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Measure Convex and Measure Extremal Sets

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Abstract. We prove that convex sets are measure convex and extremal sets are measure extremal provided they are of low Borel complexity. We also present examples showing that the positive results cannot be strengthened.

1 Introduction

Let X be a nonempty compact convex subset of a locally convex space. The aim of the paper is to show under what topological conditions imposed on a convex subset F of X the set F is measure convex. (We recall that a Borel subset F of X is said to be *measure convex* if F contains the barycenter $r(\mu)$ of every probability Radon measure μ on X supported by F.) Namely, we are interested in whether a convex set of low Borel complexity is measure convex. Since there are examples of convex G_δ-sets or F_σ-sets which are not measure convex (see Propositions 4.1, 4.3 and Theorem 4.8), we must restrict ourselves to the case of convex sets which are closed, open or ambivalent. The affirmative answer to our question for closed sets easily follows from the Hahn–Banach separation theorem, the case of open sets was solved by H. von Weizsäcker [10] and the case of ambivalent sets is answered in Theorem 3.5. (We recall that a subset A of a metrizable space X is called ambivalent if A is both an F_σ-set and a G_δ-set in X. We refer the reader to Section 2 where we define ambivalent sets in general topological spaces.)

An analogous question to the one above can be stated for extremal subsets of X. We recall that a set $F \subset X$ is extremal if $x, y \in F$ whenever $x, y \in X$, $\alpha \in (0, 1)$ and $\alpha x + (1 - \alpha)y \in F$. We investigate whether an extremal subset of X is measure extremal. A Borel set $F \subset X$ is called *measure extremal* if every probability Radon measure μ on X is supported by F whenever $r(\mu) \in F$. As Propositions 4.5, 4.7 and Theorem 4.9 show, extremal F_σ- or G_δ-sets need not be measure extremal. But it is not difficult to verify that closed or ambivalent extremal sets are measure extremal (see Corollary 3.8).

One of the main tools we use is a characterization of measure convex sets which is due to D. H. Fremlin and J. D. Pryce (see Theorem 2.2). This theorem enables us to characterize measure convex sets.

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Counterexamples contained in Proposition 4.5 and Theorem 4.8 are suitable modifications of the example by G. Choquet [3] (*cf.* [1, Example I.2.10], [8, Example, p. 91] or [7, Remark 1]). We remark that in Proposition 4.4 we present a simplified version of the proof of [1, Proposition I.2.8]. The examples of Proposition 4.7 and Theorem 4.9 partially use a construction of H. von Weiszäcker (see [10]).

We refer the reader to [11, §1.2] where a thorough investigation of measure convex sets is given.

2 Preliminaries

All topological spaces will be considered as Hausdorff. If K is a compact space, we denote by $\mathcal{C}(K)$ the space of all continuous functions on K. We identify the dual of $\mathcal{C}(K)$ with the space $\mathcal{M}(K)$ of all signed finite Radon measures on K and consider $\mathcal{M}(K)$ endowed with the w^*-topology.

By a measure μ on K we always understand a positive Radon measure on K. Let $\mathcal{M}^1(K)$ denote the set of all probability Radon measures on K and $\mathcal{M}^+(K)$ the set of all positive Radon measures on K. Let ε_x stand for the Dirac measure at $x \in K$. For a measure $\mu \in \mathcal{M}^+(K)$ we write $\operatorname{spt}\mu$ for the support of μ.

We recall that a measure $\mu \in \mathcal{M}^+(K)$ on K is said to be *atomic* if there exists a set $S \subset K$ such that $\mu(K \setminus S) = 0$ and $\mu(\{x\}) > 0$ for every $x \in S$. A measure μ is *continuous* if $\mu(\{x\}) = 0$ for every $x \in K$. Every measure μ can be uniquely expressed in the form $\mu = \mu_a + \mu_c$, where μ_a is atomic and μ_c is continuous.

Given a couple of measures μ and ν in $\mathcal{M}^+(K)$, we say that μ and ν are *singular*, which is denoted by $\mu \perp \nu$, if there exists a Borel set $B \subset K$ such that $\mu(B) = \nu(K \setminus B) = 0$. A measure ν is *absolutely continuous* with respect to μ, denoted by $\nu \ll \mu$, if $\nu(E) = 0$ whenever E is a Borel set and $\mu(E) = 0$. If ν and μ are measures on K, the measure ν can be uniquely expressed as $\nu = \nu_s + \nu_{ac}$, where $\nu_s \perp \mu$ and $\nu_{ac} \ll \mu$.

If X is a compact convex subset of a locally convex space E and μ is a probability measure on X, a point $x \in X$ is said to be the *barycenter of* μ if $\mu(\varphi) = \varphi(x)$ for every continuous functional $\varphi \in E^*$. It is well known that every $\mu \in \mathcal{M}^1(X)$ has a unique barycenter (see [8, Proposition 1.1]) which we denote by $r(\mu)$.

It is easy to see that any measure convex set is also convex and that any measure extremal set is extremal. A singleton $\{x\}$ is an extremal subset of X if and only if x is an *extreme* point of X. We write $\operatorname{ext}(X)$ for the set of all extreme points of X.

A set $F \subset X$ is called a *face* of X if F is convex and extremal. If x is a point of X, the smallest face, $\operatorname{face}(x)$, containing x consists of all points $y \in X$ for which the line segment joining x and y extends in X beyond x. We say that $\operatorname{face}(x)$ is *generated by* x. It easily follows from the definition that $\operatorname{face}(x)$ is of type F_σ (see [1, Proposition II.5.22]).

For a compact space K, let ε denote the homeomorphic embedding of K into $\mathcal{M}^1(K)$, *i.e.*, ε assigns to $x \in K$ the Dirac measure $\varepsilon_x \in \mathcal{M}^1(K)$. As is well known, $\varepsilon(K) = \operatorname{ext}(\mathcal{M}^1(K))$.

If λ is a probability measure on K, we define the measure $\varepsilon\lambda$ as the image of λ under the mapping ε, *i.e.*, $\varepsilon\lambda(B) := \lambda(\varepsilon^{-1}(B))$ for any Borel set $B \subset \mathcal{M}^1(K)$.

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Proposition 2.1 *The measure $\varepsilon\lambda$ is supported by the (closed) set $\varepsilon(\operatorname{spt}\lambda)$ and its barycenter equals λ.*

Proof Let $\Lambda := \varepsilon\lambda$. Since

$$\Lambda(\varepsilon(\operatorname{spt}\lambda)) = \lambda(\varepsilon^{-1}(\varepsilon(\operatorname{spt}\lambda))) = \lambda(\operatorname{spt}\lambda) = 1,$$

we see that Λ is supported by $\varepsilon(\operatorname{spt}\lambda)$.

Pick $\varphi \in (\mathcal{M}(K), w^*)^*$. By duality theory, there is $f \in \mathcal{C}(K)$ such that

$$\varphi(\mu) = \mu(f) \quad \text{for any } \mu \in \mathcal{M}(K).$$

Then

$$\begin{aligned}\Lambda(\varphi) = (\varepsilon\lambda)(\varphi) = \lambda(\varphi \circ \varepsilon) = \int_K \varphi(\varepsilon_x)\, d\lambda(x) = \int_K f(x)\, d\lambda(x) \\ = \lambda(f) = \varphi(\lambda).\end{aligned}$$

■

In order to obtain results on measure convex and measure extremal sets also in nonmetrizable compact spaces, we use a class of ambivalent sets which is larger than the usual family of both F_σ- and G_δ-sets. Namely, for a topological space K we denote by $\operatorname{alg}(\mathcal{F})$ the smallest algebra containing the family of all closed sets in K. Let $(\operatorname{alg}(\mathcal{F}))_\sigma$ stand for the family of all countable unions of sets from $\operatorname{alg}(\mathcal{F})$ and let $(\operatorname{alg}(\mathcal{F}))_\delta$ stand for the family of all countable intersections of sets from $\operatorname{alg}(\mathcal{F})$. These families are suitable substitutes for the families of F_σ-sets and G_δ-sets since $(\operatorname{alg}(\mathcal{F}))_\sigma$ coincides with the system of F_σ-sets and $(\operatorname{alg}(\mathcal{F}))_\delta$ with the system of G_δ-sets if K is metrizable.

Sets belonging to $(\operatorname{alg}(\mathcal{F}))_\sigma \cap (\operatorname{alg}(\mathcal{F}))_\delta$ are called *ambivalent*. We refer the reader to [9, Definition 1.1], [6, §3] or [5, §1] for further information on the Borel hierarchy in topological spaces.

Later on we will need some useful properties of sets defined above. First, we realize that any $(\operatorname{alg}(\mathcal{F}))_\delta$ set F dense in K is even residual in K, *i.e.,* the complement of F in K is of the first category in K. This assertion easily follows from the observation that

$$F = \bigcap_{n=1}^{\infty} (G_n \cup F_n),$$

where each G_n is open and each F_n is closed in K (see [6, §3]). Supposing that F is dense in K, it is easy to verify that every $G_n \cup \operatorname{Int} F_n$ is a dense set in K. Thus F contains a dense G_δ-subset of K.

From this observation we easily get the assertion that a dense ambivalent subset F of a nonempty compact space K has a nonempty interior. Indeed, assuming the contrary, we would obtain that F and $K \setminus F$ form a couple of residual sets in a Baire space K, which is impossible. (We recall that a topological space is a *Baire space* if the intersection of a sequence of dense open sets is dense.)

Now we quote the aforementioned theorem of D. H. Fremlin and J. D. Pryce [4, Theorem 2E; Proposition 2G].

Theorem 2.2 (Fremlin–Pryce) *Let A be a Borel subset of a compact set X. Then A is measure convex if and only if the closed convex hull $\overline{\mathrm{co}}K \subset A$ for any compact set $K \subset A$.*

We also recall the following well-known facts.

Proposition 2.3

(a) *If K is a compact set in a finite dimensional space, then its convex hull* $\mathrm{co}\, K$ *is compact.*

(b) *The convex hull of finitely many convex compact sets is compact.*

3 Positive Results

In what follows we will assume that X is a compact convex set in a locally convex space E.

3.1 Measure Convex Sets

Proposition 3.1 *Every closed convex subset of X is measure convex.*

Proof Let F be a closed convex subset of X and μ be a probability measure supported by F. Supposing that $r(\mu) \notin F$, the Hahn–Banach separation theorem asserts the existence of an affine continuous function f on X such that $\sup f(F) < f(r(\mu))$. Obviously, $\mu(f) \neq f(r(\mu))$, a contradiction. ■

Proposition 3.2 *Any open convex subset of X is measure convex.*

Proof Let $G \subset X$ be an open convex set. According to Theorem 2.2, it is enough to show that $\overline{\mathrm{co}}\, K \subset G$ whenever $K \subset G$ is a compact set. Given a compact set $K \subset G$, for every $x \in K$ there exists a closed convex neighbourhood V_x of x such that $V_x \subset G$. By compactness of K, the set K can be covered by finitely many compact convex sets $V_{x_1}, \dots, V_{x_n}$. By Proposition 2.3(b),

$$\overline{\mathrm{co}}K \subset \overline{\mathrm{co}}(V_{x_1} \cup \cdots \cup V_{x_n}) = \mathrm{co}(V_{x_1} \cup \cdots \cup V_{x_n}) \subset G,$$

which is the required inclusion. ■

Lemma 3.3 *Let λ be a probability measure on X. If*

$$\mathcal{T} := \{\mu \in \mathcal{M}^+(X) : \mu \leq \lambda, \mu \neq 0\}$$

and

$$S := \left\{ r\Big(\frac{\mu}{\|\mu\|}\Big) : \mu \in \mathcal{T} \right\},$$

then the closure of S equals $\overline{\mathrm{co}}\,\mathrm{spt}\,\lambda$.

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Proof It is easy to see that

$$S = \{r(\mu) : \mu \in \mathcal{M}^1(X),\ \text{there exists } c \in \mathbb{R} \text{ so that } \mu \leq c\lambda\},$$

from which it follows that S is convex.

Set $L := \overline{\mathrm{co}}\,\mathrm{spt}\,\lambda$. To show that $\overline{S} \subset L$, let μ be a nontrivial measure on X with $\mu \leq \lambda$. Then μ is supported by L and thus $r\big(\frac{\mu}{\|\mu\|}\big) \in L$ because L is a closed convex set. Thus $S \subset L$ and consequently $\overline{S} \subset L$.

Conversely, assuming that $\lambda(\overline{S}) < 1$, we can find a compact set $K \subset X \setminus \overline{S}$ such that $\lambda(K) > 0$. For every $x \in K$ we choose its closed convex neighbourhood V_x not intersecting $\overline{S}$. Using a compactness argument we select finitely many points $x_1, \dots, x_n$ of K so that $V_{x_1} \cup \dots \cup V_{x_n}$ covers K. As $\lambda(K) > 0$, there is $i \in \{1, \dots, n\}$ so that $\lambda(V_{x_i}) > 0$. We set $V := V_{x_i}$ and $\mu := \lambda \restriction_V$. Then μ is nontrivial and $\mu \leq \lambda$. Hence the barycenter of $\frac{\mu}{\|\mu\|}$ belongs to S. On the other hand, $r\big(\frac{\mu}{\|\mu\|}\big) \in V$ because V is a closed convex set. This contradiction shows that $\lambda(\overline{S}) = 1$. Thus $\mathrm{spt}\,\lambda \subset \overline{S}$ which gives $L \subset \overline{S}$. ∎

Lemma 3.4 *Let $F \subset X$ be an ambivalent convex set and let $\lambda \in \mathcal{M}^1(X)$ be supported by F. Then there exists a nonempty set $G \subset F \cap \overline{\mathrm{co}}\,\mathrm{spt}\,\lambda$ which is open in $\overline{\mathrm{co}}\,\mathrm{spt}\,\lambda$.*

Proof Let $L := \overline{\mathrm{co}}\,\mathrm{spt}\,\lambda$. In order to find the required set G we note that $L = \overline{F \cap L}$ because the latter set is a closed convex set containing the support of λ. In particular, $F \cap L$ is a dense ambivalent set in L. Due to the preliminary considerations in Section 2, $F \cap L$ has a nonempty interior (relative to L). Hence, the interior of $F \cap L$ is the sought set G. ∎

Theorem 3.5 *Any ambivalent convex subset of X is measure convex.*

Proof Let F be an ambivalent convex subset of X and let λ be a probability measure on X supported by F. We set $\lambda_0 := \lambda$ and let $L_0 := \overline{\mathrm{co}}\,\mathrm{spt}\,\lambda_0$. Let S_0, $\mathcal{T}_0$ and G_0 be sets obtained from Lemmas 3.3 and 3.4 when applied to the measure λ_0. Since S_0 is dense in L_0 and G_0 is nonempty and open in L_0, there is a measure $\mu_0 \in \mathcal{T}_0$ with

$$r\Big(\frac{\mu_0}{\|\mu_0\|}\Big) \in G_0 \subset F.$$

We set $\lambda_1 := \lambda_0 - \mu_0$ and construct by transfinite induction a sequence $\{\lambda_\alpha\}$ of positive measures on X such that, for every ordinal number $\alpha \geq 1$,

(i) $\lambda_{\alpha+1} \leq \lambda_\alpha$;
(ii) either $\lambda_\alpha = 0$ or $\|\lambda_{\alpha+1}\| < \|\lambda_\alpha\|$;
(iii) if $\lambda_\alpha - \lambda_{\alpha+1} \neq 0$, then

$$r\Big(\frac{\lambda_\alpha - \lambda_{\alpha+1}}{\|\lambda_\alpha - \lambda_{\alpha+1}\|}\Big) \in F.$$

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Suppose that the construction has been completed up to an ordinal α. If $\lambda_\alpha = 0$, we set $\lambda_{\alpha+1} = 0$. If λ_α is nontrivial, we apply Lemmas 3.3 and 3.4 to the measure $\frac{\lambda_\alpha}{\|\lambda_\alpha\|}$ (which is carried by F) and get relevant sets L_α, $\mathcal{T}_\alpha$, S_α and G_α with the properties described there. In particular, we have $G_\alpha \subset F \cap L_0$. As in the first step of the proof, we choose a nontrivial measure $\nu \in \mathcal{T}_\alpha$ such that

$$r\Big(\frac{\nu}{\|\nu\|}\Big) \in G_\alpha.$$

By setting $\lambda_{\alpha+1} := \lambda_\alpha - \nu$ we finish the inductive step for an isolated ordinal number.

Let α be a limit ordinal number. Assume that λ_β has been defined for every $\beta < \alpha$. Since $\{\lambda_\beta\}_{\beta<\alpha}$ is a decreasing sequence of positive measures, by the Riesz representation theorem, the mapping

$$\lambda_\alpha \colon g \longmapsto \inf_{\beta<\alpha} \lambda_\beta(g), \quad g \in \mathcal{C}(X), g \geq 0,$$

defines the measure λ_α. This step finishes the inductive construction.

Let γ be the first ordinal number for which $\lambda_\gamma = 0$. Since $\{\|\lambda_\alpha\| : \alpha < \gamma\}$ is a strictly decreasing transfinite sequence, the ordinal number γ is countable. We enumerate $\{\lambda_\alpha - \lambda_{\alpha+1}\}_{1\leq\alpha<\gamma}$ into a (possibly finite) sequence $\{\mu_n\}$, and obtain that $\lambda = \mu_0 + \sum_{n\geq1} \mu_n$ and $\|\lambda\| = \|\mu_0\| + \sum_{n\geq1} \|\mu_n\|$. If the sequence $\{\mu_n\}$ is finite, the equality

$$\lambda = \|\mu_0\| \cdot \frac{\mu_0}{\|\mu_0\|} + \sum_{n\geq1} \|\mu_n\| \cdot \frac{\mu_n}{\|\mu_n\|}$$

yields that λ is a finite convex combination of probability measures having their barycenters in F. Thus, in this case, $r(\lambda) \in F$.

Now, assume that the sequence $\{\mu_n\}$is infinite. For every $k \in \mathbb{N}$ we set

$$c_0 := \|\mu_0\|, \quad c_k := \sum_{n\geq k} \|\mu_n\|$$

and

$$\omega_k := \frac{c_0}{c_0 + c_k} \cdot \frac{\mu_0}{c_0} + \frac{c_k}{c_0 + c_k} \cdot \frac{\sum_{n\geq k} \mu_n}{c_k}.$$

Then $\{\omega_k\}$ is a sequence of probability measures tending to $\frac{\mu_0}{c_0}$. Moreover, $\mu_0 + \sum_{n\geq k} \mu_n$ is obviously an element of $\mathcal{T}_0$, and thus the barycenter $r(\omega_k)$ of ω_k is contained in L_0. As $r(\frac{\mu_0}{c_0}) \in G_0$, which is a relatively open subset of L_0, we can find a sufficiently large $k \in \mathbb{N}$ such that $r(\omega_k) \in G_0 \subset F$. Then

$$\begin{aligned}\lambda &= c_0 \frac{\mu_0}{c_0} + \sum_{n=1}^{k-1} \|\mu_n\| \frac{\mu_n}{\|\mu_n\|} + \sum_{n\geq k} \|\mu_n\| \frac{\mu_n}{\|\mu_n\|} \\ &= \sum_{n=1}^{k-1} \|\mu_n\| \frac{\mu_n}{\|\mu_n\|} + (c_0 + c_k) \cdot \omega_k,\end{aligned}$$

and the last formula shows that λ is a finite convex combination of measures which have their barycenters in F. Since F is convex, the barycenter of λ belongs to F as well.

■

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Proposition 3.6 *Let X be a compact convex subset of a finite dimensional space. Then any Borel convex set $A \subset X$ is measure convex.*

Proof We use again Theorem 2.2. If $K \subset X$ is a compact set, then

$$\overline{\mathrm{co}}K = \mathrm{co}\, K \subset A$$

according to Proposition 2.3(a). ∎

3.2 Measure Extremal Sets

The following simple proposition enables one to derive results on measure extremal sets using the assertions of the previous section.

Proposition 3.7 *Let F be a Borel extremal subset of X. Then F is measure extremal if and only if $X \setminus F$ is measure convex.*

Proof Assume that F is measure extremal and $\mu \in \mathcal{M}^1(X)$ is supported by $X \setminus F$. According to the hypothesis, $r(\mu) \in X \setminus F$ which gives that $X \setminus F$ is measure convex.

Conversely, let $X \setminus F$ be measure convex. Pick $\mu \in \mathcal{M}^1(X)$ with $r(\mu) \in F$. Note that $\mu(F) > 0$ since otherwise $r(\mu)$ would be contained both in F and in $X \setminus F$. If $\mu(X \setminus F) > 0$, set

$$\mu_1 := \frac{1}{\mu(F)}\mu\restriction_F \quad \text{and} \quad \mu_2 := \frac{1}{\mu(X \setminus F)}\mu\restriction_{X\setminus F}.$$

Then

$$r(\mu_2) \in X \setminus F$$

and

$$r(\mu) = \mu(F) \cdot r(\mu_1) + \mu(X \setminus F) \cdot r(\mu_2).$$

This is a contradiction since F is assumed to be extremal. Hence $\mu(X \setminus F) = 0$ and F is measure extremal. ∎

Since $X \setminus F$ is convex if F is extremal, the results from the previous section along with Proposition 3.7 yield the following two corollaries.

Corollary 3.8 *Every closed, open or ambivalent extremal subset of X is measure extremal.*

Corollary 3.9 *If A is a Borel extremal subset of a compact convex set in a finite dimensional space, then A is measure extremal.*

Proposition 3.11 below may be of some interest. It shows that ambivalent faces are necessarily closed.

Lemma 3.10 *Any proper extremal subset of X has an empty interior in X.*

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Proof Let F be a proper extremal subset of X. Assume that the interior of F relative to X contains a point z. Let x be any point of $X \setminus F$. Thanks to the continuity of vector operations in E, there is $\alpha \in (0, 1)$ so that $\alpha x + (1 - \alpha)z$ lies in the interior of F in X. Since F is extremal, $x \in F$ likewise. Thus $F = X$ which contradicts our assumption that F is proper. ■

Proposition 3.11 *Any ambivalent face is closed and, consequently, it is measure convex.*

Proof Let F be a nonempty ambivalent face such that $\overline{F} \setminus F \neq \varnothing$. Notice that $\overline{F}$ is a nonempty convex compact set. By Lemma 3.10, the interior of F relative to $\overline{F}$ is empty. Thus F and $\overline{F} \setminus F$ are disjoint dense $(\mathrm{alg}(\mathcal{F}))_\delta$-subsets of a compact space $\overline{F}$, which is impossible since $\overline{F}$ is a Baire space. ■

4 Counterexamples

4.1 Measure Convex Sets

Proposition 4.1 *Let*

$$F := \bigcup_{n=2}^{\infty} \Big\{ \mu \in \mathcal{M}^1([0, 1]) : \operatorname{spt} \mu \subset \Big[\frac{1}{n}, 1 \Big] \Big\}.$$

Then F is an F_σ-face of $\mathcal{M}^1([0, 1])$ which is not measure convex.

Proof Clearly, F is a convex set. As each set

$$\Big\{ \mu \in \mathcal{M}^1([0, 1]) : \operatorname{spt} \mu \subset \Big[\frac{1}{n}, 1 \Big] \Big\}, \quad n \in \mathbb{N},$$

is obviously a closed face and the union of extremal sets is again extremal, F is extremal and of type F_σ.

Define a measure ω on $[0, 1]$ as

$$\omega := \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon_{\frac{1}{n}}$$

and denote by Ω the image $\varepsilon\omega$ of ω (recall that ε is a homeomorphic embedding of $[0, 1]$ into $\mathcal{M}^1([0, 1])$). Then Ω is a probability measure on $\mathcal{M}^1([0, 1])$, $\Omega(F) = 1$ and $r(\Omega) = \omega \notin F$. This shows that F is not measure convex. ■

Proposition 4.2 *Let λ be a probability measure on a compact space K and*

$$\psi \colon \mu \longmapsto \mu_s(K), \quad \mu \in \mathcal{M}^1(K),$$

(μ_s is a singular part of μ with respect to λ). Then ψ is a limit of a decreasing sequence of lower semicontinuous functions on $\mathcal{M}^1(K)$.

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Proof For $n \in \mathbb{N}$, set

$$\psi_n(\mu) := \sup\Big\{ \mu(G) : \ G \subset K \ \text{ open and } \lambda(G) < \frac{1}{n} \Big\}, \quad \mu \in \mathcal{M}^1(K).$$

Obviously, $\{\psi_n\}$ is a decreasing sequence of lower semicontinuous functions. According to the portmanteau theorem [2, Theorem 30.10], the function

$$\mu \longmapsto \mu(G), \quad \mu \in \mathcal{M}^1(K),$$

is lower semicontinuous on $\mathcal{M}^1(K)$ for any open set $G \subset K$.

Pick $n \in \mathbb{N}$ and $\mu \in \mathcal{M}^1(K)$. There exists a Borel set $B \subset K$ such that

$$\mu_s(B) = \mu_s(K) = \psi(\mu) \quad \text{and} \quad \lambda(B) = 0.$$

Let $G \subset K$ be an open set containing B for which $\lambda(G) < \frac{1}{n}$. Then

$$\psi(\mu) = \mu_s(B) \leq \mu_s(G) \leq \mu(G) \leq \psi_n(\mu).$$

Hence, $\psi \leq \psi_n$ for any $n \in \mathbb{N}$.

It remains to show that $\lim_n \psi_n = \psi$. To this end, pick $\mu \in \mathcal{M}^1(K)$ and $c > \psi(\mu)$. Since $\mu_{ac} \ll \lambda$, there exists $n \in \mathbb{N}$ so that

$$\mu_{ac}(B) < c - \psi(\mu) = c - \mu_s(K)$$

whenever B is a Borel set, $\lambda(B) < \frac{1}{n}$. Now, if $G \subset K$ is an open set, $\lambda(G) < \frac{1}{n}$, then

$$\mu(G) = \mu_s(G) + \mu_{ac}(G) \leq \mu_s(K) + c - \mu_s(K) = c.$$

Thus, $\psi_n(\mu) \leq c$, and therefore $\psi_n \to \psi$. ■

Proposition 4.3 *If λ denotes the Lebesgue measure on $[0,1]$ and*

$$G := \{\mu \in \mathcal{M}^1([0,1]) : \mu \perp \lambda\},$$

then G is a G_δ-face of $\mathcal{M}^1([0,1])$ which is not measure convex.

Proof It is easy to check that G is convex and extremal.

In the next step we show that G is a G_δ-set. Let $\{\psi_n\}$ be a sequence of functions as in the proof of Proposition 4.2. Since the functions are lower semicontinuous, the assertion follows from the following equalities

$$\begin{aligned} G &= \{\mu \in \mathcal{M}^1([0,1]) : \mu_s([0,1]) = 1\} = \bigcap_{n=1}^{\infty} \{\mu \in \mathcal{M}^1([0,1]) : \psi_n(\mu) = 1\} \\ &= \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \Big\{ \mu \in \mathcal{M}^1([0,1]) : \psi_n(\mu) > 1 - \frac{1}{k} \Big\}. \end{aligned}$$

To show that G is not measure convex, let $\Lambda := \varepsilon\lambda$ denote the image of the Lebesgue measure λ. Then Λ is a probability measure on $\mathcal{M}^1([0,1])$ and $r(\Lambda) = \lambda \notin G$. Since

$$\varepsilon([0,1]) = \{\varepsilon_x : x \in [0,1]\} \subset G,$$

$\Lambda(G) = 1$. Hence, the set G is not measure convex. ■

4.2 Measure Extremal Sets

Proposition 4.4 *Let K be a compact space. Then the function*

$$\varphi\colon \mu \longmapsto \mu_a(K), \quad \mu \in \mathcal{M}^1(K),$$

is a limit of an increasing sequence of upper semicontinuous functions on $\mathcal{M}^1(K)$.

Proof First of all, note that

$$\mu_a(K) = \sum_{x\in K} \mu(\{x\})$$

for every $\mu \in \mathcal{M}^1(K)$.

Pick $b > 0$ and define the function

$$\varphi_b\colon \mu \longmapsto \mu\big(\{x \in K : \mu(\{x\}) \geq b\}\big), \quad \mu \in \mathcal{M}^1(K).$$

We claim that φ_b is an upper semicontinuous function on $\mathcal{M}^1(K)$.

Indeed, let $c > 0$ and

$$G := \{\nu \in \mathcal{M}^1(K) : \varphi_b(\nu) < c\}.$$

We have to show that G is open. To this end, pick $\mu \in G$ and set

$$L := \{x \in K : \mu(\{x\}) \geq b\}.$$

Note that L is a finite set in K. Let U be an open subset of K such that $L \subset U$ and $\mu(\overline{U}) < c$. For every $x \in K \setminus U$ there is an open neighbourhood V_x of x such that $\mu(\overline{V}_x) < b$. By compactness there must exist finitely many points $x_1, \dots, x_n$ in $K \setminus U$ so that

$$K \setminus U \subset V_{x_1} \cup \cdots \cup V_{x_n}.$$

Since the function $\nu \longmapsto \nu(H)$, $\nu \in \mathcal{M}^1(K)$, is upper semicontinuous on $\mathcal{M}^1(K)$ for every closed set $H \subset K$ due to the portmanteau theorem [2, Theorem 30.10], the set

$$W := \{\nu \in \mathcal{M}^1(K) : \nu(\overline{U}) < c, \nu(\overline{V}_{x_i}) < b, i = 1 \cdots n\}$$

is open and contains μ. It remains to prove that $W \subset G$.

To verify this, pick $\nu \in W$ and set $L_\nu := \{x \in K : \nu(\{x\}) \geq b\}$. Since $L_\nu \subset U$,

$$\varphi_b(\nu) = \nu(L_\nu) \leq \nu(U) \leq \nu(\overline{U}) < c.$$

Since the functions $\varphi_{\frac{1}{n}}$, $n \in \mathbb{N}$, form an increasing sequence converging to φ, the proof is finished. ■

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Proposition 4.5 *Let K be a compact space which admits a continuous probability measure. Then the set*

$$G := \{\mu : \mu \in \mathcal{M}^1(K), \mu = \mu_c\}$$

is a G_δ-face of $\mathcal{M}^1(K)$ which is not measure extremal.

Proof Let $\varphi\colon \mu \longmapsto \mu_a(K)$, $\mu \in \mathcal{M}^1(K)$. According to the proof of Proposition 4.4, the functions

$$\varphi_n\colon \mu \longmapsto \mu\Big(\Big\{x \in K : \mu\big(\{x\}\big) \geq \frac{1}{n}\Big\}\Big), \quad \mu \in \mathcal{M}^1(K),$$

form a sequence of positive upper semicontinuous functions such that $\varphi_n \nearrow \varphi$ on $\mathcal{M}^1(K)$. Since

$$G = \{\mu \in \mathcal{M}^1(K) : \varphi(\mu) = 0\} = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \Big\{\mu \in \mathcal{M}^1(K) : \varphi_n(\mu) < \frac{1}{k}\Big\},$$

G is a G_δ-set which is obviously a face.

Let ω be a continuous probability measure on K and $\Omega := \varepsilon\omega$. Then $r(\Omega) = \omega \in G$ whereas $\Omega(G) = 0$ since Ω is supported by $\varepsilon(K)$. Thus G is not measure extremal. ■

Remark 4.6 *We recall a well-known fact that a metrizable compact space K admits a continuous measure if and only if K contains a nonempty perfect subset,* i.e., *a closed set without isolated points, and this is the case if and only if K is uncountable.*

Proposition 4.7 *Let F be the face of $\mathcal{M}^1([0,1])$ generated by the Lebesgue measure λ. Then F is an F_σ-face which is not measure extremal.*

Proof As was mentioned in Section 2, F is an F_σ-set. Assume that $\mu \in F \cap \varepsilon([0,1])$. Hence, $\mu = \varepsilon_x$ for some $x \in [0,1]$ and by the definition of F, there exist $\nu \in \mathcal{M}^1([0,1])$ and $\alpha \in [0,1)$ so that

$$\lambda = \alpha\nu + (1-\alpha)\varepsilon_x.$$

Then

$$0 = \lambda(\{x\}) = \alpha\nu(\{x\}) + (1-\alpha),$$

which implies that $\alpha = 1$, a contradiction. Therefore, $F \cap \varepsilon([0,1]) = \varnothing$. We see that $\Lambda(F) = 0$ whereas $r(\Lambda) = \lambda \in F$. Thus the face F is not measure extremal. ■

4.3 Combined Examples

Theorem 4.8 *There exists a G_δ-face in $\mathcal{M}^1([0,1])$ which is neither measure convex nor measure extremal.*

Proof We combine examples of Propositions 4.3 and 4.5. Let λ denote again the Lebesgue measure on $[0,1]$ and C the Cantor ternary set. Set $G := G_1 \cap G_2$ where

$$G_1 := \{\mu \in \mathcal{M}^1([0,1]) : \mu \perp \lambda\}$$

and

$$G_2 := \{\mu \in \mathcal{M}^1([0,1]) : \mu \restriction_C \text{ is continuous}\}.$$

It follows from Proposition 4.3 and the proof of Proposition 4.4 that both sets G_1 and G_2 are G_δ-faces in $\mathcal{M}^1([0,1])$. Thus G is a G_δ-face as well.

Let Λ denote the image $\varepsilon\lambda$. Then $r(\Lambda) = \lambda$ by Proposition 2.1 and the barycenter $r(\Lambda)$ is not in G although

$$\Lambda(G) = \lambda(\varepsilon^{-1}(G)) = \lambda([0,1] \setminus C) = 1.$$

Thus G is not measure convex.

Let $\Omega := \varepsilon\nu$, where ν is a continuous probability measure supported by C. Then Ω is supported by $\varepsilon(C)$, and consequently $\Omega(G) = 0$. On the other hand, $r(\Omega) = \nu \in G$, and thus G is not measure extremal. ■

Theorem 4.9 *There exists an F_σ-face F in $\mathcal{M}^1([0,1])$ which is neither measure convex nor measure extremal.*

Proof We combine examples of Propositions 4.1 and 4.7. Set $F := F_1 \cap F_2$ where $F_1 := \operatorname{face}(\lambda)$ and

$$F_2 := \bigcup_{n=2}^{\infty} \Big\{ \mu \in \mathcal{M}^1([0,1]) : \operatorname{spt}\mu \subset \Big[\frac{1}{n}, 1\Big] \Big\}$$

(here λ is again the Lebesgue measure on $[0,1]$ and $\operatorname{face}(\lambda)$ denotes the face generated by λ). According to the aforementioned examples, F is an F_σ-face in $\mathcal{M}^1([0,1])$. Let

$$\omega := 2\lambda \restriction_{[\frac{1}{2},1]} \quad \text{and} \quad \Omega := \varepsilon\omega.$$

Then $\operatorname{spt}\Omega = \varepsilon([\frac{1}{2},1])$, thus $\operatorname{spt}\Omega \cap F = \varnothing$. Hence $\Omega(F) = 0$, but $r(\Omega) = \omega$ is contained in F which implies that F is not measure extremal.

To show that F is not measure convex, set for $n \in \mathbb{N}$

$$\lambda_n := \frac{n}{n-1}\lambda \restriction_{[\frac{1}{n},1]}, \quad \text{and} \quad \Omega := \sum_{n=1}^{\infty} \frac{1}{2^n}\varepsilon_{\lambda_n}.$$

Since $\lambda_n \in F$ for every $n \in \mathbb{N}$, $\Omega(F) = 1$. On the other hand,

$$r(\Omega) = \sum_{n=1}^{\infty} \frac{1}{2^n}\lambda_n$$

is not contained in F. Thus F is not measure convex and the proof is finished. ■

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