\mathbf{e}| + \sum_v 4g(v) = n-3+4g-|\mathbf{e}|$.

In total, we obtain a piecewise polynomial function, where the piecewise structure arises since the topology of the hyperplane arrangement of the expansion factors in the space with coordinates $i_1, \ldots, i_{g'}$ may vary for different choices of x_j. $\square$

Example 5.3. For this example, we fix $k=1$. The k-leaky double Hurwitz descendant is then piecewise polynomial in the x_i. Let $g=0$, $n=5$, $k=1$ and $\mathbf{e} = (1,0,0,0,0)$. We fix the inequalities $x_1, x_4 > 0$, $x_2, x_3, x_5 < 0$, $x_1 + x_4 + x + 5 - 2 > 0$, $x_1 + x_4 + x_2 - 2 > 0$, $x_1 + x_4 + x_5 - 2 > 0$, $x_1 + x_3 + x_5 - 2 > 0$, $x_1 + x_3 + x_4 - 2 > 0$, $x_1 + x_2 + x_5 - 2 > 0$.

The unique bounded edge of each of the 6 labeled trees with label 1 adjacent to a 4-valent vertex can be oriented in a unique way producing a leaky cover with positive expansion factors in the chamber defined by these inequalities; see Figure 5. In total, we obtain for all $\mathbf{x}$ satisfying the inequalities above the polynomial

$$\begin{aligned} \mathrm{H}_0(\mathbf{x}, (1,0,0,0,0)) &= (x_1+x_2+x_3-2) + (x_1+x_4+x_5-2) + (x_1+x_3+x_5-2) \\ &\quad + (x_1+x_3+x_4-2) + (x_1+x_2+x_5-2) + (x_1+x_2+x_4-2) \\ &= 6x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 - 12 = 3x_1 - 3. \end{aligned}$$

Lemma 5.4. *Let $g=0$, $n \geq 3$ and let $\mathcal{X} = \{\mathbf{x} \in \mathbb{Z}^n \mid |\mathbf{x}| = k(-2+n)$ for some $k \in \mathbb{Z}\}$. Let $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ such that $0 \leq |\mathbf{e}| \leq -3+n$. The walls separating the areas of polynomiality of the piecewise polynomial function $\mathrm{H}_g(\mathbf{x}, \mathbf{e})$ are given by vanishing expansion factors – that is, by expressions of the form*

$$\sum_{i \in I} x_i - k \cdot (\sharp I - 1) = 0,$$

where $I \subset \{1, \ldots, n\}$, $2 \leq \sharp I \leq n-2$.

Proof. The walls separating chambers of polynomiality are given by expressions as above, as this is the expansion factor of the edge separating the ends with labels in I from the ends with labels not in I. Trees with an edge of this weight correspond to the polynomial on one side – on the other side, the tree with the edge reversed contributes. $\square$

Remark 5.5. For higher genus, the walls separating chambers of polynomiality are given by the same equations. The walls arise because bridge edges can change their direction, as above, or because the

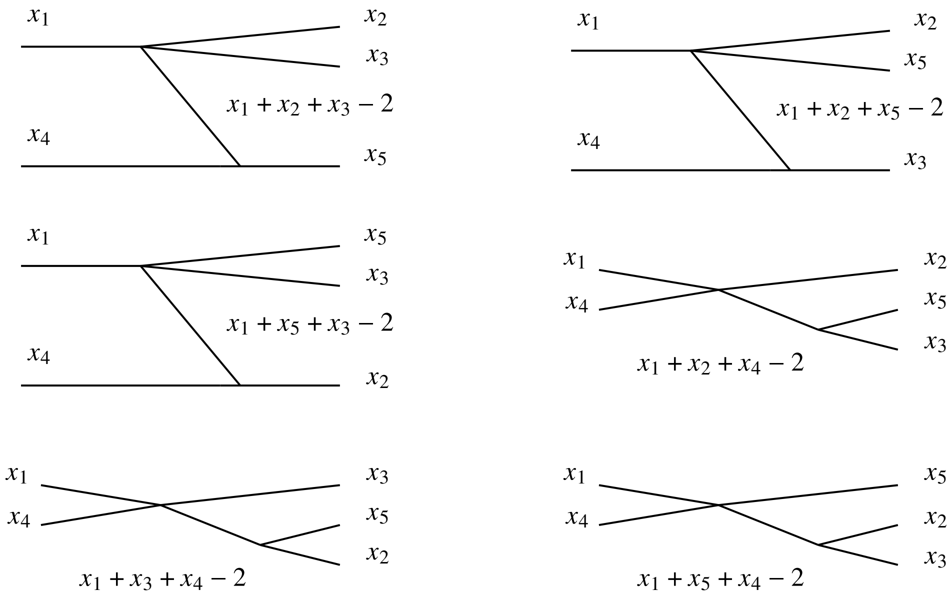

Figure 5. *Six* 1*-leaky covers (all vertices of genus* 0*) of degree* $\mathbf{x}$ *satisfying the inequalities in Example 5.3 yield a nonzero contribution to the count* $\mathrm{H}_0(\mathbf{x}, (1, 0, 0, 0, 0))$*. For each, its multiplicity equals the expansion factor of its unique bounded edge.*

hyperplane arrangement given by the egdes as discussed in the proof of Theorem 5.1 becomes degenerate. As for the case of double Hurwitz numbers, the hyperplane arrangements become degenerate precisely at the walls described above. One can view the degeneracy as arising because several edges go to zero and change their direction together. To obtain wall-crossing formulas for higher genus, one thus has to take contributions from graphs for which we need to cut more than just one edge into account, which is why the formulas become more tedious [CJM11].

Proof of Theorem 1.4. The proof follows the ideas presented in [CJM10] for tropical double Hurwitz numbers; we have to include leaking and Psi-conditions.

Recall from the proof of Theorem 5.1 that the polynomial expression equals a sum over oriented labeled trees such that the valence of a vertex V adjacent to the ends with labels in I_V equals $\sum_{i \in I_V} e_i + 3$. Each tree contributes either 0 or $\mathrm{mult}(\pi)$ for the leaky cover π we can build from it. We build a cover by adding expansion factors to the bounded edges, satisfying the leaky condition, and a metric. If an edge in such a cover has expansion factor $\delta > 0$, the corresponding oriented tree yields 0 on the other side of the wall (as then $\delta < 0$); however, the tree with the orientation of the edge reversed yields a nonzero contribution. Vice versa, that tree does not yield a contribution on the first side of the wall.

Trees that do not produce covers with an expansion factor being $\pm\delta$ contribute the same to both sides of the wall and thus do not contribute to the wall-crossing.

We produce a weighted bijection between ‘cut-and-reglued’ covers and covers contributing to the wall-crossing. Given a cover contributing to the wall-crossing, cut the edge with expansion factor δ. We obtain two covers, one that contributes to $\mathrm{H}_0(\mathbf{x}_I \cup \{\delta\}, \mathbf{e}_I)$ and one that contributes to $\mathrm{H}_0(\mathbf{x}_{I^c} \cup \{-\delta\}, \mathbf{e}_{I^c})$. Vice versa, if we have a pair of leaky covers, one contributing to $\mathrm{H}_0(\mathbf{x}_I \cup \{\delta\}, \mathbf{e}_I)$ and one to $\mathrm{H}_0(\mathbf{x}_{I^c} \cup \{-\delta\}, \mathbf{e}_{I^c})$, how can we reglue the ends labeled $\pm\delta$? First, we have to interlace the images of the vertices. There are $\binom{r}{r_1, r_2}$ choices for this, as r is the number of vertices of the whole tree, while r_i are the numbers of vertices of the two pieces.

For a fixed such choice, the orientation of the reglued edge labeled δ is determined by the images of the vertices and the compatibility with the cover of the line graph. Depending on this orientation, the reglued cover yields a nonzero contribution to precisely one of the sides of the wall. If it lives on side

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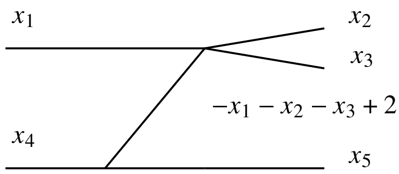

Figure 6. *The cover that arises when crossing the wall δ in Example 5.6.*

1, the only missing ingredient to count it with its correct multiplicity is the expansion factor of the cut and reglued edge, which is $\delta > 0$. If it lives on side 2, it appears with negative sign in the difference for the wall-crossing, and we have so far missed the expansion factor of its cut and reglued edge, which is $-\delta$. As the two minus signs cancel, we can treat both cases in the same way and thus obtain the claimed equality. □

Example 5.6. We continue Example 5.3. There, $n = 5$, $k = 1$ and $\mathbf{e} = (1, 0, 0, 0, 0)$. We computed the polynomial $\mathrm{H}_0(\mathbf{x}, (1, 0, 0, 0, 0))$ for $\mathbf{x}$ satisfying the inequalities $x_1, x_4 > 0, x_2, x_3, x_5 < 0, x_1+x_4+x+5-2 > 0, x_1+x_4+x_2-2 > 0, x_1+x_2+x_3-2 > 0, x_1+x_3+x_5-2 > 0, x_1+x_3+x_4-2 > 0, x_1+x_2+x_5-2 > 0$. Let us now cross the wall $\delta = x_1 + x_2 + x_3 - 2 = 0$. In Figure 5, all covers except the top left yield the same contribution on the other side of the wall, so their contributions cancel in the wall-crossing. Instead of the cover on the top left, the cover depicted in Figure 6 arises on the other side of the wall.

The wall-crossing equals $2(x_1+x_2+x_3-2) = \binom{2}{1,1}\delta$, as predicted by Theorem 1.4, as the cut 1-leaky double Hurwitz descendants are both just one.

6. Positivity and vanishing of leaky Hurwitz descendants in genus 0

In this section, we give a characterization of when genus zero leaky double Hurwitz descendant invariants vanish. By Remark 5.2, the multiplicity with which a leaky cover contributes is always positive. A leaky double Hurwitz descendant in genus 0 is thus positive if and only if we can construct a single leaky cover which contributes to the count.

If the genus is positive, it is possible to have leaky tropical covers which contribute with negative multiplicity, as well as some which contribute with positive multiplicity to a leaky double Hurwitz descendant; see Example 6.1. The question whether a leaky double descendant is positive, negative or even 0 is therefore hard to answer in general.

Example 6.1. Consider the leaky Hurwitz number $H_1(d, -(d-2k))$ for some $d > k+1 > 0$. Figure 7 shows two types of leaky tropical covers which contribute to the count. The upper cover has a vertex of genus 1 with an incoming edge of weight k and no outgoing edge. The vertex multiplicity of this genus 1 vertex equals $-\frac{1}{24}$. The upper cover thus contributes $-\frac{k}{24}$. The lower cover has only trivial vertex multiplicities. The weight i of the edge in the cycle can vary from 1 to $d-k-1$. Exchanging the two edges of the cycle yields an automorphism, so we have to divide by $\frac{1}{2}$. Altogether, the lower picture accounts for

$$\frac{1}{2} \cdot \sum_{i=1}^{d-k-1} i \cdot (d-k-i) = \frac{1}{12} \cdot (d-k) \cdot (d-k-1) \cdot (d-k+1).$$

Consequently, while the k-leaky number is positive for large d, there are regions of the parameter space where it becomes negative (e.g., $d = k+2$, $k > 12$, where the number is $\frac{1}{2} - \frac{k}{24}$).

From now on, we restrict to the case $g = 0$, where leaky tropical covers have nonnegative multiplicity.

Remark 6.2. If $k = 0$, $H_0(\mathbf{x}, \mathbf{e}) > 0$ unless $\mathbf{x} = (0, \ldots, 0)$ and $n > |\mathbf{e}| + 3$: for any degree $\mathbf{x}$ which is not zero, the existence of a tropical cover follows from Proposition 6.5, since $\mathbf{x}$ must have negative entries. If $\mathbf{x}$ is zero, we must impose Psi-conditions that force any cover to consist of only one vertex,

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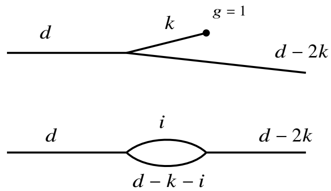

Figure 7. *Leaky tropical covers which contribute negatively resp. positively to $H_1(d, -(d-2k))$.*

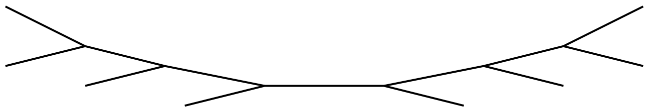

Figure 8. *A caterpillar cover: the leftmost vertices of the cover merge an end until all nonnegative ends are merged in, and the last vertices split off the negative ends.*

adjacent to all ends. Such a cover contributes positively. If we have less Psi-conditions, any cover needs to have at least one bounded edge, which must be of weight 0 by the balancing condition, leading to a contradiction. Thus, there is no cover of degree 0 and with $n > |\mathbf{e}| + 3$.

Remark 6.3. In Theorem 1.6, we consider the case $k \neq 0$. In the following, we assume without restriction that $k > 0$. That is possible, since we can 'turn around' any tropical k-leaky cover of degree $\mathbf{x}$, thus producing a tropical $(-k)$-leaky cover of degree $-\mathbf{x}$.

With the following lemma, we can deduce positivity of k-leaky double Hurwitz descendants from the positivity of k-leaky double Hurwitz numbers.

Lemma 6.4. *If the k-leaky double Hurwitz number $H_0(\mathbf{x}) > 0$ for some $\mathbf{x}$, then also the k-leaky double Hurwitz descendant $H_0(\mathbf{x}, \mathbf{e}) > 0$ for any $\mathbf{e}$.*

Proof. By definition and because of Remark 5.2, in genus 0, any leaky tropical cover contributes with positive multiplicity. Since $H_0(\mathbf{x}) > 0$, there exists a tropical leaky cover of genus 0 and degree $\mathbf{x}$. It has 3-valent vertices. We can temporarily forget the order of the images of the vertices. If $\mathbf{e} \neq 0$, we can shrink bounded edges in such a way that we produce the valencies which are required by $\mathbf{e}$. We can then order the remaining vertices again in an arbitrary way (compatible with the images of the edges). The cover we produce in this way then contributes positively to $H_0(\mathbf{x}, \mathbf{e})$ and we conclude $H_0(\mathbf{x}, \mathbf{e}) > 0$. □

To study the positivity of k-leaky double Hurwitz numbers, we first assume that $\mathbf{x}$ contains at least one entry strictly smaller than $k/2$.

Proposition 6.5. *Let $k \geq 0$, $|\mathbf{x}| = k(n-2)$ and assume $\mathbf{x}$ has at least one entry $x_i < k/2$. Then there exists a caterpillar k-leaky cover of genus 0 and degree $\mathbf{x}$ which contributes positively to the count $H_0(\mathbf{x})$, see Figure 8. As a consequence, $H_0(\mathbf{x}) > 0$.*

Proof. The case $n = 3$ is trivial, since there the trivial graph with three legs is always a leaky tropical cover with multiplicity 1 and so $H_0(\mathbf{x}) = 1 > 0$. Assume that $n \geq 4$, and without loss of generality, we order the markings such that $x_1 \geq x_2 \geq \ldots \geq x_n$ with $x_n < k/2$ by assumption. Then we have

$$\textbf{Claim} : x_1 + x_2 > k\,. \tag{6.1}$$

Indeed, assume on the contrary that $x_1 + x_2 \leq k$. Then we have that $x_2 \leq k/2$ since otherwise, $x_1 + x_2 \geq x_2 + x_2 > k$. By the ordering above, we then also have $x_3, \ldots, x_{n-1} \leq k/2$. Using these inequalities, we would have

$$\begin{aligned} &\underbrace{x_1 + x_2}_{\leq k} + \underbrace{x_3 + \ldots + x_{n-1}}_{\leq (n-3)\cdot k/2} + x_n = k(n-2) \\ \Longrightarrow\ & x_n + (n-1)\cdot k/2 \geq k(n-2) \\ \Longrightarrow\ & x_n \geq k(n-2-\frac{n-1}{2}) = k\cdot\frac{n-3}{2}. \end{aligned}$$

Since $n \geq 4$, this gives a contradiction to the assumption $x_n < k/2$.

To conclude $H_0(\mathbf{x}) > 0$ we construct a cover as follows: let v_0 be a vertex with markings $1, 2$ and an outgoing edge, whose expansion factor is $\widetilde{x} = x_1 + x_2 - k > 0$ by (6.1). Then by induction, the Hurwitz number $H_0(\widetilde{x}, x_3, \ldots, x_n)$ is positive (since still $x_n < k/2$), and so there exists an associated leaky cover. Gluing the vertex v_0 in results in a cover for $H_0(\mathbf{x})$ contributing positively, which is necessarily of the form illustrated in Figure 8. □

Lemma 6.6. *Let $k > 0$ be any number and $|\mathbf{x}| = k(n-2)$. Then the k-leaky double Hurwitz number $H_0(\mathbf{x})$ is nonnegative, and it is strictly positive if and only if $\mathbf{x}$ contains an entry which is not a positive multiple of $\frac{k}{2}$.*

Proof. We prove the theorem by induction on $n \geq 3$. For $n = 3$, the leaky Hurwitz number is always 1, as seen above. However, the condition $x_1 + x_2 + x_3 = k$ can never be satisfied if $x_i = m_i \cdot k/2$ with $m_i > 0$, so the corresponding clause of the theorem is empty, and thus, the leaky Hurwitz number is predicted to always be positive, which we verified.

For $n = 4$, we know by Proposition 6.5 that $H_0(\mathbf{x})$ is positive unless all x_i are at least $k/2$. Since they sum to $2k$, the only remaining possibility is $x_1 = \ldots = x_4 = k/2$ with k even, in which case the k-leaky Hurwitz number vanishes, since all associated tropical covers have weight $k/2 + k/2 - k = 0$ on the bounded edge.

For $n > 4$, we distinguish two cases: if k is even and all x_i are of the form $x_i = m_i \cdot k/2$ with $m_i > 0$, take any leaf vertex v_0 of a corresponding leaky cover. If the two x_i, x_j adjacent to it are equal to $k/2$, the outgoing edge has slope 0, and thus, the cover does not contribute to $H_0(\mathbf{x})$. Otherwise, the outgoing slope is a positive multiple $\widetilde{m}\cdot k/2$, so the remaining graph is one contributing to a leaky Hurwitz number $H_0(\widetilde{m}\cdot k/2, x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n)$, which vanishes by induction. Thus, the remaining graph has multiplicity zero, and thus has a bounded edge of weight 0. Hence, the original graph contributed with multiplicity 0 as well, and so, $H_0(\mathbf{x}) = 0$.

Conversely, assume that not all numbers are of the form $x_i = m_i \cdot k/2$ for $m_i \in \mathbb{N}_{>0}$, in which case we want to show $H_0(\mathbf{x}) > 0$. If any of them satisfied $x_i < k/2$, we would conclude the positivity of $H_0(\mathbf{x})$ from Proposition 6.5. Thus, we can assume $x_i \geq k/2$.

If there is at least one entry of $\mathbf{x}$, which is not a positive multiple of $k/2$, then in fact, there must be two since all the x_i are positive and their sum is $k(n-2)$. Assume the number of these entries is precisely two, and say they are given by x_1, x_2. Then necessarily $x_n = k/2$ since $n \geq 4$. Taking v_0 a vertex with markings $1, n$, its outgoing expansion factor is

$$\widetilde{x} = \underbrace{x_1}_{\geq k/2 \text{ and } \neq k/2} + x_n - k > k/2 + k/2 - k = 0.$$

By induction, the number $H_0(\widetilde{x}, x_2, \ldots, x_{n-1})$ is positive since x_2 is not a positive multiple of $k/2$, and gluing v_0 to any cover contributing to that number gives a cover with positive contribution to $H_0(\mathbf{x})$.

Finally, if there are at least three entries (say x_1, x_2, x_3) of $\mathbf{x}$ not given by positive multiples of $k/2$, then fusing x_1, x_2 at a vertex v_0 as above still has positive outgoing expansion factor (since $\widetilde{x} = x_1 + x_2 - k > 0$), and we conclude as above using the proven case $H_0(\widetilde{x}, x_3, \ldots, x_n) > 0$. □

By Lemma 6.4, we can conclude the following:

Corollary 6.7. *Let $k > 0$ be any number, and $|\mathbf{x}| = k(n-2)$. Assume $\mathbf{x}$ contains an entry which is not a positive multiple of $\frac{k}{2}$. Then the k-leaky double Hurwitz descendant $H_0(\mathbf{x}, \mathbf{e}) > 0$.*

Proposition 6.8. *Let $k > 0$ be even and $|\mathbf{x}| = k(n-2)$. Assume $\mathbf{x}$ contains only positive integer multiples of $\frac{k}{2}$, $x_i = m_i \cdot \frac{k}{2}$. Let $e \in \mathbb{Z}^n_{\geq 0}$ with $0 \leq |e| \leq n-3$ and $I \subset \{1, \ldots, n\}$ be an index set of size r such that*

$$\sum_{i \in I} e_i \geq \sum_{i \in I} m_i - r + 1. \tag{6.2}$$

Then $H_0(\mathbf{x}, \mathbf{e}) > 0$.

Proof. Since all the edge weights involved in the proof of this Proposition are integral multiples of $\frac{k}{2}$, we divide this factor out from all weights. The degree condition for a leaky cover then becomes

$$\sum m_i = 2n - 4. \tag{6.3}$$

Note that (6.3) also expresses in this normalization the leaky balancing condition at a vertex of a leaky cover, where the m_i's are the weights of the incident edges, and n is the valence of the vertex. We make the following simplifying assumptions:

1. for every $i \in I$, $e_i > 0$;
2. for every $j \notin I$, $e_j = 0$.

We prove that these two assumptions do not cause any loss of generality. If $i \in I$ has $e_i = 0$, (6.2) is satisfied by the subset $I \setminus \{i\}$; thus given any subset I satisfying (6.2), we may replace it with the subset of its elements with strictly positive e_i's, which satisfies condition (1).

Assume the vector of Psi conditions $\mathbf{e}$ is supported on the subset $I \subseteq [n]$ of size r, and $\sum_{i \in I} e_i \geq \sum_{i \in I} m_i - r + 1$. If there exists a k-leaky tropical cover of degree $\mathbf{x}$ satisfying the Psi-conditions given by $\mathbf{e}$, then for any entry-wise larger vector of Psi conditions, one can construct a k-leaky tropical cover as in Lemma 6.4 by shrinking edges. Thus, condition (2) poses no restriction.

With the simplifying assumptions in place, the strategy of proof is as follows: we construct a k-leaky cover starting from the rightmost vertex v_R, to which we attach all the ends in I; the Psi condition determines the valency of v_R, which requires adding additional s half-edges at the vertex. The leaky balancing condition determines the total weight of edges and ends incident to v_R. We can connect all but one of the s half-edges directly to ends in such a way that the weight on the last half-edge (which is determined) is still positive. To complete the picture, we are looking for a leaky cover with a bunch of left ends and exactly one right end to glue to the remaining half-edge of v_R. The existence of such a graph is guaranteed by Proposition 6.5. Now for the details.

Assume without loss of generality that $I = [1, r] \subseteq [n]$, and that $m_{r+1} \leq m_{r+2} \leq \ldots \leq m_n$.

Let $s = |\mathbf{e}| - r + 3$, and note that $3 \leq s \leq n - r$; the first inequality holds because of condition (1), the second because $|\mathbf{e}| \leq n - 3$. Let Γ_R be a graph consisting of a single vertex v_R of valence $r + s$. Assign weights $m_1, \ldots, m_{r+s-1}$ to all but one of the edges of Γ_R.

By (6.3), the weight at the last end of Γ_R equals

$$m_L := 2(r+s) - 4 - \sum_{i=1}^{r+s-1} m_i. \tag{6.4}$$

Claim. $m_L > 0$.

Assuming the claim, consider $\mathbf{x}' = (m_{r+s}, \ldots, m_n, -m_L)$, a vector of length $n - r - s + 2$. We observe that $\mathbf{x}'$ satisfies the degree condition to admit a k-leaky cover:

$$\begin{aligned}\sum_{i=r+s}^{n} m_i - m_L &= \sum_{i=1}^{n} m_i - 2(r+s) + 4 \\ &= 2n - 4 - 2(r+s) + 4 \\ &= 2(n - r - s + 2) - 4.\end{aligned}$$

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Since $-m_L$ is negative, and so in particular strictly less than one, by Proposition 6.5, there exists a caterpillar leaky cover of degree $\mathbf{x}'$. Attaching this cover to the end of weight m_L of Γ_R produces a leaky cover of degree $\mathbf{x}$, with a rightmost vertex v_R of valence

$$r + s = |\mathbf{e}| + 3. \tag{6.5}$$

By (6.5), this graph satisfies the Psi condition $\mathbf{e}$ and contributes positively to $H_0(\mathbf{x}, \mathbf{e})$. Thus, the Proposition is proved modulo proving the claim. □

Proof of Claim. Assume $m_L \leq 0$, giving

$$\begin{aligned} 2(r+s) - 4 &\leq \sum_{i=1}^{r+s-1} m_i = \sum_{i=1}^{r} m_i + \sum_{i=r+1}^{r+s-1} m_i \\ &\overset{(6.2)}{\leq} |\mathbf{e}| + r - 1 + \sum_{i=r+1}^{r+s-1} m_i \\ &\overset{(6.5)}{=} r + s - 3 + r - 1 + \sum_{i=r+1}^{r+s-1} m_i. \end{aligned}$$

Simplifying, we obtain

$$\sum_{i=r+1}^{r+s-1} m_i \geq s, \tag{6.6}$$

so one of the $s-1$ summands m_i on the left-hand side above must be at least 2. In particular, since we choose to order the weights after r in increasing order, $m_i \geq 2$ for $i \geq r+s-1$. This gives us the following contradiction:

$$\begin{aligned} 2n - 4 = \sum_{i=1}^{n} m_i &= \sum_{i=1}^{r+s-1} m_i + \sum_{i=r+s}^{n} m_i \\ &\geq 2(r+s) - 4 + 2(n-r-s+1) = 2n-2. \end{aligned}$$ □

Finally, we generalize the vanishing part $H_0(\mathbf{x}) = 0$ for $\mathbf{x}$ only containing positive multiples of $k/2$ from Lemma 6.6 to the numbers $H_0(\mathbf{x}, \mathbf{e})$, where we see that an additional condition is required (which is always satisfied for $\mathbf{e} = 0$).

Proposition 6.9. *Let $k > 0$ be even and $|\mathbf{x}| = k(n-2)$. Assume $\mathbf{x}$ contains only positive multiples of $\frac{k}{2}$, $x_i = m_i \cdot \frac{k}{2}$. Let $e \in \mathbb{Z}^n_{\geq 0}$ with $0 \leq |e| \leq n-3$ and assume that for any subset $I \subset \{1, \ldots, n\}$ with $|I| = r$, we have*

$$\sum_{i \in I} e_i < \sum_{i \in I} m_i - r + 1. \tag{6.7}$$

Then $H_0(\mathbf{x}, \mathbf{e}) = 0$.

Proof. We show that there cannot be a leaky tropical cover of genus 0 and degree $\mathbf{x}$ satisfying the Psi-conditions imposed by $\mathbf{e}$ in this case. Assume there is such a cover. Since $\mathbf{x}$ has only positive entries, the cover has no right ends. Thus, there cannot be an edge leaving the rightmost vertex v_R to the right. Denote by I the subset of ends that attach directly to v_R, and say that an additional s edges (which are

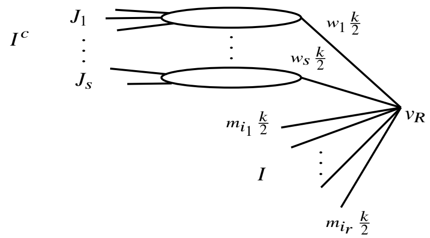

Figure 9. *The rightmost vertex of a leaky cover whose degree contains only positive multiples of $\frac{k}{2}$.*

not ends) attach to v_R, with weights $w_j \cdot \frac{k}{2}$ for $j = 1, \dots s$; see Figure 9. Denoting by $r = |I|$, the leaky balancing condition at v_R is

$$\sum_{i \in I} m_i + \sum_{j=1}^{s} w_j = 2(r+s) - 4. \tag{6.8}$$

The cover satisfying the Psi-conditions at v_R implies

$$\sum_{i \in I} e_i = r + s - 3. \tag{6.9}$$

We may subtract (6.9) from (6.8), and, using the fact that $w_j \geq 1$ for all j, obtain the inequality

$$\sum_{i \in I} m_i + s - \sum_{i \in I} e_i \leq r + s - 1. \tag{6.10}$$

It is immediate to see that (6.10) contradicts the hypothesis (6.7). Hence, no cover can exist and $H_0(\mathbf{x}, \mathbf{e}) = 0$.

□

Proof of Theorem 1.6. By Remark 6.3, we can assume $k > 0$ without restriction. The Theorem collects together the statements of Corollary 6.7, Proposition 6.8 and Proposition 6.9. □

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