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Non-discrete Frieze Groups

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Abstract. The classification of Euclidean frieze groups into seven conjugacy classes is well known, and many articles on recreational mathematics contain frieze patterns that illustrate these classes. However, it is only possible to draw these patterns because the subgroup of translations that leave the pattern invariant is (by definition) cyclic, and hence discrete. In this paper we classify the conjugacy classes of frieze groups that contain a non-discrete subgroup of translations, and clearly these groups cannot be represented pictorially in any practical way. In addition, this discussion sheds light on why there are only seven conjugacy classes in the classical case.

1 Introduction

Informally, a *frieze* is a decorative linear strip given by the regular repetition of some pattern along a line in the complex plane $\mathbb{C}$, and it is well known that (up to the identification described below) there are exactly seven distinct frieze patterns. These seven patterns are as follows:

$$
\begin{array}{lccccccc}
F_1: & \cdots & \lfloor & & \lfloor & & \cdots \\
F_2: & \cdots & \lfloor & \lceil & \lfloor & \lceil & \cdots \\
F_3: & \cdots & \sim & & \sim & & \cdots \\
F_4: & \cdots & \in & & \in & & \cdots \\
F_5: & \cdots & \vee & & \vee & & \cdots \\
F_6: & \cdots & \vee & \wedge & \vee & \wedge & \cdots \\
F_7: & \cdots & \square & & \square & & \cdots
\end{array}
$$

Each frieze pattern gives rise to a *frieze group*, namely the Euclidean symmetry group of that frieze. Mathematically speaking, it is the groups rather than the patterns that are interesting, and frieze groups are characterised as those groups of isometries of $\mathbb{C}$ that have the following two properties.

(i) There is a straight line L in $\mathbb{C}$ that is invariant under the action of G.
(ii) The subgroup of translations in G is an infinite cyclic group.

As L is invariant under G, all translations in G are in the direction of L.

Two frieze groups F and F' are *conjugate* within the group of Euclidean similarities if and only if there is a Euclidean similarity $\phi: \mathbb{C} \to \mathbb{C}$ such that $F' = \phi F \phi^{-1}$. It is usual to identify conjugate frieze groups, and the well-known classical result is that *there are exactly seven conjugacy classes of frieze groups, and these are represented by the frieze*

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groups of the seven frieze patterns illustrated above. Clearly, when discussing conjugacy classes, we can restrict our attention to frieze groups for which $L = \mathbb{R}$.

We turn now to the results in this paper. Lyndon [7, p. 43] classified the seven frieze groups and stated that *a group of real isometries whose subgroup of translations is, for example,* $\{z \mapsto z + a + b7^{1/3} : a, b \in \mathbb{Q}\}$, *is of little geometric interest*. This may be true, but it is not the real issue because the classification into conjugacy classes is algebraic, not geometric. Taking an algebraic view, we shall say that a group G is a *generalised frieze group* if it is a group of real isometries whose subgroup of translations is non-trivial *but not necessarily discrete*. Now let T be a given non-trivial group of real translations, and let $\mathcal{G}(T)$ be the class of generalised frieze groups whose subgroup of translations is T. Our aim is to classify and count the conjugacy classes in $\mathcal{G}(T)$ and in this notation the classical result is as follows.

Theorem 1.1 *If T is cyclic, then there are exactly seven conjugacy classes in $\mathcal{G}(T)$.*

We shall also prove (among others) the following three theorems, the last of which relates to Lyndon's remark.

Theorem 1.2 *Let T be the group $\{z \mapsto z + a : a \in \mathbb{F}\}$, where $\mathbb{F}$ is a subfield of $\mathbb{R}$. Then there are exactly five conjugacy classes in $\mathcal{G}(T)$.*

Theorem 1.3 *If $T = \{z \mapsto z + a : a \in \mathbb{Z} + \mathbb{Z}\sqrt{2}\}$, then there are nine conjugacy classes in $\mathcal{G}(T)$.*

Theorem 1.4 *If $T = \{z \mapsto z + a + b7^{1/3} : a, b \in \mathbb{Q}\}$, then there are five conjugacy classes in $\mathcal{G}(T)$.*

Each of these results is a special case of the following general theorem which we state and prove in a more definitive form later in the paper.

Theorem 1.5 *For each non-trivial group T of real translations there is a positive integer $N(T)$ associated with T (which will be defined in Section 5) such that $\mathcal{G}(T)$ has exactly $3 + 2N(T)$ conjugacy classes with respect to similarities.*

Finally, we mention that frieze patterns and frieze groups can be defined (in the obvious way) in the hyperbolic plane, see [6]. Similar questions arise in this case, and we leave these for the interested reader to pursue.

2 Some Preliminary Remarks

We shall be considering groups of *real isometries*, *i.e.*, isometries of $\mathbb{C}$ that leave $\mathbb{R}$ invariant, and we begin by introducing notation that will be used throughout the paper. First, for much of the time we shall be using the real isometries

$$\tau(z) = z + 1, \quad \sigma(z) = \bar{z}, \quad \rho(z) = -z, \quad \mu(z) = -\bar{z}, \quad \nu(z) = \bar{z} + \tfrac{1}{2}.$$

We also use I for the identity map, $\mathcal{I}$ for the group of real isometries, and $\langle a, b, \dots\rangle$ for the group generated by $a, b, \dots$.

Each real isometry of $\mathbb{C}$ is of the form $z \mapsto az + b$ (a *direct* isometry), or $z \mapsto a\bar{z} + b$ (an *indirect* isometry), where $a = \pm 1$ and b is real. Apart from I, there are five types of real isometries, namely

translations: $z \mapsto z + b = I(z) + b, \quad b \neq 0,$
rotations of order two: $z \mapsto -z + b = \rho(z) + b,$
reflections across a vertical line: $z \mapsto -\bar{z} + b = \mu(z) + b,$
the reflection across $\mathbb{R}$: $z \mapsto \bar{z} = \sigma(z),$
glide reflections: $z \mapsto \bar{z} + b = \sigma(z) + b, b \neq 0,$

where, in each case, b is real.

Consider for a moment the list $F_1, \dots, F_7$ above, and let us use F_i for both the pattern and the group. If we assume (as we may) that the subgroup of translations that leaves each F_i invariant is generated by τ, then the seven representative frieze groups are $F_1 = \langle\tau\rangle$, $F_2 = \langle\nu\rangle$, $F_3 = \langle\tau, \rho\rangle$, $F_4 = \langle\tau, \sigma\rangle$, $F_5 = \langle\tau, \mu\rangle$, $F_6 = \langle\nu, \mu\rangle$, and $F_7 = \langle\tau, \rho, \sigma\rangle$ or, informally,

$$\langle z+1\rangle,\ \langle \bar{z} + \tfrac{1}{2}\rangle,\ \langle z+1, -z\rangle,\ \langle z+1, \bar{z}\rangle,\ \langle z+1, -\bar{z}\rangle,\ \langle \bar{z} + \tfrac{1}{2}, -\bar{z}\rangle,\ \langle z+1, -z, \bar{z}\rangle.$$

Proofs of this can be found in many texts, *e.g.*, [2, p. 164], [4, p. 516], [5, Chapter 3], [7, pp. 40–42], and [8, pp. 78–84].

Our analysis of generalised frieze groups will be simplified if we write the five different types of isometries in a common form. So let $\mathcal{K} = \{I, \rho, \sigma, \mu\}$ (the Klein 4-group). Then each real isometry g takes the form $g(z) = \phi_g(z) + g(0)$, where $\phi_g \in \mathcal{K}$. Further, the map $\Theta\colon \mathcal{I} \to \mathcal{K}$ given by $\Theta(g) = \phi_g$, is a surjective homomorphism whose kernel is the group $\mathcal{T}$ of all real translations. The main idea behind the classification of conjugacy classes in $\mathcal{G}(T)$ is as follows. Suppose that $G \in \mathcal{G}(T)$. Then T is a normal subgroup of G, and the restriction Θ_G of Θ to G is a homomorphism of G into $\mathcal{K}$. Thus the quotient group G/T is either the trivial group, a cyclic group of order two, or the Klein group of order four, and every non-trivial element of G/T has order two. With a little work we can use these facts to identify the conjugacy classes of groups in $\mathcal{G}(T)$. We know that the quotient group $\mathcal{I}/\mathcal{T}$ is isomorphic to $\mathcal{K}$, so that $\mathcal{I}$ is the disjoint union of the four cosets

(i) $\Theta^{-1}(I)$ (the group $\mathcal{T}$ of real translations),
(ii) $\Theta^{-1}(\rho)$ (all rotations of order two with a real fixed point),
(iii) $\Theta^{-1}(\mu)$ (all reflections across some vertical line),
(iv) $\Theta^{-1}(\sigma)$ (the reflection σ across $\mathbb{R}$, and all glide reflections).

This observation raises a question about terminology. As our discussion will be in terms of Θ and $\mathcal{K}$, from now on we shall *classify the map σ as a glide reflection*, and not as a reflection. Admittedly this conflicts with standard terminology but it is simply terminology, and it is consistent with the view that the identity map is a trivial translation. More importantly, it greatly simplifies our algebraic approach which concentrates on the coset decomposition rather than the geometric action. To summarise: in this revised terminology, an isometry g in $\mathcal{I}$ is a *reflection* if and only if $\Theta(g) = \mu$ and it is a *glide reflection* if and only if $\Theta(g) = \sigma$. When we want to distinguish between $z \mapsto \bar{z}$ and $z \mapsto \bar{z} + a$, where $a \neq 0$, we shall say that $z \mapsto \bar{z}$ is the *trivial glide reflection*

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and that $z \mapsto \bar{z} + a$ is a *non-trivial glide reflection*. For the benefit of the reader we emphasize that from now on, a *reflection* will always mean a reflection across a vertical line, and *not* across the real axis. Thus a reflection preserves each of the upper and lower half-planes, whereas a glide reflection interchanges them, and this perhaps adds support in favour of the revised terminology.

Now let T be a non-trivial group of real translations, and suppose that $G \in \mathcal{G}(T)$. If G contains a rotation and a reflection, or a rotation and a glide reflection, or a reflection and a glide reflection, then it contains isometries of all three types because, in the notation above, $\rho\sigma = \mu$, $\rho\mu = \sigma$, and $\sigma\mu = \rho$. It follows from this that G is one of the following distinct types of groups.

(i) G contains only translations.
(ii) G contains translations and rotations (and only these).
(iii) G contains translations and reflections (and only these).
(iv) G contains translations and glide reflections (and only these).
(v) G contains translations, rotations, reflections and glide reflections.

We shall need the following lemma.

Lemma 2.1 *Let f be a similarity that preserves $\mathbb{R}$, and g a real isometry. Then g and fgf^{-1} are simultaneously either* (1) *a translation,* (2) *a rotation,* (3) *a reflection across a vertical line,* (4) *the trivial glide reflection $z \mapsto \bar{z}$,* (5) *a non-trivial glide reflection.*

Proof The five classes of isometries described in the lemma can be identified by (a) the number of fixed points in $\mathbb{R}$, and (b) whether they are direct or indirect isometries. For example, the isometry is a reflection across a vertical line if and only if it is an indirect isometry with one fixed point in $\mathbb{R}$. Since (a) and (b) are invariant under conjugation, the result follows immediately. ∎

Lemma 2.1 implies that each of the classes (i)–(v) described above is a union of conjugacy classes; thus our problem reduces to describing and counting the conjugacy classes in each of these five cases separately. It is also clear that for any choice of the translation group T, there are at least five conjugacy classes of groups, namely

$$T, \quad \langle T, \rho\rangle, \quad \langle T, \mu\rangle, \quad \langle T, \sigma\rangle, \quad \langle T, \rho, \mu, \sigma\rangle.$$

Thus Theorem 1.2 suggests (informally, and not surprisingly) that the richer the algebraic structure of T, the fewer the number of conjugacy classes.

3 The Proof of Theorem 1.2

Theorem 1.2 is much easier to prove than the classical Theorem 1.1 because the Euclidean translation lengths form a field rather than an integral domain. The fact that this is never mentioned in texts is testament to the (perhaps excessive) concentration on geometric illustrations to the exclusion of algebraic ideas.

We suppose then that G is a generalised frieze group that leaves $\mathbb{R}$ invariant, and whose subgroup of translations contains $z \mapsto z + a$ if and only if $a \in \mathbb{F}$. The first possibility is that G contains only these translations and there is nothing more to be said about this case. Now suppose that G contains only translations and rotations.

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Then G contains some rotation $r(z) = -z + a$. Let $t(z) = z - a/2$ and note that here we are not assuming that $a \in \mathbb{F}$. Then $tTt^{-1} = T$ (because translations commute), and $trt^{-1}(z) = -z = \rho(z)$. It follows easily that tGt^{-1} has the coset decomposition $T \cup T\rho$, so that G is conjugate to $\langle T, \rho \rangle$. This shows that there is only one conjugacy class of groups that contains only translations and rotations. A similar argument holds for groups that contain only translations and reflections across some vertical line, and these groups are conjugate to $\langle T, \mu \rangle$.

We need a slightly different argument for groups that contain only translations and glide reflections. Let G be such a group, and suppose that $h \in G$, where $h(z) = \bar{z} + a$ with a real. Then $h^2(z) = z + 2a$, so that $2a$, and hence a, is in $\mathbb{F}$. Let $f(z) = z + a$. Then $f \in G$ so that $hf^{-1} \in G$. As $hf^{-1}(z) = \bar{z} = \sigma(z)$, we now see that $G = \langle T, \sigma \rangle$.

The argument so far accounts for four conjugacy classes of groups. The remaining case that we have not yet discussed is when G contains translations, and at least two of the other maps considered above. In this case G must contain a rotation, a reflection across a vertical line, and a glide reflection. Then, by conjugating with a translation (as above) we may assume that the rotation is ρ. Also as above, we see that G must contain the map σ. It follows that G contains $\rho\sigma$, which is μ. It follows that after a conjugation with a suitable translation G has coset decomposition $T \cup T\rho \cup T\sigma \cup T\mu$. Thus if G contains all four types of maps, then G is conjugate to $\langle T, \rho, \sigma, \mu \rangle$, and this gives us exactly one more conjugacy class.

4 Groups Without Glide Reflections

In this section we show that given a non-trivial, real translation group T, each of the subclasses (i), (ii), and (iii) described above is a single conjugacy class. Thus the groups in $\mathcal{G}(T)$ that do not contain glide reflections are partitioned into exactly three conjugacy classes. This result follows immediately from the following lemma.

Lemma 4.1 *Suppose that T is a non-trivial group of real translations and that G in $\mathcal{G}(T)$ does not contain any glide reflections. Then G is conjugate to exactly one of the groups T, $\langle T, \rho \rangle$, and $\langle T, \mu \rangle$.*

Proof First, it is clear from Lemma 2.1 that no two of the three given groups are conjugate. As G does not contain any glide reflections, the quotient group G/T is either trivial or a group of order two. Obviously, if G/T is trivial, then $G = T$. Now suppose that G/T is of order two. Then either G contains some map $g(z) = -z + b$ or some map $h(z) = -\bar{z} + b$. Let $\phi(z) = z - b/2$, so that $\phi g \phi^{-1} = \rho$ and $\phi h \phi^{-1} = \mu$. If $g \in G$, then $G = \langle T, g \rangle$ so that $\phi G \phi^{-1} = \langle T, \rho \rangle$. If $h \in G$, then $G = \langle T, h \rangle$ so that $\phi G \phi^{-1} = \langle T, \mu \rangle$. ∎

Lemma 4.1 shows that there is one conjugacy class in $\mathcal{G}(T)$ for each of the groups that contain only translations, only translations and rotations, and only translations and reflections. Later we shall see that for some positive integer $N(T)$ (which will be defined later) there are $N(T)$ conjugacy classes of groups that contain only translations and glide reflections, and also $N(T)$ conjugacy classes of groups that contain isometries of all four types. Thus the total number of conjugacy classes in $\mathcal{G}(T)$ is

$3 + 2N(T)$. We shall also see that if T is cyclic then $N(T) = 2$, and this will confirm the classical division into seven conjugacy classes.

5 An Analysis of Glide Reflections

It is clear from the preceding remarks that the number of conjugacy classes in $\mathcal{G}(T)$ depends in an essential way on the glide reflections in G. We shall now explain the basic mechanism that allows us to compute the number of conjugacy classes in $\mathcal{G}(T)$ when glide reflections are present. The key question to ask is that if the translation $g(z) = z + g(0)$ is in G, *is the glide reflection* $z \mapsto \bar{z} + g(0)/2$ *(whose second iterate is* g*) in* G? In order to discuss this question we let

$$T_0 = \{g(0) : g \in T\}, \qquad T_1 = \{\tfrac{1}{2}g(0) : g \in T\},$$

so that $T_0 \subset T_1$. Of course, $z \mapsto z + a$ is in T if and only if a is in T_0. However, although $g \mapsto g(0)$ is an isomorphism from T onto T_0, our arguments will be more transparent (and easier to follow) if we use a notation that carefully preserves the distinction between T and T_0.

Next, let U_T be the multiplicative group of non-zero real numbers u such that $uT_0 = T_0$. We call U_T the *group of units of* T_0 (or of T). If $T_0 = \mathbb{Z}$ (the classical case), then $U_T = \{1, -1\}$. However, if $T_0 = \mathbb{R}$, then U_T is the infinite multiplicative group of non-zero real numbers. The groups T_0, T_1, and U_T will play a crucial role in what follows and, in particular, we have the following lemma.

Lemma 5.1 *Suppose that T is a non-trivial group of real translations and that f is a similarity whose restriction to $\mathbb{R}$ is, say, $f(x) = ax + b$, where $a, b, x \in \mathbb{R}$. Then $fTf^{-1} = T$ if and only if $aT_0 = T_0$ (equivalently, $a \in U_T$).*

Proof If $g(z) = z + k$, then $fgf^{-1}(x) = x + ak$. Thus

$$fTf^{-1} = \{z \mapsto z + ak : k \in T_0\} = \{z \mapsto z + k' : k' \in aT_0\}.$$

It follows that $fTf^{-1} = T$ if and only if $aT_0 = T_0$. ∎

Lemma 5.1 shows that if two generalised frieze groups (both with invariant line $\mathbb{R}$) are conjugate, then the conjugating map $z \mapsto az + b$ is such that $a \in U_T$. The group U_T also plays another important role in our analysis. For each u in U_T we have a natural operation of "multiplication by u" acting on the quotient group T_1/T_0, namely

$$u(x + T_0) = ux + uT_0 = ux + T_0, \quad x \in T_1,$$

and this provides the "multiplication" map $m_u \colon T_1/T_0 \to T_1/T_0$ defined by

$$m_u(x + T_0) = u(x + T_0) = ux + T_0.$$

The next lemma describes in detail how the maps m_u act on the quotient group T_1/T_0.

Lemma 5.2 *Let $M = \{m_u : u \in U_T\}$. Then M is a group of permutations of T_1/T_0, and $u \mapsto m_u$ is a homomorphism of U_T into M.*

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With this available, we can finally state our main result, which is alluded to in Theorem 1.5 and which gives a precise formula for the number of conjugacy classes in $\mathcal{G}(T)$.

Theorem 5.3 *Let T be a non-trivial group of real translations, and let $N(T)$ be the number of orbits in T_1/T_0 under the action of the group M. Then $\mathcal{G}(T)$ has exactly $3 + 2N(T)$ conjugacy classes with respect to similarities.*

We give the proof of Lemma 5.2 now, but as it may be helpful for the reader to gain familiarity with these ideas before seeing the proof of Theorem 5.3, we give some applications of Theorem 5.3 in the next section and defer its proof until Section 7.

The Proof of Lemma 5.2 First, m_u is properly defined on the quotient group T_1/T_0. Indeed, suppose that $x + T_0 = y + T_0$. Then $x - y \in T_0$ so that $u(x - y) \in T_0$. Thus $ux + T_0 = uy + T_0$ so that m_u is properly defined on T_1/T_0. Next, m_u maps T_1/T_0 into itself because if $x \in T_1$, then $2x \in T_0$. Thus $2ux \in T_0$, so that $ux \in T_1$. Next, m_u is injective because if $ux + T_0 = uy + T_0$, then $u(x - y) \in T_0$, so that (as $u^{-1} \in U_T$) $x - y \in T_0$ and hence $x + T_0 = y + T_0$. Finally, m_u is surjective since given any element $y + T_0$ of T_1/T_0 is $m_u(u^{-1}y + T_0)$. We conclude that m_u is a permutation of T_1/T_0. It is obvious that M is a group, and that $u \mapsto m_u$ is a homomorphism of U_T into M. ■

6 Some Examples

First, suppose that T_0 is finitely generated, say $T_0 = \langle a_1, \dots, a_r\rangle$. Then

$$\begin{aligned} T_0 &= \langle a_1, \dots, a_r\rangle = \{m_1a_1 + \cdots + m_ra_r : m_1, \dots, m_r \in \mathbb{Z}\},\\ T_1 &= \{\tfrac{1}{2}m_1a_1 + \cdots + \tfrac{1}{2}m_ra_r : m_1, \dots, m_r \in \mathbb{Z}\},\\ T_1/T_0 &= \{\varepsilon_1a_1 + \cdots + \varepsilon_ra_r + T_0 : \varepsilon_1, \dots, \varepsilon_r = 0, \tfrac{1}{2}\}. \end{aligned}$$

Thus, in this case T_1/T_0 is isomorphic to $C_2 \times \cdots \times C_2$ with r factors. For example, if $T_0 = \mathbb{Q}$, then $T_1 = T_0$ so that T_1/T_0 is the trivial group. If $T_0 = \mathbb{Z}$, then T_1/T_0 is C_2. If $T_0 = \mathbb{Z} + \mathbb{Z}\sqrt{2}$, then T_1/T_0 is $C_2 \times C_2$. We now give some explicit examples.

Example 6.1 Let $T = \langle\tau\rangle$ (this is the classical case). Then $T_1/T_0 = \{\mathbb{Z}, \frac{1}{2} + \mathbb{Z}\}$, $U_T = \{-1, 1\}$, and M acts as the trivial group on T_1/T_0. Thus $N(T) = 2$ (the cardinality of T_1/T_0), so there are exactly seven conjugacy classes in $\mathcal{G}(T)$.

Example 6.2 Suppose that T_0 is a subfield of $\mathbb{R}$. Then $T_1 = T_0$, so that T_1/T_0 is the trivial group. As T_1/T_0 is a singleton set, $N(T) = 1$. Hence in this case there are exactly five conjugacy classes in $\mathcal{G}(T)$. This provides another proof of Theorem 1.2.

Example 6.3 Suppose that $T_0 = \mathbb{Q} + \mathbb{Q}\lambda$, where λ is irrational. Then $T_0 = T_1$, so that $N(T) = 1$. Thus, for any irrational λ, there are five conjugacy classes in $\mathcal{G}(T)$. In particular, there are five conjugacy classes in the example given by Lyndon, and this provides a proof of Theorem 1.4.

Example 6.4 Suppose that $T_0 = \mathbb{Z} + \mathbb{Z}\lambda$, where λ is irrational. Then

$$T_1/T_0 = \left\{ T_0, \tfrac{1}{2} + T_0, \tfrac{\lambda}{2} + T_0, \tfrac{1}{2} + \tfrac{\lambda}{2} + T_0 \right\}.$$

Thus $1 \leq N(T) \leq 4$, so there are 5, 7, 9, or 11 conjugacy classes in $\mathcal{G}(T)$.

In certain cases in Example 6.4 we can calculate $N(T)$, and the last two examples are of this type.

Example 6.5 Suppose that $T_0 = \mathbb{Z}+\mathbb{Z}\sqrt{2}$. We shall show that in this case, $N(T) = 3$ so that in this case there are exactly nine conjugacy classes in $\mathcal{G}(T)$, and this proves Theorem 1.3. First, it is clear that

$$T_1/T_0 = \left\{ T_0, \tfrac{1}{2} + T_0, \tfrac{\sqrt{2}}{2} + T_0, \tfrac{1+\sqrt{2}}{2} + T_0 \right\}.$$

Next, U_T is the set of units of the integral domain T_0, and it is known that

$$U_T = \{\pm\eta^n : n \in \mathbb{Z}\}, \qquad \eta = 1 + \sqrt{2}$$

(see [1, pp. 5, 264]). We must now examine the induced group action on T_1/T_0. A simple calculation shows that the map $m_\eta \colon x + T_0 \mapsto \eta x + T_0$ acts as follows:

$$T_0 \mapsto T_0, \quad \tfrac{\sqrt{2}}{2} + T_0 \mapsto \tfrac{\sqrt{2}}{2} + T_0, \quad \tfrac{1}{2} + T_0 \mapsto \tfrac{1+\sqrt{2}}{2} + T_0 \mapsto \tfrac{1}{2} + T_0.$$

It follows from this that $m_{\eta^{-1}} = m_\eta$, so that the action of M on T_1/T_0 provides exactly three orbits. Thus $N(T) = 3$ as required.

Example 6.6 Let $T_0 = \mathbb{Z} + \mathbb{Z}\sqrt{6}$. Then

$$T_1/T_0 = \left\{ 0 + T_0, \tfrac{1}{2} + T_0, \tfrac{\sqrt{6}}{2} + T_0, \tfrac{1+\sqrt{6}}{2} + T_0 \right\}.$$

Again, U_T is the set of units in the integral domain T_0 and it is known that here $U_T = \{\pm\eta^n : n \in \mathbb{Z}\}$, where $\eta = 5+2\sqrt{6}$. A straightforward calculation shows that m_η is the identity map on T_1/T_0. Thus $N(T) = 4$ and there are exactly eleven conjugacy classes in $\mathcal{G}(T)$.

For more details of this discussion see, for example, [1, Chapters 1 and 11], [3, Chapter 7], [9, Section 4.8], and [10, Chapters 5 and 7] (and many other texts). The element η in Examples 6.5 and 6.6 is called the *fundamental unit* of $\mathbb{Z} + \mathbb{Z}\sqrt{m}$, $m = 2, 6$, and Table 4 in [1, p.280] gives the fundamental units of $\mathbb{Z} + \mathbb{Z}\sqrt{m}$ for all squarefree m with $2 \leq m \leq 39$. With this, the reader can easily provide many more examples and it may be that a deeper investigation of these algebraic ideas would would reveal some further structure here.

7 The Proof of Theorem 5.3

Let T be a given group of real translations. In view of our discussion above, it is sufficient to show that there are exactly $2N(T)$ conjugacy classes of groups in $\mathcal{G}(T)$ that contain glide reflections. In order to analyse these groups, we select a complete set of coset representatives of T_0 in T_1, say $c_j + T_0$, where $j \in J$; thus,

$$T_1/T_0 = \{c_j + T_0 : j \in J\}, \qquad T_1 = \bigcup_{j \in J}(c_j + T_0),$$

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where $c_i+T_0 = c_j+T_0$ if and only if $i = j$. Corresponding to these coset representatives, we have the glide reflections $\sigma_j(z) = \bar{z} + c_j$, $j \in J$.

We begin our analysis by considering groups that contain translations and glide reflections, and only these.

Lemma 7.1 *Suppose that $G \in \mathcal{G}(T)$ and that G contains glide reflections, but no rotations or reflections. Then G is one of the groups $\langle T, \sigma_j\rangle$. Moreover, the groups $\langle T, \sigma_i\rangle$ and $\langle T, \sigma_j\rangle$ are conjugate if and only if $c_i + T_0$ and $c_j + T_0$ are in the same orbit under the action of M on T_1/T_0. In particular, there are exactly $N(T)$ conjugacy classes of such groups in $\mathcal{G}(T)$*

Proof Suppose that g is a glide reflection in G, say $g(z) = \bar{z} + b$. Then $b \in T_1$ so that $b \in c_j + T_0$ for some j. We conclude that $g(z) = \bar{z} + c_j + a$, where $a \in T_0$ or, equivalently, there is some t in T with $t(z) = z + a$ and

$$g(z) = \bar{z} + c_j + a = t\sigma_j(z) = \sigma_j t(z).$$

This implies that $gT = \sigma_j T$, and this proves that $G = \langle T, \sigma_j\rangle$ since in this case we have $G = T \cup gT = T \cup \sigma_j T = \langle T, \sigma_j\rangle$.

We now prove the second assertion in Lemma 7.1. Suppose that $c_i + T_0$ and $c_j + T_0$ are in the same M-orbit in T_1/T_0. Then there is some a in U_T such that $ac_i + T_0 = c_j + T_0$ or, equivalently, $\eta\sigma_i\eta^{-1}T = \sigma_j T$, where $\eta(z) = az$. It follows that

$$\eta\langle T, \sigma_i\rangle\eta^{-1} = \langle \eta T\eta^{-1}, \eta\sigma_i\eta^{-1}\rangle = \langle T, \eta\sigma_i\eta^{-1}\rangle = \langle T, \sigma_j\rangle,$$

so that $\langle T, \sigma_i\rangle$ and $\langle T, \sigma_j\rangle$ are conjugate.

Now suppose that $\langle T, \sigma_i\rangle$ and $\langle T, \sigma_j\rangle$ are conjugate, say $f\langle T, \sigma_i\rangle f^{-1} = \langle T, \sigma_j\rangle$, where f is the real similarity whose restriction to $\mathbb{R}$ is $f(x) = ax + b$. By Lemma 2.1, we see that $fTf^{-1} = T$ so that from Lemma 5.1 $aT_0 = T_0$. Now $f\sigma_i f^{-1} \in \langle T, \sigma_j\rangle = T \cup T\sigma_j$, so that $f\sigma_i f^{-1} \in T\sigma_j$. This implies that there is some k in T_0 such that for all real x, $x + ac_i = x + c_j + k$. Hence $ac_i \in c_j + T_0$, so that

$$m_a(c_i + T_0) = ac_i + T_0 = c_j + T_0.$$

This shows that $c_i + T_0$ and $c_j + T_0$ are in the same M-orbit in T_1/T_0. ∎

We now consider groups in $\mathcal{G}(T)$ that contain maps from at least two of the classes (i) rotations, (ii) reflections, and (iii) glide reflections. As such a group G must then contain maps from all three classes, we may assume that G in $\mathcal{G}(T)$ contains rotations, reflections, and glide reflections.

Lemma 7.2 *Suppose that $G \in \mathcal{G}(T)$ and that G contains rotations, reflections, and glide reflections. Then G is conjugate to one of the groups $\langle \tau, \sigma_j, \rho\rangle$. Moreover, the groups $\langle T, \sigma_i, \rho\rangle$ and $\langle T, \sigma_j, \rho\rangle$ are conjugate if and only if the groups $\langle T, \sigma_i\rangle$ and $\langle T, \sigma_j\rangle$ are conjugate.*

Proof By Lemma 7.1, the subgroup of translations and glide reflections in G is one of the groups $\langle T, \sigma_j\rangle$ where $j \in J$. By assumption, G contains a rotation, say r, where $r(z) = -z + a$, and a reflection, say s. Now let $t(z) = z - a/2$. Then $trt^{-1} = \rho$ and $t\sigma_j t^{-1} = \sigma_j$. Thus tGt^{-1} contains the rotation ρ, the glide reflection σ_j, and the two reflections s_1 (= tst^{-1}) and $\rho\sigma_j$. As $s_1^{-1}\rho\sigma_j$ is a translation in T, we see that $s_1 T = \rho\sigma_j T$. Thus $tGt^{-1} = T \cup \rho T \cup \sigma_j T \cup s_1 T = T \cup \rho T \cup \sigma_j T \cup \rho\sigma_j T$. Hence G is conjugate to $\langle T, \sigma_j, \rho\rangle$.

We now prove the second statement in Lemma 7.2. Suppose that the groups $\langle T, \sigma_i, \rho\rangle$ and $\langle T, \sigma_j, \rho\rangle$ are conjugate. Then from Lemma 2.1 we see that the subgroups $\langle T, \sigma_i\rangle$ and $\langle T, \sigma_j\rangle$ are conjugate.

Now suppose that $\langle T, \sigma_i\rangle$ and $\langle T, \sigma_j\rangle$ are conjugate, say $\langle T, \sigma_j\rangle = f\langle T, \sigma_i\rangle f^{-1}$, for some similarity f. Then $f\langle T, \sigma_i, \rho\rangle f^{-1} = \langle T, \sigma_j, \rho_1\rangle$ where $\rho_1 = f\rho f^{-1}$ (a rotation). Now choose a translation t such that $t\rho_1 t^{-1} = \rho$. As t commutes with all translations and glide reflections, we see that $(tf)\langle T, \sigma_i, \rho\rangle(tf)^{-1} = \langle T, \sigma_j, \rho\rangle$, so that $\langle T, \sigma_i, \rho\rangle$ and $\langle T, \sigma_j, \rho\rangle$ are conjugate. ∎

In conclusion, Lemmas 4.1, 7.1, and 7.2 provide the proof of Theorem 5.3.

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