Proceedings of the Edinburgh Mathematical Society (2009) **52**, 263–272 ©
DOI:10.1017/S0013091507000697 Printed in the United Kingdom

ON APPLICATION OF KINETIC FORMULATION OF THE LE ROUX SYSTEM

ZHIXIN CHENG*

Department of Mathematics, University of Science and Technology of China, 230026 Hefei, Anhui, People's Republic of China (czx@mail.ustc.edu.cn)

(Received 1 June 2007)

Abstract We use the compensated compactness method coupled with some basic ideas of kinetic formulation developed by Lions, Perthame, Souganidis and Tadmor to give a refined proof for the existence of global bounded entropy solutions to the Le Roux system. This new method of the reduction of Young measures can be applied to solve other problems.

Keywords: strong entropy; entropy solution; hyperbolic system; kinetic formulation

2000 *Mathematics subject classification:* Primary 35L65
Secondary 35B45; 35L45

1. Introduction

We are concerned with the Cauchy problem for the following nonlinear hyperbolic system:

$$\left.\begin{aligned} u_t + \tfrac{2}{3}(u^2+v)_x &= 0,\\ v_t + \tfrac{2}{3}(uv)_x &= 0,\end{aligned}\right\} \tag{1.1}$$

with bounded measurable initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad v_0(x) \geqslant 0. \tag{1.2}$$

The scaling $x \to \frac{2}{3}x$ allows one to reduce system (1.1) to the following:

$$\left.\begin{aligned} u_t + (u^2+v)_x &= 0,\\ v_t + (uv)_x &= 0,\end{aligned}\right\} \tag{1.3}$$

which was first derived by Le Roux in [**3**] as a mathematical model and is therefore called the Le Roux system.

By simple calculations, two eigenvalues of system (1.1) are $\lambda_1 = u - \frac{1}{3}D$, $\lambda_2 = u + \frac{1}{3}D$, and two corresponding Riemann invariants are $w(u,v) = u + D$, $z(u,v) = u - D$, where

* Present address: Department of Mathematics, Yancheng Teachers College, 224002 Yancheng, People's Republic of China.

$D = (u^2 + 4v)^{1/2}$. Thus, $\lambda_1 = \lambda_2$ at point $(0, 0)$, at which strict hyperbolicity fails to hold. The singularity of entropies at this singular point is the main difficulty in dealing with system (1.1).

Lu [**8**] obtained the global existence of weak solutions to the Cauchy problem (1.3), (1.2) by constructing four families of entropies and entropy fluxes of Lax type, while, in the current paper, we apply the compensated compactness method and kinetic formulation to give a refined proof for the following theorem.

Theorem 1.1. *Let the initial data $(u_0(x), v_0(x))$ be bounded measurable and let $v_0(x) \geqslant 0$. Then the Cauchy problem (1.1), (1.2) has a global bounded entropy solution (u, v) with the property $v \geqslant 0$.*

Remark 1.2. A pair of functions $(u(x,t), v(x,t))$ is called an entropy solution of the Cauchy problem (1.1), (1.2) if

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + \tfrac{2}{3}(u^2 + v)\phi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\,\mathrm{d}x = 0,$$
$$\int_0^\infty \int_{-\infty}^\infty v\phi_t + \tfrac{2}{3}uv\phi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^\infty v_0(x)\phi(x,0)\,\mathrm{d}x = 0$$

for any test function $\phi(x,t) \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$ and

$$\eta(u(x,t), v(x,t))_t + q(u(x,t), v(x,t))_x \leqslant 0$$

in the sense of distributions for any convex entropy $\eta(u, v)$ of system (1.1), where $q(u, v)$ is the entropy flux associated with $\eta(u, v)$.

2. Proof of Theorem 1.1

First consider the Cauchy problem for the related parabolic system:

$$\left.\begin{aligned} u_t^\varepsilon + \tfrac{2}{3}((u^\varepsilon)^2 + v^\varepsilon)_x &= \varepsilon u_{xx}^\varepsilon, \\ v_t^\varepsilon + \tfrac{2}{3}(u^\varepsilon v^\varepsilon)_x &= \varepsilon v_{xx}^\varepsilon \end{aligned}\right\} \tag{2.1}$$

with bounded measurable initial data

$$(u^\varepsilon(x,0), v^\varepsilon(x,0)) = (u_0^\varepsilon(x), v_0^\varepsilon(x)), \tag{2.2}$$

where $(u_0^\varepsilon(x), v_0^\varepsilon(x)) = (u_0(x), v_0(x) + \varepsilon) * J^\varepsilon$ and J^ε is a mollifier.

Lemma 2.1. *Let $(u^\varepsilon(x,t), v^\varepsilon(x,t)) \in C^\infty(\mathbb{R} \times (0, T])$ be a local solution of the Cauchy problem (2.1), (2.2) and let $v_0(x) \geqslant 0$. Then $v^\varepsilon(x,t) \geqslant c(\varepsilon, t) > 0$, where the positive function $c(\varepsilon, t)$ could tend to 0 as $\varepsilon \to 0$ or $t \to \infty$.*

Proof. We rewrite the second equation of system (2.1) as follows:

$$W_t + \tfrac{2}{3}uW_x + \tfrac{2}{3}u_x = \varepsilon(W_{xx} + W_x^2),$$

where we omit the superscript ε and $W = \log v$. Then

$$W_t = \varepsilon W_{xx} + \varepsilon\left(W_x - \frac{u}{3\varepsilon}\right)^2 - \frac{2}{3}u_x - \frac{u^2}{9\varepsilon}. \tag{2.3}$$

The solution W of (2.3) with initial data $W_0(x) = \log v_0^\varepsilon(x)$ can be represented by a Green function

$$G^\varepsilon(x-y,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \exp\left\{-\frac{(x-y)^2}{4\varepsilon t}\right\}$$

as follows:

$$W = \int_{-\infty}^{\infty} G^\varepsilon(x-y,t)W_0(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty}\left[\varepsilon\left(w_x - \frac{u}{3\varepsilon}\right)^2 - \frac{2}{3}u_x - \frac{u^2}{9\varepsilon}\right]G^\varepsilon(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s. \tag{2.4}$$

Since $v_0^\varepsilon(x) \geqslant \varepsilon$ and

$$\int_{-\infty}^{\infty} G^\varepsilon(x-\xi,t)\,\mathrm{d}\xi = 1, \qquad \int_0^t \int_{-\infty}^{+\infty} |G_y^\varepsilon(x-y,t-s)|\,\mathrm{d}y\,\mathrm{d}s = 2\sqrt{\frac{t}{\pi\varepsilon}}, \quad t > 0,$$

it follows from (2.4) that

$$\begin{aligned} W &\geqslant \log\varepsilon + \int_0^t \int_{-\infty}^{\infty}\left(-\frac{2}{3}u_x - \frac{u^2}{9\varepsilon}\right)G^\varepsilon(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \log\varepsilon + \int_0^t \int_{-\infty}^{\infty}\left(\frac{2}{3}uG_y^\varepsilon(x-y,t-s) - \frac{u^2}{4\varepsilon}G^\varepsilon(x-y,t-s)\right)\mathrm{d}y\,\mathrm{d}s \\ &\geqslant \log\varepsilon - C_1\sqrt{\frac{t}{\varepsilon}} - \frac{C_2 t}{\varepsilon} \\ &= -C(\varepsilon,t) \\ &> -\infty. \end{aligned}$$

Thus, $v^\varepsilon(x,t)$ has a positive lower bound $c(\varepsilon,t)$. □

Lemma 2.2. *The viscous solutions $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ of the Cauchy problem (2.1), (2.2) exist globally and satisfy $|u^\varepsilon(x,t)| \leqslant M$, $0 < c(\varepsilon,t) \leqslant v^\varepsilon(x,t) \leqslant M$, where M is a positive constant independent of ε and $c(\varepsilon,t)$ is a positive function, which could tend to 0 as $\varepsilon \to 0$ or $t \to \infty$.*

Proof. Multiply system (2.1) by $\nabla w(u^\varepsilon, v^\varepsilon)$ and $\nabla z(u^\varepsilon, v^\varepsilon)$, respectively. Then, by simple calculations, we have

$$\left.\begin{aligned} w_t + \lambda_2 w_x &= \varepsilon w_{xx} - \varepsilon(w_{uu}(u_x^\varepsilon)^2 + w_{uv}u_x^\varepsilon v_x^\varepsilon + w_{vv}(v_x^\varepsilon)^2) = \varepsilon w_{xx} - \frac{\varepsilon}{D}w_x z_x, \\ z_t + \lambda_1 z_x &= \varepsilon z_{xx} - \varepsilon(w_{uu}(u_x^\varepsilon)^2 + w_{uv}u_x^\varepsilon v_x^\varepsilon + w_{vv}(v_x^\varepsilon)^2) = \varepsilon z_{xx} + \frac{\varepsilon}{D}w_x z_x. \end{aligned}\right\} \tag{2.5}$$

Thus, we obtain $-M_1 \leqslant z(u^\varepsilon, v^\varepsilon) \leqslant w(u^\varepsilon, v^\varepsilon) \leqslant M_1$ by applying the maximum principle to (2.5), where M_1 is a suitable large constant independent of ε. This shows that the region $\Sigma = \{(u,v) : -M_1 \leqslant z(u,v) \leqslant w(u,v) \leqslant M_1,\ v \geqslant 0\}$ is an invariant region by Lemma 2.1, so we get the *a priori* estimates

$$|u^\varepsilon(x,t)| \leqslant M, 0 < c(\varepsilon,t) \leqslant v^\varepsilon(x,t) \leqslant M, \tag{2.6}$$

and hence obtain the proof of Lemma 2.2 by [**6**, Theorem 1.0.2]. □

In view of the estimates (2.6), there exists a subsequence $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ of the viscous solutions such that

$$(u(x,t), v(x,t)) = w^\star - \lim(u^\varepsilon(x,t), v^\varepsilon(x,t)), \quad v(x,t) \geqslant 0.$$

We shall show that the weak-$\star$ convergence is indeed pointwise almost-everywhere convergence.

Now we use the kinetic formulation to give three families of entropy–entropy flux pairs of system (1.1). Our idea is motivated partly by the argument in [**2**, **10**], where some other hyperbolic systems are treated by the kinetic formulation. Let $\rho = D^3$, $u = u$. Then for smooth solutions, system (1.1) is equivalent to the following:

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + (\tfrac{1}{2}u^2 + \tfrac{1}{6}\rho^{2/3})_x &= 0, \end{aligned}\right\} \tag{2.7}$$

which is just the Euler equations of one-dimensional, compressible fluid flow with $\gamma = \frac{5}{3}$. Any entropy–entropy flux pair $(\bar\eta(\rho,u), \bar q(\rho,u))$ of system (2.7) satisfies the additional system

$$\bar q_\rho = u\bar\eta_\rho + \tfrac{1}{9}\rho^{-1/3}\bar\eta_u, \qquad \bar q_u = \rho\bar\eta_\rho + u\bar\eta_u. \tag{2.8}$$

Eliminating $\bar q$ from (2.8), we have the entropy equation

$$\bar\eta_{\rho\rho} = \tfrac{1}{9}\rho^{-4/3}\bar\eta_{uu}. \tag{2.9}$$

It is well known that there exists a non-positive bounded measure $m(x,t,\xi)$ such that the fundamental solution $G(x,t,\xi) = G(\rho, \xi - u)$ of Equation (2.9) satisfies

$$\partial_t G + \partial_x[\theta\xi + (1-\theta)u]G = \partial_{\xi\xi} m(x,t,\xi),$$

especially $m(x,t,\xi) = 0$ for smooth solutions of system (2.7), and the entropies of system (2.7) are generated by the following fundamental solutions (see [**2**, **4**, **5**, **7**]):

$$\begin{aligned} G_0(\rho, \xi - u) &= [(w-\xi)(\xi - z)]_+, \\ G_+(\rho, \xi - u) &= (\xi - z)(\xi - w)_+, \\ G_-(\rho, \xi - u) &= (w - \xi)(z - \xi)_+, \end{aligned}$$

since $\theta = \frac{1}{2}(\gamma - 1) = \frac{1}{3}$ and $\lambda = (3-\gamma)/2(\gamma-1) = 1$. Here we use the notation $x_+ = \max\{0, x\}$.

Precisely, one family of weak entropy of system (2.7) is given by

$$\bar{\eta}^0(\rho,u)=\int_R g(\xi)G_0(\rho,\xi-u)\,\mathrm{d}\xi,$$

and the corresponding weak entropy flux is

$$\bar{q}^0(\rho,u)=\int_R g(\xi)\frac{\xi+2u}{3}G_0(\rho,\xi-u)\,\mathrm{d}\xi.$$

Two families of strong entropies of system (2.7) are given as follows:

$$\bar{\eta}^\pm(\rho,u)=\int_R g(\xi)G_\pm(\rho,\xi-u)\,\mathrm{d}\xi,$$

and the strong entropy fluxes $\bar{q}^\pm$ associated with $\bar{\eta}^\pm$ are

$$\bar{q}^\pm(\rho,u)=\int_R g(\xi)\frac{\xi+2u}{3}G_\pm(\rho,\xi-u)\,\mathrm{d}\xi,$$

where $g(\xi)$ is a non-negative smooth function with a compact support set in $(-\infty,\infty)$.

Because $g(\xi)\in C_0^\infty(\mathbb{R})$, the Lebesgue integrals in the kinetic formulations make sense, and hence the entropies given above touch the singular point $(0,0)$ and are compatible with this singularity. More precisely, we have the following.

Lemma 2.3. *Let $\eta^0(u,v)=\bar{\eta}_0(\rho,u)$, $\eta^1(u,v)=\bar{\eta}^+(\rho,u)$ and $\eta^2(u,v)=\bar{\eta}^-(\rho,u)$. Then $\eta^i(u^\varepsilon,v^\varepsilon)_t+q^i(u^\varepsilon,v^\varepsilon)_x$, $i=0,1,2$, are compact in H^{-1} with respect to the viscous solutions $(u^\varepsilon,v^\varepsilon)$ obtained in Lemma 2.2, where $q^i(u,v)$ are the entropy fluxes associated with $\eta^i(u,v)$.*

Proof. Let $\tau=\xi-w$. Then

$$\begin{aligned}\bar{\eta}^+(\rho,u)&=\int_w^\infty g(\xi)(\xi-z)(\xi-w)\,\mathrm{d}\xi\\&=\int_0^\infty g(\tau+w)(\tau+2\rho^{1/3})\tau\,\mathrm{d}\tau,\\ \bar{\eta}^+_\rho&=\frac{\rho^{-2/3}}{3}\int_0^\infty g'(\tau+w)(\tau+2\rho^{1/3})\tau\,\mathrm{d}\tau+\frac{2\rho^{-2/3}}{3}\int_0^\infty g(\tau+w)\tau\,\mathrm{d}\tau,\end{aligned}$$

since $w(\rho,u)=u+\rho^{1/3}$, $z(\rho,u)=u-\rho^{1/3}$. Thus,

$$\bar{\eta}^+_\rho=O(\rho^{-2/3})\quad\text{as }\rho\to0.$$

Integrating by parts, we obtain that

$$\int_0^\infty g'(\tau+w)(\tau+2\rho^{1/3})\tau\,\mathrm{d}\tau=-\int_0^\infty g(\tau+w)(2\tau+2\rho^{1/3})\,\mathrm{d}\tau$$

and hence

$$\bar{\eta}^+_\rho=-\frac{2\rho^{-1/3}}{3}\int_0^\infty g(\tau+w)\,\mathrm{d}\tau\leqslant0.\tag{2.10}$$

https://doi.org/10.1017/S0013091507000697 Published online by Cambridge University Press

By the chain rule, we have

$$\eta^1_{uu} = \bar{\eta}^+_{\rho\rho}\rho_u^2 + 2\bar{\eta}^+_{\rho u}\rho_u + \bar{\eta}^+_{\rho}\rho_{uu} + \bar{\eta}^+_{uu},$$
$$\eta^1_{uv} = \bar{\eta}^+_{\rho\rho}\rho_u\rho_v + \bar{\eta}^+_{\rho u}\rho_v + \bar{\eta}^+_{\rho}\rho_{uv},$$
$$\eta^1_{vv} = \bar{\eta}^+_{\rho\rho}\rho_v^2 + \bar{\eta}^+_{\rho}\rho_{vv}.$$

It is not difficult to see that $\bar{\eta}^+(\rho, u)$ is smooth on the variable u, so by the entropy Equation (2.9), $\bar{\eta}^+_{\rho\rho}\rho_u^2 = \bar{\eta}^+_{uu}(u^2/(u^2+4v))$ is bounded; similarly, $\bar{\eta}^+_{\rho\rho}\rho_v^2 v$ and $\bar{\eta}^+_{\rho\rho}\rho_u\rho_v\sqrt{v}$ are also bounded. Since $\bar{\eta}^+_{\rho} = O(\rho^{-2/3})$ as $\rho \to 0$, we can see that $\bar{\eta}^+_{\rho u}\rho_u$ and $\bar{\eta}^+_{\rho u}\rho_v\sqrt{v}$ are both bounded by direct calculations.

Obviously, system (1.1) has a convex entropy

$$\eta^\star(u, v) = \frac{u^2}{2} + \int_0^v \log v \,\mathrm{d}v$$

and the corresponding entropy flux

$$q^\star(u, v) = \frac{4u^3}{9} + \frac{2}{3}uv\log v.$$

We multiply system (2.1) by $\nabla\eta^\star(u^\varepsilon, v^\varepsilon)$ to obtain

$$\eta^\star_t + q^\star_x = \varepsilon\eta^\star_{xx} - \varepsilon(\eta^\star_{uu}(u^\varepsilon)^2_x + 2\eta^\star_{uv}u^\varepsilon_x v^\varepsilon_x + \eta^\star_{vv}(v^\varepsilon_x)^2) = \varepsilon\eta^\star_{xx} - \varepsilon\biggl((u^\varepsilon_x)^2 + \frac{(v^\varepsilon_x)^2}{v^\varepsilon}\biggr).$$

Hence, $\varepsilon(u^\varepsilon_x)^2$ and $\varepsilon(v^\varepsilon_x)^2/v^\varepsilon$ are bounded in L^1_{loc}. For simplicity, we will drop the superscript ε.

Therefore, multiplying system (2.1) by $\nabla\eta^1(u, v)$, we have

$$\begin{aligned}\eta^1_t + q^1_x &= \varepsilon\eta^1_{xx} - \varepsilon(\eta^1_{uu}u_x^2 + 2\eta^1_{uv}u_x v_x + \eta^1_{vv}v_x^2)\\ &= \varepsilon\eta^1_{xx} - \varepsilon A(\rho, u)u_x^2 - \varepsilon B(\rho, u, v)u_x\frac{v_x}{\sqrt{v}} - \varepsilon C(\rho, u, v)v_x^2 - \varepsilon I,\end{aligned} \tag{2.11}$$

where $A(\rho, u)$, $B(\rho, u, v)$ and $C(\rho, u, v)$ are bounded and

$$I = \bar{\eta}^+_{\rho}(\rho_{uu}u_x^2 + 2\rho_{uv}u_x v_x + \rho_{vv}v_x^2) = 3\bar{\eta}^+_{\rho}(u^2+4v)^{-1/2}((u^2+4v)u_x^2 + (uu_x + 2v_x)^2)$$

is non-positive from (2.10).

Multiplying equality (2.11) by a test function ϕ, where $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ satisfies $\phi_K = 1$, $0 \leqslant \phi \leqslant 1$, $K \subset S = \operatorname{supp}\phi$ is an arbitrary compact set and integrating over $\mathbb{R} \times \mathbb{R}^+$, we then have

$$\begin{aligned}\iint_S 2\varepsilon|I|\phi \,\mathrm{d}x\,\mathrm{d}t &= -\iint_S \varepsilon\eta^1\phi_{xx} - \varepsilon\biggl[A(\rho, u)u_x^2 + B(\rho, u, v)u_x\frac{v_x}{\sqrt{v}} + C(\rho, u, v)\frac{v_x^2}{v}\biggr]\phi \,\mathrm{d}x\,\mathrm{d}t\\ &\qquad - \iint_S \eta^1\phi_t + q^1\phi_x \,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant M(\phi).\end{aligned}$$

https://doi.org/10.1017/S0013091507000697 Published online by Cambridge University Press

Thus, $\varepsilon(\eta^1_{uu}u_x^2 + 2\eta^1_{uv}u_x v_x + \eta^1_{vv}v_x^2)$ is bounded in L^1_{loc} and hence compact in $W^{-1,\alpha}$ for a constant $\alpha \in (1, 2)$. Since

$$|\eta^1_x| = |\bar{\eta}^+_\rho(\rho_u u_x + \rho_v v_x) + \bar{\eta}^+_u u_x| \leqslant C\left(|u_x| + \left|\frac{v_x}{\sqrt{v}}\right|\right),$$

the part η^1_{xx} is compact in H^{-1}. Noting the boundedness of $\eta^1_t + q^1_x$ in $W^{-1,\infty}$, we obtain the H^{-1} compactness of $\eta^1_t + q^1_x$ by Murat's lemma (see [**9**, **11**]). A similar treatment gives the proof for η^2. Since

$$\bar{\eta}^0(\rho, u)_\rho = \int_{-1}^{1} (g(u + \rho^{1/3}s) + \frac{\rho^{1/3}}{3} g'(u + \rho^{1/3}s))(1 - s^2)\,\mathrm{d}s$$

(see [**7**]), we can easily obtain the H^{-1} compactness of $\eta^0_t + q^0_x$ by similar treatment. Thus, we obtain the proof of Lemma 2.3. □

Finally, we use a new technique to reduce Young measures. Since the viscous solutions $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ are uniformly bounded in L^∞ space, by using the representation theorem of Young measures we may consider the family of compactly supported probability measures $\nu_{x,t}$. Without loss of generality we may fix $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and consider only one measure, ν. We apply the measure equation to obtain

$$\begin{aligned}
&\int g(\xi_1)\overline{G_i(\xi_1)}\,\mathrm{d}\xi_1 \int h(\xi_2)\overline{\frac{\xi_2 + 2u}{3} G_j(\xi_2)}\,\mathrm{d}\xi_2 \\
&\quad - \int h(\xi_2)\overline{G_j(\xi_2)}\,\mathrm{d}\xi_2 \int g(\xi_1)\overline{\frac{\xi_1 + 2u}{3} G_i(\xi_1)}\,\mathrm{d}\xi_1 \\
&\qquad = \int g(\xi_1)h(\xi_2)\overline{G_i(\xi_1)\frac{\xi_2 + 2u}{3} G_j(\xi_2)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
&\qquad\quad - \int g(\xi_1)h(\xi_2)\overline{G_i(\xi_1)\frac{\xi_1 + 2u}{3} G_j(\xi_2)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,
\end{aligned}$$

where G_i is any one of the three fundamental solutions. Here and below we use the bar to indicate the usual integration with respect to the Young measure; for example, $\overline{G(\xi)} = \int G(\rho, \xi - u)\,\mathrm{d}\nu(\rho, u)$.

The above equality holds for any non-negative smooth functions g, h with compact support sets and this yields

$$\begin{aligned}
&\overline{G_i(\xi_1)}\;\overline{\frac{\xi_2 + 2u}{3} G_j(\xi_2)} - \overline{G_j(\xi_2)}\;\overline{\frac{\xi_1 + 2u}{3} G_i(\xi_1)} \\
&\qquad = \overline{G_i(\xi_1)\frac{\xi_2 + 2u}{3} G_j(\xi_2)} - \overline{G_i(\xi_1)\frac{\xi_1 + 2u}{3} G_j(\xi_2)} \\
&\qquad = \frac{\xi_2 - \xi_1}{3}\,\overline{G_i(\xi_1)G_j(\xi_2)}.
\end{aligned} \tag{2.12}$$

In fact, if we choose

$$g_n(x) \in C_0^\infty\left(\xi_1 - \frac{1}{n}, \xi_1 + \frac{1}{n}\right)$$

https://doi.org/10.1017/S0013091507000697 Published online by Cambridge University Press

such that $g_n \geqslant 0$, $\int_R g_n(x)\,\mathrm{d}x = 1$, and

$$h_n(x) \in C_0^\infty\left(\xi_2 - \frac{1}{n}, \xi_2 + \frac{1}{n}\right)$$

such that $h_n \geqslant 0$, $\int_R h_n(x)\,\mathrm{d}x = 1$, then letting $n \to \infty$, we obtain the desired result (2.12). It is worth mentioning that the equality (2.12) plays a crucial role later in the present paper, which is exactly [**6**, (47)] with $\theta = \frac{1}{3}$.

Let

$$z_- = \inf_{(\rho,u)\in \operatorname{supp}\nu} z(\rho,u), \qquad z_+ = \sup_{(\rho,u)\in \operatorname{supp}\nu} z(\rho,u),$$
$$w_- = \inf_{(\rho,u)\in \operatorname{supp}\nu} w(\rho,u), \qquad w_+ = \sup_{(\rho,u)\in \operatorname{supp}\nu} w(\rho,u).$$

If we choose $G_i = G_j = G_+$ and $\xi_1, \xi_2 \in (w_-, +\infty)$, then we may rewrite (2.12) as

$$\frac{1}{2}\left[\frac{\overline{G_+(\xi_1)G_+(\xi_2)}}{\overline{G_+(\xi_1)}\,\overline{G_+(\xi_2)}} - 1\right] = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}} - \frac{\overline{uG_+(\xi_1)}}{\overline{G_+(\xi_1)}}\right]. \tag{2.13}$$

Similarly, choosing $G_i = G_j = G_-$ and $\xi_1, \xi_2 \in (-\infty, z_+)$, we have

$$\frac{1}{2}\left[\frac{\overline{G_-(\xi_1)G_-(\xi_2)}}{\overline{G_-(\xi_1)}\,\overline{G_-(\xi_2)}} - 1\right] = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_-(\xi_2)}}{\overline{G_-(\xi_2)}} - \frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}}\right]. \tag{2.14}$$

As in [**5**], we define

$$f_0^\pm(\xi) = \frac{G_\pm(\xi) - \overline{G_\pm(\xi)}}{\overline{G_\pm(\xi)}}$$

so that (2.13) and (2.14) take the equivalent form:

$$\tfrac{1}{2}\overline{f_0^\pm(\xi_1)f_0^\pm(\xi_2)} = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_\pm(\xi_2)}}{\overline{G_\pm(\xi_2)}} - \frac{\overline{uG_\pm(\xi_1)}}{\overline{G_\pm(\xi_1)}}\right]. \tag{2.15}$$

Let $I_\alpha(\xi)$ be a non-negative, smooth function with compact support set in $(-1/\alpha, 1/\alpha)$ satisfying $I_\alpha(\xi) \to 1$ as $\alpha \to 0^+$, let $\psi_\alpha(\xi) \geqslant 0$ be a unit mass mollifier and define $f_\alpha^\pm = (f_0^\pm I_\alpha) * \psi_\alpha$. Then, from (2.15), we have that

$$\tfrac{1}{2}\overline{f_\alpha^\pm(\xi_1)f_\alpha^\pm(\xi_2)} = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_\pm(\xi_2)}}{\overline{G_\pm(\xi_2)}} - \frac{\overline{uG_\pm(\xi_1)}}{\overline{G_\pm(\xi_1)}}\right] I_\alpha(\xi_1)I_\alpha(\xi_2) * \psi_\alpha(\xi_1) * \psi_\alpha(\xi_2).$$

Owing to the boundedness of the left-hand side and the smoothness of the right-hand side, we may now take $\xi_2 = \xi_1 = \xi$, which shows that

$$\tfrac{1}{2}\overline{(f_\alpha^\pm(\xi)^2)} = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_\pm(\xi_2)}}{\overline{G_\pm(\xi_2)}} - \frac{\overline{uG_\pm(\xi_1)}}{\overline{G_\pm(\xi_1)}}\right] I_\alpha(\xi_1)I_\alpha(\xi_2) * \psi_\alpha(\xi_1) * \psi_\alpha(\xi_2)\bigg|_{\xi_2=\xi_1=\xi}. \tag{2.16}$$

If we now let $\alpha \to 0^+$, then the left-hand side of (2.16) yields a positive measure, whereas the right-hand side tends to $(\partial/\partial\xi)(\overline{uG_\pm(\xi)}/\overline{G_\pm(\xi)})$. Thus, $\overline{uG_+(\xi)}/\overline{G_+(\xi)}$ and $\overline{uG_-(\xi)}/\overline{G_-(\xi)}$ are non-decreasing in (w_-, ∞) and $(-\infty, z_+)$, respectively. In a similar fashion, we find that $\overline{uG_0(\xi)}/\overline{G_0(\xi)}$ is non-decreasing in (z_-, w_+).

Case 1 ($z_+ \leqslant w_-$). If $z_+ \leqslant w_-$, then we choose $G_i = G_+$, $G_j = G_0$ and $\xi_1 = \xi_2 = \xi$ in (2.12) to obtain

$$\overline{uG_+(\xi)}\ \overline{G_0(\xi)} = \overline{uG_0(\xi)G_+(\xi)}.$$

Hence,

$$\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}}$$

for $\xi \in (w_-, w_+)$. In particular,

$$\lim_{\xi \to w_- + 0} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_0(w_-)}}{\overline{G_0(w_-)}}.$$

Similarly, we have

$$\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}}$$

for $\xi \in (z_-, z_+)$. In particular,

$$\lim_{\xi \to z_+ - 0} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \frac{\overline{uG_0(z_+)}}{\overline{G_0(z_+)}}.$$

Therefore,

$$\begin{aligned}\bar{u} = \lim_{\xi \to \infty} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} &\geqslant \lim_{\xi \to w_- + 0} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_0(w_-)}}{\overline{G_0(w_-)}} \geqslant \frac{\overline{uG_0(z_+)}}{\overline{G_0(z_+)}} \\ &\geqslant \lim_{\xi \to z_+ - 0} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} \geqslant \lim_{\xi \to -\infty} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \bar{u}\end{aligned}$$

and hence $\overline{uG_+(\xi)}/\overline{G_+(\xi)}$ and $\overline{uG_-(\xi)}/\overline{G_-(\xi)}$ are constant in (w_-, ∞) and $(-\infty, z_+)$, respectively, by the monotonicity of the two functions.

Using the similar treatment in [**5**], we can finish the reduction of the Young measure in this case. Indeed, it follows from (2.16) that $\overline{(f_\alpha^\pm(\xi))^2} = 0$. Hence, $f_\alpha^\pm(\xi)$ vanishes on the support of ν and, in particular, by letting $\alpha \to 0$, so do $f_0^\pm(\xi)$ and

$$f_0^\pm(\xi) = \frac{G(\rho, \xi - u)}{\overline{G(\xi)}} - 1 = 0, \quad (\rho, u) \in \operatorname{supp} \nu.$$

This shows that the Young measure must be a Dirac mass.

Case 2 ($z_+ > w_-$). If $z_+ > w_-$, then we choose $G_i = G_+$, $G_j = G_-$ and $\xi_2 = \xi_1 = \xi$ in (2.12) to obtain $\overline{uG_+(\xi)}\ \overline{G_-(\xi)} = \overline{uG_-(\xi)G_+(\xi)}$. Hence,

$$\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}$$

for $\xi \in (w_-, z_+)$. In particular,

$$\lim_{\xi \to w_- + 0} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_-(w_-)}}{\overline{G_-(w_-)}}, \qquad \lim_{\xi \to z_+ - 0} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \frac{\overline{uG_+(z_+)}}{\overline{G_+(z_+)}}.$$

Therefore,

$$\bar{u} = \lim_{\xi\to\infty} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} \geqslant \frac{\overline{uG_+(z_+)}}{\overline{G_+(z_+)}} = \lim_{\xi\to z_+-0} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}$$
$$\geqslant \lim_{\xi\to w_-+0} \frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_-(w_-)}}{\overline{G_-(w_-)}} \geqslant \lim_{\xi\to-\infty} \frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \bar{u}$$

and hence $\overline{uG_+(\xi)}/\overline{G_+(\xi)}$ and $\overline{uG_-(\xi)}/\overline{G_-(\xi)}$ are constant in (w_-,∞) and $(-\infty,z_+)$, respectively, by the monotonicity of the two functions. Thus, the Young measure ν is also a Dirac mass from the proof of case 1. This is contrary to the assumption $z_+ > w_-$ since $w \geqslant z$. Thus, only case 1, i.e. $z_+ \leqslant w_-$ is permitted, and hence ν is a Dirac mass. According to the compensated compactness method (see [**1**]), $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ converges to $(u(x,t), v(x,t))$ almost everywhere, which is a global bounded entropy solution to the Cauchy problem (1.1), (1.2). Thus, we complete the proof of Theorem 1.1.

References

1. R. J. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Ration. Mech. Analysis* **82** (1983), 27–70.
2. F. J. James, Y.-J. Peng and B. Perthame, Kinetic formulation for chromatography and some other hyperbolic systems, *J. Math. Pures Appl.* **74** (1995), 367–385.
3. A. Y. LeRoux, Numerical stability for some equations of gas dynamics, *Math. Comput.* **37** (1981), 435–446.
4. P. L. Lions, B. Perthame and P. E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Commun. Pure Appl. Math.* **49** (1996), 599–638.
5. P. L. Lions, B. Perthame and E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-systems, *Commun. Math. Phys.* **163** (1994), 415–431.
6. Y.-G. Lu, *Hyperbolic conservation laws and the compensated compactness method*, Volume 128 (Chapman and Hall/CRC Press, New York, 2002).
7. Y.-G. Lu, Existence of global entropy solutions to a nonstrictly hyperbolic system, *Arch. Ration. Mech. Analysis* **178** (2005), 287–299.
8. Y.-G. Lu, I. Mantilla and L. Rendon, Convergence of approximated solutions to a nonstrictly hyperbolic system, *Adv. Nonlin. Studies* **1** (2001), 65–79.
9. F. Murat, Compacité par compensation, *Annali Scuola Norm. Sup. Pisa* **5** (1978), 489–507.
10. B. Perthame, *Kinetic formulations* (Oxford University Press, 2002).
11. T. Tartar, Compensated compactness and applications to partial differential equations, in *Proc. Heriot-Watt Symp. Nonlinear Analysis and Mechanics* (ed. R. J. Knops), Research Notes in Mathematics, Volume 4 (Pitman, London, 1979).