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APPLICATION OF THE BRUHAT-TITS TREE OF $SU_3(h)$ TO SOME $\tilde{A}_2$ GROUPS

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Abstract

Let K be a nonarchimedean local field, let L be a separable quadratic extension of K, and let h denote a nondegenerate sesquilinear form on L^3. The Bruhat-Tits building associated with $SU_3(h)$ is a tree. This is applied to the study of certain groups acting simply transitively on vertices of the building associated with $SL(3, F)$, $F = \mathbb{Q}_3$ or $\mathbb{F}_3((X))$.

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1. Introduction

Let L be a nonarchimedean local field, and let Δ_L denote the Bruhat-Tits building of type $\tilde{A}_2$ associated with $SL(3, L)$ (see, for example, [Ro, §9.2], [Br, §V.8] or [Ste]).

Now suppose that L is a separable quadratic extension of a local field K. Let q be the order of the residual field of K. Let h denote a nondegenerate sesquilinear form on L^3, and let $SU_3(h)$ denote the group of 3×3 matrices g of determinant 1 with entries in L which preserve h. The nontrivial Galois automorphism of L over K induces a non-type-preserving automorphism σ of Δ_L. This gives rise to a tree T, as follows. The vertex set of T is the union of two disjoint sets, Λ_0 and Λ_1, consisting, respectively, of the vertices of Δ_L fixed by σ and of the pairs of adjacent vertices of Δ_L interchanged by σ. The edges of T correspond to the chambers of Δ_L fixed by σ. That is, $v_0 \in \Lambda_0$ and $v_1 \in \Lambda_1$ are adjacent in T if the vertex of Δ_L corresponding to v_0 and the two vertices of Δ_L corresponding to v_1 form a chamber of Δ_L.

More precisely, the following result is well-known [Ti]:

THEOREM 1.1. *With the above notation, the set $\Lambda_0 \cup \Lambda_1$, together with the above adjacency relation, forms a tree T. This tree is homogeneous of degree $q+1$ when L is a ramified extension of K, and is bihomogeneous when L is an unramified extension of K, each $v \in \Lambda_0$ having q^3+1 neighbours, and each $v \in \Lambda_1$ having $q+1$ neighbours. It is isomorphic to the Bruhat-Tits building associated with $SU_3(h)$.*

This theorem is well-known (though we do not know of a complete proof in the literature) so we shall not prove it here. In Section 2, we recall the well-known concrete description of Δ_L in terms of lattices. This gives us a lattice description of T. In Section 3, we use this description to help us obtain a better realization of certain subgroups Γ of $PGL(3, F)$, $F = \mathbb{Q}_3$ or $\mathbb{F}_3((X))$, described in [CMSZ] which act simply transitively on the vertices of Δ_F. In particular, certain pairs of these groups Γ are commensurable in $PGL(3, F)$, and the commensurability index is explained in terms of the groups' action on the tree T (for suitable K, L and h). Note that the groups Γ have property (T) [CMS], and so must have subgroups of index at most 2 which fix a vertex of T (see [HV, Proposition 6.4]). Location of this vertex is an important step in obtaining the realizations referred to above.

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2. The tree of $SU_3(h)$

For a local field L, denote the valuation on L by $\omega : L \to \mathbb{Z} \cup \{\infty\}$, and let $\mathfrak{o}_L = \{x \in L : \omega(x) \geq 0\}$ be the valuation ring of L. Let $\mathfrak{p}_L = \{x \in L : \omega(x) > 0\}$ be the maximal ideal of $\mathfrak{o}_L$, and let $\bar{L} = \mathfrak{o}_L/\mathfrak{p}_L$ denote the residual field of L. Let $\pi = \pi_L$ be a generator of $\mathfrak{p}_L$. We assume that ω is normalized so that $\omega(L^\times) = \mathbb{Z}$, and hence $\omega(\pi) = 1$.

A lattice in L^3 is a subset $\mathscr{L}$ of L^3 of the form

$$\mathscr{L} = \{a_1 v_1 + a_2 v_2 + a_3 v_3 : a_1, a_2, a_3 \in \mathfrak{o}_L\}, \tag{2.1}$$

where $\{v_1, v_2, v_3\}$ is a basis of L^3. Let **Lat** denote the set of lattices. Two lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ are called *equivalent* if $\mathscr{L}_2 = t\mathscr{L}_1$ for some non-zero $t \in L$. The vertices of Δ_L consist of the equivalence classes $[\mathscr{L}]$ of lattices. Two vertices $[\mathscr{L}_1]$ and $[\mathscr{L}_2]$ are called *adjacent* if representatives $\mathscr{L}_1$ and $\mathscr{L}_2$ can be found so that $\pi\mathscr{L}_1 \subsetneqq \mathscr{L}_2 \subsetneqq \mathscr{L}_1$. A *chamber* in Δ_L consists of three vertices, any two of which are adjacent.

Taking the usual basis of L^3 in (2.1), the lattice $\mathscr{L}$ is

$$\mathscr{L}_0 = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathfrak{o}_L \right\}. \tag{2.2}$$

The group $GL(3, L)$ acts naturally on **Lat**. In fact, $GL(3, L)$ acts transitively on **Lat**, for if $\mathcal{L}$ is the lattice (2.1), and if g is the matrix whose columns are v_1, v_2 and v_3, then $\mathcal{L} = g(\mathcal{L}_0)$. The *stabilizer* $\{g \in GL(3, L) : g(\mathcal{L}_0) = \mathcal{L}_0\}$ of $\mathcal{L}_0$ in $GL(3, L)$ is $GL(3, \mathfrak{o}_L) = \{g \in GL(3, L) : g \text{ and } g^{-1} \text{ have entries in } \mathfrak{o}_L\}$. We define the *type* $\tau([\mathcal{L}])$ of a vertex $[\mathcal{L}]$ to be $\omega(\det(g)) \bmod 3$ if $\mathcal{L} = g(\mathcal{L}_0)$.

If we fix a vertex v_1 of Δ_L and a lattice $\mathcal{L}_1$ in the class v_1, then the vertices $v_2 = [\mathcal{L}_2]$ adjacent to v_1 are in one to one correspondence ($\mathcal{L}_2 \leftrightarrow \mathcal{L}_2/\pi\mathcal{L}_1$) with the nonzero proper vector subspaces of the 3-dimensional vector space $\mathcal{L}_1/\pi\mathcal{L}_1$ over $\bar{L}$. Thus if $q_L = |\bar{L}|$, there are $q_L^2 + q_L + 1$ v_2's corresponding to the 2-dimensional subspaces of $\mathcal{L}_1/\pi\mathcal{L}_1$, and $q_L^2 + q_L + 1$ v_2's corresponding to the 1-dimensional subspaces of $\mathcal{L}_1/\pi\mathcal{L}_1$. These neighbours of v_1 form a projective plane, with incidence being adjacency.

Let $A \subset \mathfrak{o}_L$ denote a set of representatives of $\bar{L}$, that is, a set such that $a \mapsto a + \mathfrak{p}_L$ is a bijection $A \to \bar{L}$. We shall assume that $0 \in A$. Let us take $\mathcal{L}_1$ to be the $\mathcal{L}_0$ of (2.2). Then the $q_L^2 + q_L + 1$ $\mathcal{L}_2$'s corresponding to the 2-dimensional subspaces of $\mathcal{L}_0/\pi\mathcal{L}_0$ are the lattices $g(\mathcal{L}_0)$, where for $a, b \in A$,

$$(2.3) \qquad g = \begin{pmatrix} \pi & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & a \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix}.$$

The $q_L^2 + q_L + 1$ $\mathcal{L}_2$'s corresponding to the 1-dimensional subspaces of $\mathcal{L}_0/\pi\mathcal{L}_0$ are the lattices $g(\mathcal{L}_0)$, where for $a, b \in A$,

$$(2.4) \qquad g = \begin{pmatrix} 1 & 0 & 0 \\ a & \pi & 0 \\ b & 0 & \pi \end{pmatrix}, \quad g = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & \pi \end{pmatrix} \quad \text{or} \quad g = \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now suppose that L is a separable quadratic extension of a local field K, and that the valuation on K is the restriction of the valuation ω on L. Let $\mathfrak{o}_K$, $\mathfrak{p}_K$, $\bar{K}$ and π_K be defined as for L, and let $q = |\bar{K}|$.

Consider the natural map $x + \mathfrak{p}_K \mapsto x + \mathfrak{p}_L$ embedding $\bar{K}$ into $\bar{L}$. There are two cases (see, for example, [Cas, p. 127]): either L is an *unramified* extension of K — this means that $\omega(K^\times) = \mathbb{Z}$, and that $\bar{L}$ is a degree 2 extension of $\bar{K}$ (so that $q_L = |\bar{L}| = q^2$); or L is an *ramified* extension of K — this means that $\omega(K^\times) = 2\mathbb{Z}$, and that the above embedding is an isomorphism $\bar{K} \to \bar{L}$.

Let $x \mapsto \bar{x}$ denote the non-trivial Galois automorphism of L over K. As the extension to L of the valuation on K is unique, $\omega(\bar{x}) = \omega(x)$ for all $x \in L$.

Let h denote a *nondegenerate sesquilinear form* on L^3, that is, $h : L^3 \times L^3 \to L$ is a map such that

(a) $x \mapsto h(x, y)$ is a linear (over L), for each fixed $y \in L^3$;

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(b) $h(y, x) = \overline{h(x, y)}$ for each $x, y \in L^3$;

(c) if $y \neq 0$, the linear map in (a) is not the zero map.

If $\{v_1, v_2, v_3\}$ is a basis of L^3, and if H is the 3×3 matrix with (i, j)-th entry $h(v_j, v_i)$, then we can write $h(x, y) = y^* H x$, where x and y are the coordinate column vectors of x and y with respect to the basis, and where for any matrix M, M^* denotes the matrix obtained from M by applying $x \mapsto \bar{x}$ to each element of the transpose of M. Let $U(h) = U_3(h)$ denote the group of 3×3 matrices with entries in L which preserve h (equivalently, $g^* H g = H$), and let $SU(h) = SU_3(h) = \{g \in U(h) : \det(g) = 1\}$.

If $\mathcal{L} \in$ **Lat**, then

$$\mathcal{L}' = \{x \in L^3 : h(x, y) \in \mathfrak{o}_L \text{ for all } y \in \mathcal{L}\}$$

is again a lattice. For if $\mathcal{L}$ is as in (2.1) and if v_1', v_2', v_3' is the dual basis with respect to h, that is, $h(v_i, v_j') = \delta_{i,j}$, then $\mathcal{L}'$ is the $\mathfrak{o}_L$-span of v_1', v_2', v_3'. This also shows that $(\mathcal{L}')' = \mathcal{L}$. If $t \in L^\times$, then $(t\mathcal{L})' = \bar{t}^{-1}\mathcal{L}'$. Hence we may define an involution $\sigma : \Delta_L \to \Delta_L$ by $\sigma([\mathcal{L}]) = [\mathcal{L}']$. Note that σ does not preserve types. For if $\mathcal{L} = g(\mathcal{L}_0) \in$ **Lat**, then $\mathcal{L}'$ is the lattice $(g^* H)^{-1}(\mathcal{L}_0)$. Hence $\tau(\sigma([\mathcal{L}])) = -\tau([\mathcal{L}]) - \omega(\det(H)) \bmod 3$. Because $\mathcal{L}_1 \subset \mathcal{L}_2$ implies that $\mathcal{L}_2' \subset \mathcal{L}_1'$, we see that σ preserves adjacency, and maps chambers to chambers.

Now suppose that σ stabilizes a chamber. Then it must fix one of the vertices (the one of type i, where $2i = -\omega(\det(H)) \bmod 3$) and interchange the other two vertices of the chamber. This motivates the following definitions:

Let

$$\Lambda_0 = \{[\mathcal{L}] : \mathcal{L} \in \textbf{Lat} \text{ and } [\mathcal{L}'] = [\mathcal{L}]\},$$

and let

$$\Lambda_1 = \{([\mathcal{M}], [\mathcal{M}']) : \mathcal{M} \in \textbf{Lat} \text{ and } [\mathcal{M}] \text{ is adjacent to } [\mathcal{M}']\}.$$

We shall call $[\mathcal{L}] \in \Lambda_0$ and $([\mathcal{M}], [\mathcal{M}']) \in \Lambda_1$ *adjacent* if $\{[\mathcal{L}], [\mathcal{M}], [\mathcal{M}']\}$ form a chamber in Δ_L (equivalently, $[\mathcal{L}]$ is adjacent in Δ_L to either of $[\mathcal{M}]$ or $[\mathcal{M}']$). The set $\Lambda_0 \cup \Lambda_1$, with this adjacency relation, forms a *graph* T, and Theorem 1.1 states, amongst other things, that T is a tree.

Notice that if $\mathcal{L} = g(\mathcal{L}_0)$, then $[\mathcal{L}] \in \Lambda_0$ if and only if $g^* H g$ is a multiple of a matrix in $GL(3, \mathfrak{o})$. When $\omega(\det(H))$ is even, $2r$ say, as happens in our applications below (and can always be arranged by multiplying h by a suitable element of K), then for each $v \in \Lambda_0$ there is a unique $\mathcal{L} \in$ **Lat** such that $v = [\mathcal{L}]$ and $\mathcal{L}' = \mathcal{L}$. Indeed, if $v = [g(\mathcal{L}_0)]$ and if $t(g^* H g) \in GL(3, \mathfrak{o})$, let $\mathcal{L} = cg(\mathcal{L}_0)$ for $c = 1/(\pi^r t \det(g))$. Similarly, if $v \in \Lambda_1$, there is a unique $\mathcal{M} \in$ **Lat** such that $v = ([\mathcal{M}], [\mathcal{M}'])$ and $\pi\mathcal{M}' \subsetneq \mathcal{M} \subsetneq \mathcal{M}'$. Thus we may work with lattices rather than lattice classes.

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3. Application to some $\tilde{A}_2$ groups

An $\tilde{A}_2$ group is a group which acts simply transitively and in a type-rotating way on the set of vertices of a thick building Δ of type $\tilde{A}_2$ (see [CMSZ]). Amongst the results of [CMSZ], all $\tilde{A}_2$ groups were found for the case when Δ was the building Δ_F, $F = \mathbb{Q}_3$ or $\mathbb{F}_3((X))$, and all were realized as co-compact lattice subgroups of $PGL(3, F)$. Two of these groups, named Groups 7.1 and 8.1, were realized in $PGL(3, \mathbb{Q}_3)$ in a rather messy way, using certain simple algebras of dimension 9 over $\mathbb{Q}(\sqrt{-23})$. Five other groups, named 4.1, ... , 4.4 and 5.1 were all realized in $PGL(3, \mathbb{Q}(\sqrt{-2})) \subset PGL(3, \mathbb{Q}_3)$, but it was not shown how to realize Group 5.1 in a way so that it was commensurable with the Groups $4.j$, though general results guaranteed that this was possible. A similar situation held for four other groups, numbered 2.1, 2.2, 3.1 and 3.2, which were all realized in $PGL(3, \mathbb{F}_3((X)))$.

In this section, we show how the building T of $SU_3(h)$ was used to understand better the relationship between these groups. First of all, some pairs of these of groups Γ, Γ' were realized as commensurable subgroups of $PGL(3, F)$ as follows. Suppose that the natural (see [CMSZ]) generators of Γ and Γ' are represented by 3×3 matrices a_j and b_j over F, respectively, $j = 0, \dots, 12$. The quantity $\mathrm{Inv}(g) = \mathrm{Trace}(g)^3/\det(g)$, for $g \in GL(3, F)$, is an invariant with respect to conjugation and multiplication by nonzero numbers. The invariants $\mathrm{Inv}(g_1^i g_2^j)$, $i, j = 0, 1, 2$, were calculated for noncommuting pairs (g_1, g_2) of short words in the a_j's and the b_k's. If $\mathrm{Inv}(g_1^i g_2^j) = \mathrm{Inv}(h_1^i h_2^j)$ for $i, j = 0, 1, 2$, where g_1 and g_2 are words in the a_j's, while h_1 and h_2 are words in the b_j's, then we sought to conjugate all the b_j's by some matrix, and multiply the b_j's by various constants so that h_1 coincided with g_1 and h_2 with g_2. This achieved, $\Gamma \cap \Gamma'$ contains the images in $PGL(3, F)$ of both g_1 and g_2, and closer investigation showed that, for some g_1, g_2, h_1 and h_2, $\Gamma \cap \Gamma'$ had small finite index in Γ and Γ'. The package MAGMA was useful for verifying this.

Groups 4.1, ... , 4.4 **and** 5.1. These groups were realized in [CMSZ] by exhibiting matrices a_j and b_j in $GL(3, \mathbb{Q}(S))$, $j = 0, \dots, 12$, where $S^2 = -2$, for Groups 4.1 and 5.1, respectively. These matrices had entries in $\mathbb{Z}[S, 1/2, 1/3]$, and preserved the form $h(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$, where $y \mapsto \bar{y}$ is the nontrivial automorphism of $\mathbb{Q}(S)$. Groups 4.1–4.4 are all normal index 4 subgroups of a group $\tilde{\Gamma}_{4.1}$ which is generated by a_6 and an element f of order 4 (see [CMSZ]). Similarly, Group 5.1 is a normal index 4 subgroup of a group $\tilde{\Gamma}_{5.1}$ which is generated by b_6 and an element f' of order 4.

We found that $\mathrm{Inv}(a_0^i a_{10}^j) = \mathrm{Inv}(b_0^i b_{10}^j)$ for $i, j = 0, 1, 2$, and could conjugate the b_j's by a suitable matrix so that the new b_0 and b_{10} coincided with a_0 and a_{10}, respectively. It turns out that also $b_4 b_5^{-1} = a_3 a_{11}^{-1}$, and MAGMA told us that a_0, a_{10} and $a_3 a_{11}^{-1}$ generate a subgroup of index 8 in each of Groups 4.1 and 5.1.

Take $K = \mathbb{Q}_2$ and $L = K(\sqrt{-2})$. By mapping $S \in \mathbb{Q}(S)$ to $\sqrt{-2} \in L$, we can regard the matrices representing the elements of our groups as elements of $U(h)$. These matrices are determined only up to multiplication by elements $a + bS \in \mathbb{Q}(S)$ satisfying $a^2 + 2b^2 = 1$, but as $\omega_L(a + b\sqrt{-2}) = 0$, these groups act on the tree T of Theorem 1.1, which is homogeneous of degree 3 in this case. Because of the 2's in the denominators of some of the a_j's and b_j's, the groups Γ did not fix $\mathscr{L}_0$. However, starting from $\mathscr{L}_0$, and using the matrices (2.3) and (2.4), it was easy to move around the vertices at a small distance from $\mathscr{L}_0$, and a vertex was found at distance 3 from $\mathscr{L}_0$ which was fixed by all five Γ's. Conjugating these groups by a suitable matrix, the groups were all realized in $G(\mathbb{Z}[1/3])$, where G is the projective unitary group with respect to the form $y^* Hx$, where

$$H = \begin{pmatrix} 2 & -S & 0 \\ S & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The matrices a_6 and f generating $\tilde{\Gamma}_{4.1}$ are now

$$a_6 = \begin{pmatrix} -1 & 0 & 0 \\ (2S+1)/3 & -(S-1)/3 & (S+2)/3 \\ (S-1)/3 & (S+2)/3 & -(S-1)/3 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} -S & -1 & 0 \\ -1 & S & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now consider the action of $G(\mathbb{Q})$ on $\Delta_{\mathbb{Q}_3}$. It is easy to calculate the stabilizer $G_0(\mathbb{Z}[1/3])$ of $[(\mathbb{Z}_3)^3]$ in $G(\mathbb{Z}[1/3])$. For if $g \in M_{3\times 3}(\mathbb{Q}(S))$ and $g^* Hg = H$, then $g^{-1} = H^{-1} g^* H$. Now H and H^{-1} have entries in $\mathbb{Z}[S, 1/2] \subset \mathbb{Z}_3$. So g and g^{-1} have entries in $\mathbb{Z}_3$ if and only if g and g^* have entries in $\mathbb{Z}_3$. Since $\mathbb{Z}[1/3] \cap \mathbb{Z}_3 = \mathbb{Z}$, we need only find matrices g with entries in $\mathbb{Z}[S]$ such that $g^* Hg = H$. Routine calculations show that (up to multiplication by ± 1), there are precisely 16 such matrices. Hence $G_0(\mathbb{Z}[1/3])$ has order 16, and is generated by f and g, where

$$g = \begin{pmatrix} 0 & -1 & 0 \\ -1 & S & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and f is as above. These generators, with the relations $f^4 = g^8 = 1$, $fgf^{-1} = g^3$ and $f^2 = g^4$, give a presentation of $G_0(\mathbb{Z}[1/3])$.

We obtain generators $b_0, \dots, b_{12}$ and f' for $\tilde{\Gamma}_{5.1}$ in $G(\mathbb{Z}[1/3])$ by setting $b_0 = a_0$, $b_1 = a_{12}g$, $b_2 = a_2 g^4$, $b_3 = a_7 g^7$, $b_4 = a_3 g$, $b_5 = a_{11} g$, $b_6 = a_5 g^5$, $b_7 = a_4 g^7$, $b_8 = a_1 g^5$, $b_9 = a_9 g$, $b_{10} = a_{10}$, $b_{11} = a_6 g^2$, $b_{12} = a_8 g^2$ and $f' = fg^2$.

We obtain a presentation of all of $G(\mathbb{Z}[1/3])$ from the generators a_i, $i = 0, \dots, 12$, plus f and g, together the relations of the form $a_i a_j a_k = 1$ and $f a_i f^{-1} = a_{i'}$ given in [CMSZ, p. 184], and the relations $ga_0 = a_1 g^3$, $ga_1 = a_8 g^2$, $ga_2 = a_2 g$, $ga_3 = a_4 g$,

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$ga_4 = a_9g^5$, $ga_5 = a_6g^2$, $ga_6 = a_{12}g$, $ga_7 = a_3g^5$, $ga_8 = a_{11}g$, $ga_9 = a_7g$, $ga_{10} = a_5g^3$, $ga_{11} = a_{10}g^6$ and $ga_{12} = a_0g^6$.

The situation is summarized by the diagram in Figure 1.

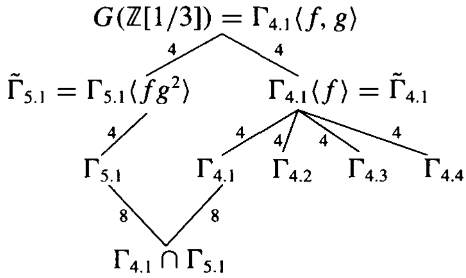

FIGURE 1

Groups 7.1 **and** 8.1. In [CMSZ], Group 7.1 was realized as a subgroup of $\mathrm{Aut}(\mathscr{A}) \cong \mathscr{A}^\times/Z(\mathscr{A}^\times)$ for a central simple algebra $\mathscr{A}$ of dimension 9 over $K = \mathbb{Q}(S)$, where $S^2 = -23$. Group 8.1 had a similar realization, involving another 9 dimensional algebra $\mathscr{B}$ over K. These algebras had definitions in terms of messy structure constants. Calculations of Hasse invariants told us that $\mathscr{A}$ and $\mathscr{B}$ were not isomorphic to $M_{3\times3}(K)$, but, instead, isomorphic or anti-isomorphic to a cyclic simple algebra $\mathscr{A}_\theta$ defined as follows: Let $L = K(\theta) = \mathbb{Q}(S, \theta)$, where $\theta^3 = \theta + 1$. Then L is a normal extension of $\mathbb{Q}$ of degree 6 over $\mathbb{Q}$. Let φ generate the Galois group of L over K. We adjoin to L an element σ satisfying $\sigma^3 = 2$ and $\sigma x \sigma^{-1} = \varphi(x)$ for all $x \in L$, and obtain $\mathscr{A}_\theta = L[\sigma]$, which consists of expressions $a + b\sigma + c\sigma^2$, where $a, b, c \in L$. Thus

$$\mathscr{A}_\theta = \{a + b\sigma + c\sigma^2 : a, b, c \in L,\ \sigma^3 = 2,\ \sigma x \sigma^{-1} = \varphi(x) \text{ for all } x \in L\}.$$

The algebra $\mathscr{A}$ has an involutive semilinear anti-automorphism *, and Group 7.1 embedded in the associated projective unitary group $\{\alpha \in \mathrm{Aut}(\mathscr{A}) : \alpha(\xi^*) = \alpha(\xi)^* \text{ for all } \xi \in \mathscr{A}\}$; similarly for $\mathscr{B}$ and Group 8.1. Here "semilinear" refers to the nontrivial field automorphism $x = a + bS \mapsto \bar{x} = a - bS$ of K.

We would like to find an involutive semilinear anti-automorphism * on $\mathscr{A}_\theta$, and embeddings of Groups 7.1 and 8.1 as arithmetic subgroups the associated projective unitary group. There is a simple involutive semilinear anti-automorphism $\tilde{}$ on $\mathscr{A}_\theta$, determined by $\tilde{\sigma} = \sigma$ and $\tilde{x} = \tau(x)$ for $x \in L$, where τ is the field automorphism of L fixing θ and mapping S to $-S$:

$$(a + b\sigma + c\sigma^2)\tilde{} = \tau(a) + \sigma\tau(b) + \sigma^2\tau(c) = \tau(a) + \varphi(\tau(b))\sigma + \varphi^2(\tau(c))\sigma^2.$$

Note that $\tau^2 = id$ and $\tau\varphi\tau^{-1} = \varphi^2$.

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As explained below, the anti-automorphism $\tilde{}$ is not quite suitable for our needs, and will be modified below, giving us the anti-automorphism $*$.

Consider the following basis of $\mathbb{Q}(\theta, S)$ over $\mathbb{Q}(S)$: $\{\xi_0, \xi_1, \xi_2\} = \{1, \theta, \theta^2\}$. The dual basis (with respect to Trace : $\mathbb{Q}(\theta, S) \to \mathbb{Q}(S)$) is

$$\{\eta_0, \eta_1\eta_2\} = \{(5 - 6\theta + 4\theta^2)/23, (-6 - 2\theta + 9\theta^2)/23, (4 + 9\theta - 6\theta^2)/23\}.$$

Form the 3×3 matrix Q whose (i, j) entry is $\varphi^j(\xi_i)$ for $i, j = 0, 1, 2$. Then Q^{-1} has (i, j) entry $\varphi^i(\eta_j)$ for each i, j.

Now let w denote an element satisfying $w^3 = w^2 + 1$ in some extension of $\mathbb{Q}$. Let $K' = K(w) = \mathbb{Q}(S, w)$. The algebra $\mathscr{A}_\theta \otimes K'$ splits. For we can map $x \in K'(\theta) = L'$ to

$$\Psi(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & \varphi(x) & 0 \\ 0 & 0 & \varphi^2(x) \end{pmatrix},$$

where φ denotes the extension to an automorphism of L' over K' of the automorphism φ of L over K; we also map σ to

$$\Psi(\sigma) = \begin{pmatrix} 0 & 1 + \theta w & 0 \\ 0 & 0 & 1 + \varphi(\theta)w \\ 1 + \varphi^2(\theta)w & 0 & 0 \end{pmatrix}.$$

Then $\xi \mapsto Q\Psi(\xi)Q^{-1}$ gives an isomorphism of $\mathscr{A}_\theta \otimes K'$ onto $M_{3\times3}(K')$. Explicitly, $x \in L'$ is mapped to the matrix with (i, j) entry $\mathrm{Trace}(\xi_i x \eta_j)$, and σ is mapped to the matrix with (i, j) entry $\mathrm{Trace}(\xi_i w\varphi(\eta_j))$ for $i, j = 0, 1, 2$. The semilinear anti-automorphism $\tilde{}$ extends to a semilinear anti-automorphism of $\mathscr{A}_\theta \otimes K'$, semilinear now referring to the extension to an automorphism τ of L' over $\mathbb{Q}(\theta, w)$ of the automorphism τ of L over $\mathbb{Q}(\theta)$. By the Skolem-Noether Theorem [We, p. 166], this anti-automorphism corresponds to an anti-automorphism $M \mapsto PM^*P^{-1}$ of $M_{3\times3}(K')$, for some $P \in GL(3, K')$, where for $M \in M_{3\times3}(K')$, M^* is obtained from M by applying τ to each entry of the transpose of M. In fact, a simple calculation shows that P must be a multiple of

$$\begin{pmatrix} 3 & 2w & 3w + 2 \\ 2w & 3w + 2 & 2w + 3 \\ 3w + 2 & 2w + 3 & 5w + 2 \end{pmatrix}.$$

Unfortunately, this matrix is not positive definite. So if G denotes the projective unitary group associated with $\tilde{}$, which we regard as defined over $\mathbb{Q}$, then $G(\mathbb{R})$ is not compact.

So we must replace $\tilde{}$ by another involutive semilinear anti-automorphism. Again by the Skolem-Noether Theorem, this must be of the form $\xi \mapsto u\tilde{\xi}u^{-1}$, with $\tilde{u} = u$ to ensure that this is an involution. The $u \in \mathscr{A}_\theta$ satisfying $\tilde{u} = u$ are the elements $a + \varphi^2(b)\sigma + \varphi(c)\sigma^2$, where $a, b, c \in \mathbb{Q}(\theta)$. A little experimentation led to the choice

$$u = \theta^2 + \varphi^2(\theta^2)\sigma + \varphi(\theta^2)\sigma^2.$$

The anti-automorphism $\xi^* = u\tilde{\xi}u^{-1}$ corresponds as above to the anti-automorphism $M \mapsto H^{-1}M^*H$ of $M_{3\times3}(K')$, where $H = 138(UP)^{-1}$ equals

$$\begin{pmatrix} 28w^2+27w+39 & 17w^2+(9-7S)w+36-3S & -19w^2+(2S-29)w-S-24 \\ 17w^2+(9+7S)w+36+3S & -2w^2+3w+104 & 9w^2-(48+6S)w-S-31 \\ -19w^2-(2S+29)w+S-24 & 9w^2-(48-6S)w+S-31 & 17w^2+32w+13 \end{pmatrix},$$

and where $U \in M_{3\times3}(K')$ corresponds to $u \in \mathscr{A}_\theta \otimes K'$ under the above isomorphism. Now H is positive definite, and so if G is the projective unitary group associated with *, regarded as an algebraic group defined over $\mathbb{Q}$, then $G(\mathbb{R})$ is compact.

Our aim is to exhibit Groups 7.1 and 8.1 as subgroups of $G(\mathbb{Q})$ commensurable with $G(\mathbb{Z}[1/3])$. To do this, we must first specify a basis of $\mathscr{A}_\theta$ over $\mathbb{Q}(S)$. A convenient basis is $\{m_1, \dots, m_9\}$, where

$$m_1 = u\varphi(\theta^2)\sigma^2, \quad m_2 = u\varphi^2(\theta^2)\sigma, \quad m_3 = u\varphi(\theta)\sigma^2, \quad m_4 = u\varphi^2(\theta)\sigma,$$
$$m_5 = u\theta^2, \quad m_6 = u\theta, \quad m_7 = u\sigma^2, \quad m_8 = u\sigma, \quad m_9 = 1.$$

(The m_j satisfy $m_j^* = m_j$, because they are of the form $u\xi$, where $\tilde{\xi} = \xi$.) By the Skolem-Noether Theorem, the automorphisms α of $\mathscr{A}_\theta$ satisfying $\alpha(\xi^*) = \alpha(\xi)^*$ for all ξ are the maps $\xi \mapsto a\xi a^{-1}$, where $a \in \mathscr{A}_\theta$ satisfies $a^*a = c$, for some $c \in \mathbb{Q}^\times$. Note that $a^*a = c$ for some $c \in \mathbb{Q}^\times$ if and only if $(ta)^*(ta) = 1$ for some $t \in \mathbb{Q}(S)^\times$. Thus to say that $a \in \mathscr{A}_\theta^\times$ corresponds to an element of $G(\mathbb{Z}[1/3])$ means that $(ta)^*(ta) = 1$ for some $t \in \mathbb{Q}(S)^\times$, and that $am_ia^{-1} = \sum_{j=1}^{9} c_{i,j}m_j$ for some $c_{i,j} \in \mathbb{Z}[1/3]$. Note that $(ta)^*(ta) = 1$ and $m_i^* = m_i$ implies that the $c_{i,j}$ must be in $\mathbb{Q}$.

To exhibit Groups 7.1 and 8.1 in $G(\mathbb{Q})$, we next needed to find some elements of $G(\mathbb{Z}[1/3])$, in fact enough to generate a finite index subgroup thereof. To find elements satisfying $a^*a = 1$, we first took elements x satisfying $x^* = x$, and then let a be the (modified) Cayley transform of x: $a = (S + x)(S - x)^{-1}$. A program was run which took x of the form $\sum_{i=1}^{9} x_jm_j$, where the x_j's were small integers, and checked whether the corresponding automorphism $\xi \mapsto a\xi a^{-1}$ was in $G(\mathbb{Z}[1/3])$. Forming suitable words in the elements a found in this way, we obtained more elements. We then sought non-commuting pairs a, a' found this way and pairs b, b' of elements in Group 7.1 so that $\mathrm{Inv}(a^i(a')^j) = \mathrm{Inv}(b^i(b')^j)$ for $i, j = 0, 1, 2$. Here $\mathrm{Inv}(\xi) = \mathrm{Trace}(\xi)^3/\det(\xi)$ for ξ in either algebra, regarding ξ as a 3×3 matrix with entries in $\mathbb{Q}(S, w)$ for

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$\xi \in \mathscr{A}_\theta$, and with entries in $\mathbb{Q}(S, \alpha)$ (see [CMSZ, p. 193]) for $\xi \in \mathscr{A}$. For if there is an isomorphism or anti-isomorphism $\mathscr{A}_\theta \to \mathscr{A}$ mapping a to a multiple of b and a' to a multiple of b', then the above equations between the invariants must hold.

After much effort, elements $a, a' \in \mathscr{A}_\theta^\times$ were found so that the corresponding automorphisms belonged to $G(\mathbb{Z}[1/3])$, and so that

$$a \mapsto \frac{(11+S)}{12} g_{11}^{-1} \quad \text{and} \quad a' \mapsto \frac{(11+S)}{12} g_6^{-1} g_3 g_1^{-1}$$

induced an anti-isomorphism from $\mathscr{A}_\theta$ to the algebra of Group 7.1 (see below). Explicitly, if we let $x = \sum_{i=1}^9 x_i m_i$ and $a = (S+x)(S-x)^{-1}$ for

$$(x_1, \dots, x_9) = \frac{1}{3}(-8, -10, 2, 6, 0, -8, 6, 6, -9)$$

we obtain an element $a \in \mathscr{A}_\theta^\times$ satisfying $a m_i a^{-1} = \sum_{j=1}^9 c_{i,j} m_j$ for certain $c_{i,j} \in \mathbb{Z}[1/3]$. We can express a explicitly as a $\mathbb{Q}(S)$-linear combination of the m_j's: $a = \sum_{i=1}^9 t_i m_i$ for

$$\begin{aligned}(t_1, \dots, t_9) = \frac{1}{4 \cdot 23 \cdot 27}&(3(41S-23), 2(97S+23), 3(-5S+23),\\ &2(-61S-23), 8(7S+23), 12(5S-23), 4(-34S-23),\\ &2(-103S-161), 12(29S-23)).\end{aligned}$$

Similarly, if we let $x' = \sum_{i=1}^9 x_i' m_i$ and $a' = (S+x')(S-x')^{-1}$ for

$$(x_1', \dots, x_9') = \frac{1}{3}(162, 268, 14, -120, 236, 156, -98, -132, -471)$$

we obtain an element $a' \in \mathscr{A}_\theta^\times$ satisfying $a' m_i (a')^{-1} = \sum_{j=1}^9 c_{i,j}' m_j$ for certain $c_{i,j}' \in \mathbb{Z}[1/3]$. Again, we can express a' explicitly as a $\mathbb{Q}(S)$-linear combination of the m_j's: $a' = \sum_{i=1}^9 t_i' m_i$ for

$$\begin{aligned}(t_1', \dots, t_9') = \frac{1}{4 \cdot 27 \cdot 23}&(81(S+23), 211S+2921, 7(S+23), -203S-1081,\\ &6(27S+437), 72(2S+23), 28(S-46),\\ &2(-55S-713), 48(-S-92)).\end{aligned}$$

Using the above anti-isomorphism, we can realise Group 7.1 in $\mathscr{A}_\theta^\times / Z(\mathscr{A}_\theta^\times)$. To specify generators a_j, $j = 0, 1, \dots, 12$, of Group 7.1 in $\mathscr{A}_\theta$ in a very compact way, we first give 13 elements h_j of $\mathscr{A}_\theta$ satisfying $h_j^* = h_j$. Then we let a_j be the Cayley transform $(S+h_j)(S-h_j)^{-1}$ of h_j for each j. We let

$$h_j = \frac{1}{9} \sum_{k=1}^9 t_{j,k} m_k \quad \text{for } j = 0, \dots, 12,$$

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where $t_{j,k}$ is the (j, k) entry of the matrix

$$\begin{pmatrix}
-12 & -60 & -24 & 54 & -72 & -12 & 36 & 36 & -63 \\
-4 & -20 & -2 & 12 & -12 & -4 & 6 & 24 & -33 \\
-18 & -36 & -6 & 24 & -36 & -24 & 6 & 24 & 27 \\
30 & 56 & 0 & -20 & 32 & 24 & -4 & -56 & -9 \\
-132 & -168 & 24 & 48 & -192 & -96 & 114 & 120 & 63 \\
36 & 72 & 6 & -36 & 60 & 48 & -24 & -48 & -45 \\
114 & 198 & 30 & -114 & 168 & 96 & -66 & -114 & -279 \\
24 & 72 & 12 & -24 & 60 & 24 & 6 & -60 & -117 \\
-42 & -60 & 24 & 48 & -12 & -132 & -30 & 48 & 207 \\
-42 & -54 & 6 & 12 & -48 & -36 & 24 & 36 & 57 \\
-60 & -132 & -30 & 48 & -120 & -24 & 24 & 84 & 135 \\
-12 & -30 & -6 & -12 & -24 & 24 & -6 & 42 & 9 \\
48 & 90 & 24 & -12 & 84 & -24 & -24 & -66 & -27
\end{pmatrix}.$$

It is convenient to multiply a_1, a_2, a_7 and a_{10} by $-(S+11)/12$. This ensures that all the a_j's have entries in $\mathbb{Z}_3$, when S and w are regarded as 3-adic numbers (S being chosen so that $S \equiv 1 \bmod 3$). Moreover, the 3-adic valuation of each of the a_j's is 2, so that if $L_0 = (\mathbb{Z}_3)^3$ and $v_0 = [L_0]$ is the corresponding vertex in $\Delta_{\mathbb{Q}_3}$, then the $a_j v_0$, $j = 0, \dots, 12$, are vertices which are neighbours of v_0, and all of the same type. One may check that they are distinct.

If we let $g_1' = ((S+11)/12)a_1^{-1}$ and $g_9' = ((S+11)/12)a_9^{-1}$, then one can check that $(g_1')^3 = \sum_{j=0}^{2} c_j (g_1')^j$, that $(g_9')^3 = \sum_{j=0}^{2} d_j (g_9')^j$, and that $g_1' g_9' = \sum_{i,j=0}^{2} \gamma_{i,j} (g_9')^j (g_1')^i$, where the c_j's, d_j's and $\gamma_{i,j}$'s are as in [CMSZ, p. 193]. Thus there is an anti-isomorphism $\mathscr{A} \to \mathscr{A}_\theta$ determined by mapping g_1 to g_1' and g_9 to g_9'.

The automorphisms $\xi \mapsto a_j \xi a_j^{-1}$ are in $G(\mathbb{Z}[1/2, 1/3, 1/23])$. The elements $a, a' \in \mathscr{A}_\theta$ defined above are $a_2^{-1} a_{11} a_2$ and $a_2^{-1}(a_1 a_3^{-1} a_6) a_2$, respectively. The fact that a, a' correspond to elements of $G(\mathbb{Z}[1/3])$ means that a_{11} and $a_1 a_3^{-1} a_6$ are in $G(\mathbb{Z}[1/3])$ if we replace $\{m_1, \dots, m_9\}$ by $\{a_2 m_j a_2^{-1} : j = 1, \dots, 9\}$. MAGMA tells us that $\langle g_{11}, g_1 g_3^{-1} g_6 \rangle$ is an index 24 subgroup of Group 7.1. This exhibits the arithmeticity of Group 7.1.

For Group 8.1, we let

$$h_j' = \frac{1}{9 \cdot 23} \sum_{k=1}^{9} t_{j,k} m_k \quad \text{for } j = 0, \dots, 12,$$

where $t_{j,k}$ is the (j, k) entry of the matrix

$$\begin{pmatrix} 1668 & 2598 & -444 & -324 & 2736 & 372 & -1026 & -2520 & -3069 \\ -106 & -332 & 130 & 300 & -60 & -244 & 114 & 348 & -999 \\ -96 & -212 & 54 & 68 & -332 & -36 & -86 & 176 & -87 \\ -216 & 876 & 1224 & -912 & 516 & -360 & 210 & -516 & -2115 \\ 780 & 1092 & -162 & -756 & 732 & 1260 & -216 & -468 & -783 \\ -1124 & -1892 & -88 & 724 & -1432 & -1172 & 422 & 1072 & 3597 \\ 860 & 1444 & -32 & -318 & 1380 & 812 & -330 & -996 & -2865 \\ 630 & 720 & -108 & 324 & 1800 & 936 & 288 & -1296 & -747 \\ 1104 & 2070 & 552 & -276 & 1932 & -552 & -552 & -1518 & -621 \\ -1962 & -2088 & -144 & -396 & -2808 & 288 & 1170 & 1764 & 1737 \\ -1560 & -3024 & -474 & 1560 & -2712 & -1632 & 780 & 2064 & 3105 \\ -276 & -690 & -138 & -276 & -552 & 552 & -138 & 966 & 207 \\ 732 & 2034 & 258 & -1326 & 768 & 1200 & -636 & -1326 & -2097 \end{pmatrix}.$$

Then we let $a'_j = (S + h'_j)(S - h'_j)^{-1}$ for each j. Again, it is convenient to multiply a'_1, a'_2, a'_3 and a'_{10} by $-(S+11)/12$. This achieves the same normalizations as described above for Group 7.1.

The algebra $\mathscr{B}$ associated with Group 8.1 is isomorphic to $\mathscr{A}_\theta$. If we let $x'_1 = -((7 + 3 * S)/16)a'_1$ and $x'_4 = a'_4$, then one can check that $(x'_1)^3 = \sum_{j=0}^{2} c_j (x'_1)^j$, that $(x'_4)^3 = \sum_{j=0}^{2} d_j (x'_4)^j$, and that $x'_4 x'_1 = \sum_{i,j=0}^{2} \gamma_{i,j} (x'_1)^i (x'_4)^j$, where the c_j's, d_j's and $\gamma_{i,j}$'s are as in [CMSZ, p. 196]. Thus there is an isomorphism $\mathscr{B} \to \mathscr{A}_\theta$ determined by mapping x_1 to x'_1 and x_4 to x'_4.

Then $a'_{11} = a_{11}$ and $a'_4(a'_0)^{-1}a'_{12} = a_1 a_3^{-1} a_6$. Hence, with these realizations $\Gamma_{7.1}$ and $\Gamma_{8.1}$ of Groups 7.1 and 8.1 in $\mathscr{A}_\theta^\times / Z(\mathscr{A}_\theta^\times)$, the two groups have in common the index 24 subgroup generated by a_{11} and $a_1 a_3^{-1} a_6$. The automorphisms $\xi \mapsto a'_j \xi (a'_j)^{-1}$ are in $G(\mathbb{Z}[1/2, 1/3, 1/23])$. So we have the situation shown in Figure 2.

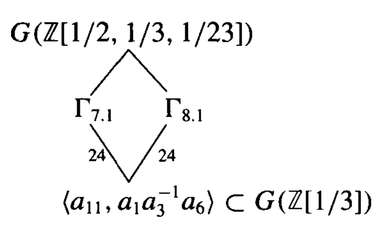

FIGURE 2

The situation is therefore rather more complicated than in the case of Groups 4.1 and 5.1, say. We now use the tree T associated with $\mathbb{Q}_{23}(\sqrt{-23})$ to show that we cannot simplify the embeddings.

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Because $L = \mathbb{Q}_{23}(\sqrt{-23})$ is a ramified extension of $\mathbb{Q}_{23}$, T is homogeneous of degree 24. The group G defined above acts on T. For if $\alpha \in \mathrm{Aut}(\mathscr{A}_\theta)$ and $\alpha(\xi^*) = \alpha(\xi)^*$ for all $\xi \in \mathscr{A}_\theta$, then there is an $a \in \mathscr{A}_\theta^\times$ such that $\alpha(\xi) = a\xi a^{-1}$ for all ξ, and such that $a^*a = 1$. Let $A \in M_{3\times 3}(K')$ be the matrix corresponding to a under the isomorphism $\mathscr{A}_\theta \otimes K') \cong M_{3\times 3}(K')$ defined above. Then $A^*HA = H$. Moreover, $w^3 = w^2 + 1$ has a solution in $\mathbb{Q}_{23}$ (with $w \equiv 17 \bmod 23$). So we can regard K' as a subfield of L. Thus $A \in U_3(h)$ for the form h on L^3 corresponding to H. So for $v \in \Lambda_0 \cup \Lambda_1$, we define $\alpha.v = A.v$. This is well-defined, for if a is replaced by ta, where $t = t_1 + t_2 S \in \mathbb{Q}(S)$ and $\bar{t}t = 1$, then $t_1^2 + 23t_2^2 = 1$, and so t, regarded as an element of L, has valuation 0, so that $t\mathscr{L} = \mathscr{L}$ for any $\mathfrak{o}_L$-lattice $\mathscr{L}$.

Hence our groups $\Gamma_{7.1}$ and $\Gamma_{8.1}$, being subgroups of $G(\mathbb{Z}[1/2, 1/3, 1/23]) \subset G(\mathbb{Q})$, act on T. Let $\mathscr{L}_0 = \mathfrak{o}_L^3$, and let $\mathscr{L}_1 = g_1(\mathscr{L}_0)$ for

$$g_1 = S^{-1}\begin{pmatrix} S & 15 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathscr{L}_1' = \mathscr{L}_1$, so that $\mathscr{L}_1 \in \Lambda_0$. Let $\mathscr{M}_2 = g_2(\mathscr{L}_0)$ for

$$g_2 = g_1\begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $S\mathscr{M}_2' \subsetneqq \mathscr{M}_2 \subsetneqq \mathscr{M}_2'$, so that $(\mathscr{M}_2, \mathscr{M}_2') \in \Lambda_1$ is a neighbour of $\mathscr{L}_1$ in T. One may verify that $\Gamma_{7.1}$ fixes $\mathscr{M}_2$ (and therefore also $\mathscr{M}_2'$), and hence the vertex $(\mathscr{M}_2, \mathscr{M}_2')$ of T. Now let $\mathscr{L}_3 = g_3(\mathscr{L}_0)$ for

$$g_3 = S^{-1}g_2\begin{pmatrix} 1 & 0 & 0 \\ 19 & S & 0 \\ 15 & 0 & S \end{pmatrix}.$$

Then $\mathscr{L}_3 \in \Lambda_0$ is a neighbour of $(\mathscr{M}_2, \mathscr{M}_2')$ in T, and is fixed by $\Gamma_{8.1}$.

Moreover, $\Gamma_{7.1}$ acts transitively on the 24 neighbours of $(\mathscr{M}_2, \mathscr{M}_2')$. Indeed, the 24 elements $\{1, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_9, a_{10}, a_1^{-1}, a_3^{-1}, a_5^{-1}, a_6^{-1}, a_{10}^{-1}, a_0a_1^{-1}, a_0a_2^{-1}, a_0a_{10}^{-1}, a_1a_7^{-1}, a_1a_8^{-1}, a_1a_9^{-1}, a_3a_6^{-1}, a_4a_{10}^{-1}\}$ of $\Gamma_{7.1}$ move $\mathscr{L}_3$ to these neighbours.

Also, $\Gamma_{8.1}$ acts transitively on the 24 neighbours of $\mathscr{L}_3$. Indeed, the 24 elements $\{a_0', a_1', a_2', a_3', a_4', a_5', a_6', a_7', a_8', a_{10}', (a_1')^{-1}, (a_2')^{-1}, (a_3')^{-1}, (a_4')^{-1}, (a_{12}')^{-1}, a_0'a_3', a_0'a_5', a_0'a_7', a_1'a_2', a_1'a_3', a_1'a_{12}', a_2'a_4', a_2'a_{10}'\}$ of $\Gamma_{8.1}$ move $(\mathscr{M}_2, \mathscr{M}_2')$ to these neighbours.

There is no realization of $\Gamma_{7.1}$ and $\Gamma_{8.1}$ in $G(\mathbb{Q})$ for which $\Gamma_{7.1} \cap \Gamma_{8.1}$ has index strictly less than 24 in $\Gamma_{7.1}$ and $\Gamma_{8.1}$. For if there were, then there would be isomorphic subgroups $H_{7.1}$ and $H_{8.1}$ of the realizations of $\Gamma_{7.1}$ and $\Gamma_{8.1}$ given above, for which $[\Gamma_{8.1} : H_{8.1}] = n < 24$. Applying [Mar, Theorem (5), p. 5], our isomorphism

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$f : H_{7.1} \to H_{8.1}$ would be induced by conjugation α_x by some element x of $G(\mathbb{Q})$. See also [Hum, Section 27.4]; the fact that f is not induced from an automorphism involving the nontrivial Dynkin diagram automorphism may be deduced, for example, from the fact that $\mathscr{A}_\theta$ admits no (linear) anti-automorphism. For let $\mathscr{A}_\theta'$ denote the algebra defined as was $\mathscr{A}_\theta$, but with the element σ replaced by an element σ' satisfying $(\sigma')^3 = 4$. Then there is an anti-isomorphism $\mathscr{A}_\theta \to \mathscr{A}_\theta'$ mapping σ to $(\sigma')^2/2$ and mapping each $x \in \mathbb{Q}(S,\theta)$ to x. If $\mathscr{A}_\theta$ has an anti-automorphism, then $\mathscr{A}_\theta$ and $\mathscr{A}_\theta'$ would be isomorphic, which is impossible because $4/2 = 2$ is not the norm of any element of $\mathbb{Q}(S,\theta)$ (see, for example, [Deu, p. 65]).

Thus, modulo scalars, f is the restriction of α_x to $H_{7.1}$. Let $u \in \Lambda_1$ denote the vertex $(\mathscr{M}_2, \mathscr{M}_2') \in T$ fixed by $\Gamma_{7.1}$, and let $v \in \Lambda_0$ denote the vertex $\mathscr{L}_3 \in T$ fixed by $\Gamma_{8.1}$. Then $\alpha_x \Gamma_{7.1} \alpha_x^{-1}$ fixes $\alpha_x.u \in \Lambda_1$. Thus $\Gamma_{8.1} \cap \alpha_x \Gamma_{7.1} \alpha_x^{-1}$, and therefore $H_{8.1}$, fixes the geodesic in T from v to $\alpha_x.u$. But $\Gamma_{8.1}$ moves this geodesic to 24 different paths. Hence $n = [\Gamma_{8.1}, H_{8.1}] \geq 24$.

Similar considerations show that there is no semi-linear involutory anti-automorphism $\xi \mapsto \xi^\dagger$ of $\mathscr{A}_\theta$ such that, if $G^\dagger$ denotes the corresponding projective unitary group, then $G^\dagger(\mathbb{R})$ is compact and Groups 7.1 and 8.1 embed in $G^\dagger(\mathbb{Z}[1/3])$.

Groups 2.1, 2.2, 3.1 and 3.2. In [CMSZ], these groups were exhibited in $\mathscr{A}^\times/Z(\mathscr{A}^\times)$ for the following cyclic simple algebra $\mathscr{A}$, defined over $\mathbb{F}_3(P)$, P an indeterminate: $\mathscr{A} = \mathbb{F}_{27}(P)[\sigma]$, where $\sigma^3 = P$, and $\sigma x \sigma^{-1} = \varphi(x)$ for $x \in \mathbb{F}_{27}(P)$. Here φ is a generator of the Galois group of $\mathbb{F}_{27}(P)$ over $\mathbb{F}_3(P)$; if we think of $\mathbb{F}_{27}$ as $\mathbb{F}_3(\theta)$, where $\theta^3 = \theta + 1$, then we can assume that $\varphi(\theta) = \theta + 1$ and $\varphi(P) = P$. We regarded $\mathbb{F}_3(P)$ as a quadratic extension of $\mathbb{F}_3(R)$, where $R = P - 1/P$. We exhibited an involutive semilinear antiautomorphism $*$ of $\mathscr{A}$. Groups 2.1 and 2.2 are normal index 3 subgroups of a group $\tilde{\Gamma}_{2.1}$ generated by elements $a_j \in \mathscr{A}$, $j = 0, \ldots, 12$, and σ. Similarly, Groups 3.1 and 3.2 are normal index 3 subgroups of a group $\tilde{\Gamma}_{3.1}$ generated by elements $b_j \in \mathscr{A}$, $j = 0, \ldots, 12$, and σ. We embedded $\tilde{\Gamma}_{2.1}$ in $PU(\mathbb{F}_3[1/R])$, where

$$PU(\mathbb{F}_3(R)) = \{\alpha \in \mathrm{Aut}(\mathscr{A}) : \alpha(\xi^*) = \alpha(\xi)^* \text{ for all } \xi \in \mathscr{A}\}.$$

If we replace the generators of Group $\widetilde{3.1}$ given in [CMSZ] by $b_i' = \gamma^{-1} b_i \gamma$ and $\sigma' = \gamma^{-1}\sigma\gamma$ ($= \sigma$), where $\gamma = P\sigma + \sigma^2$, then the elements $b_i'/(P+1)$ are unitary, so that Group $\widetilde{3.1}$ is now realized in $PU(\mathbb{F}_3(R))$. Moreover, the intersection $\Gamma_{2.1} \cap \Gamma_{3.1}$ of these realizations $\Gamma_{2.1}$ and $\Gamma_{3.1}$ of Groups 2.1 and 3.1 has index 10 in each of $\Gamma_{2.1}$ and $\Gamma_{3.1}$. Indeed, $b_0' = a_0$, $b_4' b_9' = a_4 a_1$ and $b_4' b_5' = a_4 a_6$; MAGMA tells us that the subgroup of $\Gamma_{2.1}$ generated by a_0, $a_4 a_1$ and $a_4 a_6$ has index 10 in $\Gamma_{2.1}$ (and the subgroup of $\Gamma_{3.1}$ generated by b_0', $b_4' b_9'$ and $b_4' b_5'$ has index 10 in $\Gamma_{3.1}$). Group $\widetilde{3.1}$ is now generated by σ and b'_2, where

$$\begin{aligned} b'_2 = & \frac{(P-1)R^2-(P+1)R}{R^2+1} - \theta + (P-1)\theta^2 \\ & + \left(\frac{R^2+(P-1)R-P}{R^2+1} + \theta + \theta^2\right)\sigma \\ & + \left(\frac{R^2+(P+1)R+P}{R^2+1} - \theta^2\right)\sigma^2. \end{aligned}$$

Let K denote the completion of $\mathbb{F}_3(R)$ with respect to the valuation associated with the irreducible polynomial R^2+1. Thus $q=9$, and we can take $\pi_K = R^2+1$. Let $L = K(P)$. Then L is a ramified quadratic extension of K containing $\mathbb{F}_3(P)$; we can take $\pi_L = R+P$, which satisfies $\pi_L^2 = R^2+1$. The antiautomorphism * of $\mathscr{A}$ gives rise to a sesquilinear form $(x, y) \mapsto y^*Hx$ on L^3, where

$$H = \begin{pmatrix} R & R-1 & R-1 \\ R-1 & -R+1 & -R-1 \\ R-1 & -R-1 & R-1 \end{pmatrix} \mod R^2+1,$$

and the associated tree is homogeneous of degree 10. One may readily check that $\tilde{\Gamma}_{2.1}$ fixes the vertex $u = \mathscr{L}_0 = \mathfrak{o}_L^3 \in \Lambda_0$. On the other hand, $\tilde{\Gamma}_{3.1}$ fixes the neighbouring vertex $v = (\mathscr{M}, \mathscr{M}') \in \Lambda_1$, where $\mathscr{M} = g(\mathscr{L}_0)$ for

$$g = \begin{pmatrix} \pi_L & 0 & 1-R \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One may verify that Group 2.1 acts transitively on the 10 neighbours of u. Indeed, the elements $1, a_1, a_2, a_3, a_4, a_7, a_{11}, a_1^{-1}, a_3^{-1}$ and a_5^{-1} move v to these 10 neighbours. Similarly, Group 3.1 acts transitively on the 10 neighbours of v. Indeed, the elements $1, b'_1, b'_3, b'_4, b'_5, b'_8, b'_{10}, (b'_1)^{-1}, (b'_3)^{-1}$ and $(b'_6)^{-1}$ move u to these 10 neighbours.

Considerations similar to those in the last subsection show that there is no realization of $\Gamma_{2.1}$ and $\Gamma_{3.1}$ in $G(\mathbb{Q})$ for which $\Gamma_{2.1} \cap \Gamma_{3.1}$ has index strictly less than 10 in $\Gamma_{2.1}$ and $\Gamma_{3.1}$.

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