OSCILLATION CRITERIA FOR MATRIX DIFFERENTIAL EQUATIONS

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1. Introduction. We shall be concerned at first with some properties of the solutions of the matrix differential equation

$$Y''(x) + P(x)Y(x) = 0 \tag{1.1}$$

where

$$P(x) \equiv (p_{ij}(x)), \qquad i, j, = 1, 2, \ldots, n,$$

is an $n \times n$ symmetric matrix whose elements are continuous real-valued functions for $0 < x < \infty$, and $Y(x) \equiv (y_{ij}(x))$, $Y''(x) \equiv (y''_{ij}(x))$ are $n \times n$ matrices. It is clear such equations possess solutions for $0 < x < \infty$, since one can reduce them to a first-order system and then apply known existence theorems **(6**, Chapter 1**)**.

We shall be primarily interested in giving sufficient conditions for solutions Y of (1.1), or for solutions Y of generalizations of (1.1), to *oscillate*, in the sense that the equation

$$\text{determinant } Y(x) \equiv |Y(x)| = 0$$

possesses an infinite number of roots in $a < x < \infty$ for all $a > 0$. We shall also give sufficient conditions for *non-oscillation* of solutions of matrix equations, that is, for a solution Y to be such that the equation $|Y(x)| = 0$ has a finite number of roots in $(0, \infty)$.

The subject of oscillation criteria for matrix differential equations is not a new one. The reader is referred to the recent book of Atkinson **(1)**, and in particular Chapter 10 and the Notes on Chapter 10, for an exposition of the subject of matrix oscillation criteria, together with many references to the literature. It may be noted here that the definitions given in much of the pertinent literature for "oscillation" and "non-oscillation" of solutions are different from the ones used here. Indeed, in Reid's work **(11–14)** and other papers **(7, 9, 16–18)** in which there is a calculus of variations background, a vector solution y of a differential system is said to be non-oscillatory in an interval I, if $y(x_1) = 0$ $(y \not\equiv 0)$ implies $y(x_2) \neq 0$, for any $x_2 \neq x_1$, $x_2 \in I$. Barrett **(2, 3)** has obtained results similar in content to those given here, while Hunt **(8)** has combined results of Barrett **(4)** and Reid **(14)**, to discuss certain types of oscillation problems for higher-order scalar equations, which are shown to be equivalent to a system of matrix equations.

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I acknowledge with pleasure a number of discussions with Dr. Lincoln Bragg concerning Theorem 1, which have resulted in significant improvements of the theorem.

Standard matrix notation will be used. If

$$A(x) \equiv (a_{ij}(x)), \qquad i, j = 1, \ldots, n,$$

then $A^*(x)$ and $A^{-1}(x)$ denote the transpose and inverse of $A(x)$, respectively, while $A(x) \geqslant 0$ means that $A(x)$ is a positive semi-definite matrix, and $A(x) \geqslant B(x)$ means $A(x) - B(x)$ is a positive semi-definite matrix. Dropping the equality sign in the symbol $\geqslant$ of the last line will mean that the matrices are positive definite. Finally, $A^{(k)}(x)$ is the matrix whose ijth element is $a_{ij}^{(k)}(x)$, $k = 1, 2, \ldots$, while

$$\int_a^b A(x)\, dx$$

is the matrix whose ijth element is

$$\int_a^b a_{ij}(x)\, dx.$$

We shall be concerned with real-valued functions exclusively.

We define now what we mean when a matrix possesses property D.

Definition. The matrix $A(x)$ has *property D* if and only if

$$\inf_\xi (\xi^* A(x) \xi) \to \infty \qquad \text{as } x \to \infty,$$

where ξ represents a column vector of unit length.

2. An oscillation theorem. In this section we prove the following oscillation theorem for equation (1.1).

THEOREM 1. *Suppose*

(1) $Y(x)$ *is a solution of* (1.1) *such that* $W(Y(x), Y(x)) = 0$ *at some point* x_0, $0 < x_0 < \infty$ *where*

$$W(U(x), V(x)) \equiv U^*(x) V'(x) - (U^*(x))' V(x),$$

(2) $H(x) = \int_{x_1}^{x} P(t)\, dt$ *possesses property* D $(x_1 > 0)$.

Then $Y(x)$ *is an oscillatory solution of* (1.1).

Proof. The proof is by contradiction. Suppose the solution of Hypothesis 1 is such that $|Y(x)|$ possesses a last zero before x_2, $x_2 > 0$. Then the matrix $S(x) \equiv -Y'(x)Y^{-1}(x)$ is well defined for $x \geqslant x_2$ and equation (1.1) becomes

$$S'(x) = -Y''(x)Y^{-1}(x) + (Y'(x)Y^{-1}(x))^2 = P(x) + (S(x))^2. \tag{2.1}$$

Introducing the matrix $T(x)$ by

(2.2) $$S(x) = T(x) + \int_{x_1}^{x} P(t)\,dt, \qquad x \geqslant x_2,$$

we have from (2.1)

(2.3) $$T'(x) = (T(x) + H(x))^2, \qquad x \geqslant x_2.$$

Now

$$\begin{aligned} d(W(Y, Y))/dx &= (Y^*)'Y' + Y^*Y'' - (Y^*)''Y - (Y^*)'Y' \\ &= Y^*(-PY) - (-Y^*P^*)\,Y = 0 \end{aligned}$$

since P is symmetric. Thus $W(Y, Y) = \text{const.} = 0$ since $W = 0$ at $x = x_0$. Therefore

$$(Y'Y^{-1})^* = (Y^{-1})^*(Y')^* = (Y^*)^{-1}(Y^*Y'Y^{-1}) = Y'Y^{-1}.$$

Hence S is a symmetric matrix and, by (2.2), T is also symmetric, since P is symmetric.

But the eigenvalues of a real symmetric matrix are real, so the eigenvalues of $(T + H)^2$ are real and non-negative, being the square of the eigenvalues of $(T + H)$. Thus from (2.3) we see that $T'(x)$ is positive semi-definite, so $T(x) \geqslant T(x_2)$, $x \geqslant x_2$.

By Hypothesis 2, there exists a point $x_3 \geqslant x_2$ such that $T(x) + H(x) \geqslant I$ for $x \geqslant x_3$. Indeed, there exists a point $x'_3 \geqslant x_2$ such that $T(x_2) + H(x'_3) \geqslant I$, since as $x \to \infty$, $H(x)$ possesses property D. It remains to choose $x_3 \geqslant x'_3$ such that $H(x) \geqslant H(x'_3)$, for $x \geqslant x_3$, which is possible since $H(x)$ possesses property D. Hence

$$T(x) + H(x) \geqslant T(x) + H(x'_3) \geqslant T(x_2) + H(x'_3) \geqslant I \qquad \text{for } x \geqslant x_3.$$

Thus we can write, for $x \geqslant x_3$,

$$T(x) + H(x) \geqslant I \equiv g_1(x)I.$$

Hence $T'(x) \geqslant g_1^2(x)I$ and

$$T(x) \geqslant T(x_3) + \left(\int_{x_3}^{x} g_1^2(t)\,dt\right) I,$$

so

$$T(x) + H(x) \geqslant g_2(x)I, \qquad \text{where } g_2(x) \equiv 1 + \int_{x_3}^{x} g_1^2(t)\,dt;$$

indeed

$$T(x) + H(x) \geqslant T(x_3) + H(x) + \left(\int_{x_3}^{x} g_1^2(t)\,dt\right) I \geqslant \left(1 + \int_{x_3}^{x} g_1^2(t)\,dt\right) I$$

for $x \geqslant x_3$. By induction, we have

$$T(x) + H(x) \geqslant g_{n+1}(x), \qquad x \geqslant x_3,$$

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where

$$g_{n+1}(x) = 1 + \int_{x_3}^{x} g_n^{\,2}(t)\,dt \qquad \text{for } n \geqslant 1,$$

and $g_1(x) \equiv 1$.

We note that the g_n increase, pointwise, with n. Indeed, for $x \geqslant x_3$,

$$g_{n+1}(x) - g_n(x) = \int_{x_3}^{x} (g_n^{\,2}(t) - g_{n-1}^{\,2}(t))\,dt,$$

so $g_n \geqslant g_{n-1} \geqslant 0$ implies $g_{n+1} \geqslant g_n \geqslant 0$; moreover

$$g_2(x) \equiv 1 + (x - x_3) \geqslant g_1(x) \equiv 1 \geqslant 0,$$

to start this induction. Now suppose that the g_n are bounded on an interval $[x_3, x_4]$, $x_4 > x_3 + 1$. Then $\lim_n g_n(x) = f(x)$ exists and satisfies

$$f(x) = 1 + \int_{x_3}^{x} f^2(t)\,dt, \qquad f(x_3) = 1. \tag{2.4}$$

But the only C^1 solution of this problem is $f(x) = 1/(1 + x_3 - x)$ which is unbounded as $x \to x_3 + 1$. Hence the $g_n \to \infty$ on any interval $[x_3, x_4]$, $x_4 > x_3 + 1$. But from $T(x) + H(x) \geqslant g_n(x)I$ we reach a contradiction since the elements of T and H are continuous for $x \geqslant x_3$ and hence bounded on any compact set $[x_3, x_4]$. Thus the last inequality is impossible. This proves the theorem.

3. Other oscillation theorems for linear equations. In this section we shall outline the proof of a more general oscillation theorem for equations like (1.1), as well as the proof of an oscillation theorem for the equation

$$(QY')' + PY = 0 \tag{3.1}$$

where Q is a positive definite symmetric $n \times n$ matrix whose elements belong to C^1 for $0 < x < \infty$, and P is a symmetric $n \times n$ matrix with continuous elements for $0 < x < \infty$.

First we strengthen Theorem 1.

THEOREM 2. *Suppose*

(1) $Y(x)$ *is a solution of* (1.1) *satisfying Hypothesis* 1 *of Theorem* 1,

(2) *there exists a positive definite scalar matrix* G (*whose positive element* $g \in C^1$ *for* $0 \leqslant x < \infty$) *such that*

(3) $\displaystyle\int_{x_1}^{x} (1/g(t))\,dt \to \infty$ *as* $x \to \infty$,

(4) *the matrix*

$$K(x) \equiv \int_{x_1}^{x} \{G(t)P(t) - \tfrac{1}{4}(G'(t))^2 G^{-1}(t)\}\,dt + \tfrac{1}{2}G'(x)$$

has property D $(x_1 > 0)$.

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Then $Y(x)$ is an oscillatory solution of (1.1).

It is to be noted that Hypothesis 2 of Theorem 1 has been replaced here by a weaker requirement.

Proof. As before, the proof is by contradiction, and is very similar to the proof of Theorem 1. This time the assumption that $|Y(x)|$ possesses a last zero, and the substitution

$$S(x)G^{-1}(x) = -Y'(x)Y^{-1}(x), \tag{3.2}$$

leads to the equation

$$S'(x)G^{-1}(x) = G^{-2}(x)[S^2(x) + G'(x)S(x)] + P(x). \tag{3.3}$$

Completing the square (which is possible since G is a scalar matrix and therefore commutes with any compatible matrix), we obtain

$$S'(x) = G^{-1}(x)[S(x) + \tfrac{1}{2}G'(x)]^2 + G(x)P(x) - \tfrac{1}{4}(G'(x))^2G^{-1}(x), \tag{3.4}$$

this equation holding for all x sufficiently large. One obtains, finally, an equation in T similar to (2.3), namely,

$$T'(x) = G^{-1}(x)[T(x) + K(x)]^2, \qquad x \to \infty. \tag{3.5}$$

Once again it is easy to check that T is a symmetric matrix and that $T'(x) \geqslant 0$.

From this point a contradiction is reached using precisely the same type of reasoning as in Theorem 1. In place of (2.4) one has

$$f(x) = 1 + \int_{x_3}^{x} \frac{f^2(t)}{g(t)}\,dt, \qquad f(x_3) = 1, \tag{3.6}$$

with solution

$$f(x) = 1 \Big/ \left(1 - \int_{x_3}^{x} \frac{dt}{g(t)}\right).$$

Noting Hypothesis 3, the remaining details are readily supplied. This proves the theorem.

Finally, we have a theorem concerning the oscillatory behaviour of some of the solutions of equation (3.1).

THEOREM 3. *Suppose*

(1) *Y is a solution of* (3.1) *such that $W(Y(x), Y(x)) = 0$ at some point x_0, $0 < x_0 < \infty$, where*

$$W(U(x), V(x)) \equiv U^*(x)Q(x)V'(x) - (U^*(x))'Q(x)V(x),$$

(2) *$G(t)$ is a positive definite scalar matrix such that*

(3) *the matrix*

$$L(x) \equiv \int_{x_1}^{x} \{G(t)P(t) - \tfrac{1}{4}Q(t)(G(t))^2G^{-1}(t)\}\,dt + \tfrac{1}{2}Q(x)G'(x)$$

has property D $(x_1 > 0)$,

(4) $(G(t)Q(t))^{-1} \geqslant q(t)I$ *where* $q > 0$ *and*

$$\int_{x_1}^{x} q(t)\,dt \to \infty \qquad \text{as } x \to \infty.$$

Then $Y(x)$ *is an oscillatory solution of* (3.1).

Proof. Once again the proof is by contradiction. This time we make the substitution

$$S(x)G^{-1}(x) = -Q(x)Y'(x)Y^{-1}(x) \tag{3.7}$$

for $x \geqslant x_2$, with the last zero of the equation $|Y(x)| = 0$ lying to the left of x_2, by assumption. A calculation gives

$$S'(x)G^{-1}(x) = S(x)G^{-2}(x)G'(x) + P(x) + S(x)Q(x)S(x)G^{-2}(x), \qquad x \geqslant x_2. \tag{3.8}$$

A factorization of the right member yields

$$\begin{aligned} S'(x) = \{S(x) + \tfrac{1}{2}Q(x)G'(x)\}\{(Q(x)G(x))^{-1}\}\{S(x) + \tfrac{1}{2}Q(x)G'(x)\} \\ + G(x)P(x) - \tfrac{1}{4}Q(x)(G'(x))^2G^{-1}(x), \qquad x \geqslant x_2. \end{aligned} \tag{3.9}$$

The substitution

$$S(x) = T(x) + \int_{x_1}^{x} \{G(t)P(t) - \tfrac{1}{4}Q(t)(G'(t))^2G^{-1}(t)\}\,dt$$

gives

$$T'(x) = (T(x) + L(x))(Q(x)G(x))^{-1}(T(x) + L(x)) \qquad \text{for } x \geqslant x_2. \tag{3.10}$$

As in the proof of Theorem 1, one shows that $W(Y, Y) \equiv \text{const.} = 0$, and as a consequence of this, that $Q(x)Y'(x)Y^{-1}(x)$ is symmetric; hence $S(x)$ and $T(x)$ are also symmetric, since P and Q are assumed to be symmetric. But we may rewrite (3.10) in the following manner:

$$T'(x) = (T(x) + L(x))^*R^*(x)R(x)(T(x) + L(x)), \qquad x \geqslant x_2, \tag{3.11}$$

since by assumption $(QG)^{-1}$ is positive definite and symmetric, so there exists a non-singular matrix R such that $(QG)^{-1} = R^*R$ (see **(5)** or **(10)**). Thus

$$T'(x) = [R(x)(T(x) + L(x))]^*[R(x)(T(x) + L(x))], \qquad x \geqslant x_2, \tag{3.12}$$

so T' is a positive semi-definite matrix for $x \geqslant x_2$. Hence $T(x) \geqslant T(x_2)$ and using Hypothesis 3, as in previous theorems, we conclude that

$$T(x) + L(x) \geqslant I \equiv g_1(x)I$$

for $x \geqslant x_3$, say, where x_3 is sufficiently large. From (3.12), we obtain

$$T'(x) \geqslant g_1^2(x)q(x)I, \qquad x \geqslant x_3. \tag{3.13}$$

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Hence, arguing as before, we have, for $x \geqslant x_3$,

$$T(x) + L(x) \geqslant g_2(x)I \equiv \left(1 + \int_{x_3}^{x} q(t)g_1^2(t)\,dt\right) I$$

and by induction

$$T(x) + L(x) \geqslant g_{n+1}(x)I \equiv \left(1 + \int_{x_3}^{x} q(t)g_n^2(t)dt\right) I.$$

The proof is now concluded as in Theorems 1 and 2.

It may be noted in Theorems 1–3 that no assumption is made concerning the positive definiteness or positive semi-definiteness of the coefficient matrix P.

4. Oscillation theorems for a system of matrix equations. By much the same type of reasoning as we have used previously, one can obtain oscillation criteria for a matrix system. Indeed, suppose one has the pair of matrix equations

$$\begin{cases} U'(x) = A(x)U(x) + B(x)V(x), \\ V'(x) = C(x)U(x) - A(x)V(x). \end{cases} \tag{4.1}$$

If we assume A, B, and C are matrices with continuous elements for $0 < x < \infty$, then fundamental existence theorems (**6**, Chapter 1) tell us that non-trivial solutions exist for $0 < x < \infty$ with the property that

$$U^*(x)V(x) - V^*(x)U(x)|_{x=x_0} = 0, \qquad 0 < x_0 < \infty. \tag{4.2}$$

Indeed, one might demand that $U(x_0) = I$, $V(x_0) = 0$, for example.

Now the symmetry of U^*V at a single point implies the symmetry of U^*V everywhere on the positive real axis if B and C are symmetric. For by computing $d[U^*V - V^*U]/dx$ and substituting for the derivatives from (4.1), one finds that $U^*V - V^*U$ = const. = 0 by the use of (4.2). Moreover, if V^{-1} exists we have from $U^*V = V^*U$ that $UV^{-1} = (V^*)^{-1}U^* = (UV^{-1})^*$, so UV^{-1} is also symmetric. We now have the following theorem.

THEOREM 4. *Suppose*

(1) *U and V are solutions of* (4.1) *such that* (4.2) *holds,*

(2) *$G(t)$ is a positive definite scalar matrix such that if*

(3) *the matrices A, B, and C of* (4.1) *are symmetric, with $-C$ positive definite, and A and C are commutative matrices, and*

(4) *the matrix*

$$N(x) \equiv \int_{x_1}^{x} \{G(t)B(t) + [A(t) + \tfrac{1}{2}G'(t)G^{-1}(t)](G(t)C^{-1}(t))[A(t) + \tfrac{1}{2}G'(t)G^{-1}(t)]\}\,dt + [\tfrac{1}{2}G'(x) + G(x)A(x)](-C^{-1}(x))$$

has property D, $(x_1 > 0)$,

(5) $(-G^{-1}(t)C(t)) \geqslant c(t)I$ *where* $c \geqslant 0$ *and*

$$\int_{x_1}^{x} c(t)\,dt \to \infty \qquad \text{as } x \to \infty.$$

Then V *is an oscillatory solution of* (4.1).

Proof. The proof is by contradiction. If the equation $|V(x)| = 0$ possesses a last zero before $x_2 > 0$, then we know that $S(x)G^{-1}(x) \equiv U(x)V^{-1}(x)$ is well defined for $x \geqslant x_2$ and is also symmetric. A computation gives

$$(4.3) \qquad S'(x)G^{-1}(x) = S(x)G^{-2}(x)G'(x) + A(x)S(x)G^{-1}(x) + B(x) + S(x)G^{-1}(x)(-C(x))S(x)G^{-1}(x) + S(x)G^{-1}(x)A(x).$$

It is readily checked that the right-hand side may be factored to give

$$(4.4) \qquad S'(x)G^{-1}(x) = \{S(x) - G^2(x)[\tfrac{1}{2}G'(x)G^{-2}(x) + A(x)G^{-1}(x)]C^{-1}(x)\} \times \{G^{-1}(x)(-C(x))G^{-1}(x)\}\{S(x) - C^{-1}(x)[\tfrac{1}{2}G'(x)G^{-2}(x) + A(x)G^{-1}(x)]G^2(x)\} + B(x) + \{\tfrac{1}{2}G^{-2}(x)G'(x) + A(x)G^{-1}(x)\}\{G(x)C^{-1}(x)G(x)\} \times \{\tfrac{1}{2}G^{-2}(x)G'(x) + A(x)G^{-1}(x)\} \qquad \text{for } x \geqslant x_2.$$

By making the substitution

$$(4.5) \qquad S(x) = T(x) + N(x)$$

(after multiplying (4.4) through by G) and noting the symmetry of S and N (the commutativity of A and C is used here), we have

$$(4.6) \qquad T'(x) = (T(x) + N(x))^*(-C(x)G^{-1}(x))(T(x) + N(x)).$$

But $-C$ is a positive definite matrix and G is a positive definite scalar matrix; so we may write $-CG^{-1} = R^*R$ for some non-singular matrix R. Thus

$$(4.7) \qquad T'(x) = [T(x) + N(x))^*R(x)^*][(T(x) + N(x))R(x)]$$

for $x \geqslant x_2$. Precisely as in the proof of Theorem 3, one can show that T' has non-negative eigenvalues and, by use of Hypothesis 4, that $T + N \geqslant I$ for all x sufficiently large. The remainder of the proof is similar to the last part of Theorem 3 and is omitted.

It should be noted that an analogous theorem with the roles of U and V interchanged exists. Because the proof is similar to the proof of Theorem 4 we only state the result.

THEOREM 5. *Suppose*

(1) *Hypotheses* 1 *and* 2 *of Theorem* 4 *hold,*

(2) *the matrices* A, B, *and* C *of equation* (4.1) *are symmetric, with* B *positive definite, and* A *and* B *are commutative matrices,*

(3) *the matrix*

$$N(x) \equiv \int_{x_1}^{x} \{G(t)(-C(t)) + [\tfrac{1}{2}G^{-1}(t)G'(t) - A(t)](G(t)B^{-1}(t))[\tfrac{1}{2}G^{-1}(t)G'(t) - A(t)]\}\, dt + [\tfrac{1}{2}G'(x) - A(x)G^{-1}(x)](B^{-1}(x))$$

has property D $(x_1 > 0)$,

(4) $G^{-1}(t)B(t) \geqslant b(t)I$ *where* $b \geqslant 0$ *and*

$$\int_{x_1}^{x} b(t)\, dt \to \infty \qquad as\ x \to \infty.$$

Then U is an oscillatory solution of (4.1).

5. Non-oscillation theorems. In this section we shall obtain a non-oscillation criterion for matrix equations. The technique used is a modification of the one used to obtain oscillation criteria in the preceding sections. We first prove two lemmas that will be useful.

LEMMA 1. *Suppose*

(1) $Y(x)$ *is a solution of equation* (3.1) (*where the only restrictions now on* Q *will be that it is non-singular and symmetric for* $0 < x < \infty$),

(2) $|Y(x_0)| \neq 0$, $|Y(x_1)| = 0$, *with* x_1 *the first root of the equation* $|Y(x)| = 0$ *to the right of* $x_0 > 0$.

Then there exists a unit vector ξ *such that*

$$\limsup_{x\to x_1} |\xi^* Q(x) Y'(x) Y^{-1}(x)\xi| = \infty.$$

Proof. Since $Y(x)$ is non-singular for $x_0 \leqslant x < x_1$, we write

$$Q(x)Y'(x)Y^{-1}(x) = A(x),$$

for those x, and obtain

$$Y'(x) = Q^{-1}(x)A(x)Y(x) \equiv B(x)Y(x).$$

But, by (**6**, p. 82),

$$|Y(x)| = |Y(x_0)| \exp\left(\int_{x_0}^{x} \operatorname{tr} B(t)\, dt\right) \tag{5.1}$$

for $x_0 \leqslant x < x_1$. From this we see that

$$\limsup_{x\to x_1^-} |\operatorname{tr} B(x)| = \infty; \tag{5.2}$$

for otherwise $|\operatorname{tr} B(x)|$ would be bounded for $x_0 \leqslant x < x_1$, and the assumption $|Y(x_1)| = 0$ could not hold.

Next we note that

$$\operatorname{tr} B(x) = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} q_{ji}(x)a_{ij}(x) \right\} \tag{5.3}$$

where $a_{rs}(x)$ is a typical element of $A(x)$ and $q_{tu}(x)$ is a typical element of $Q^{-1}(x)$. But in the neighbourhood of $x = x_1$, the elements of $Q^{-1}(x)$ are all bounded, since they have been assumed continuous for $0 < x < \infty$; hence at least one of the $a_{ij}(x)$ satisfies

$$\limsup_{x \to x_1^-} |a_{ij}(x)| = \infty.$$

There are several cases to be considered to complete the proof. If there exists an unbounded a_{ij} lying on the main diagonal, a_{kk} say, then one can take as ξ the unit vector with zeros everywhere except for a one in the kth component. One then has

$$\xi^* Q(x) Y'(x) Y^{-1}(x) \xi = a_{kk}(x)$$

and the result follows. If no such unbounded diagonal element exists, suppose there exists precisely one unbounded off-diagonal element, a_{ij} say. Then one takes as ξ the unit vector with zeros everywhere except for $1/\sqrt{2}$ in the ith and jth components. One then has

$$\xi^* Q(x) Y'(x) Y^{-1}(x) \xi = \tfrac{1}{2}[a_{ii}(x) + a_{ij}(x) + a_{ji}(x) + a_{jj}(x)] \tag{5.4}$$

and the result follows since a_{ij} is the only unbounded term on the right side of (5.4), by assumption. If there is more than one unbounded off-diagonal element, then either there exists exactly one "pair" of unbounded terms, a_{ij} and a_{ji} say, or there does not exist such a pair. In the latter case one proceeds as in the case of a single unbounded off-diagonal element. In the former case it might appear that while a_{ij} and a_{ji} were each unbounded, their sum $a_{ij} + a_{ji}$, or some linear combination of a_{ij} and a_{ji}, might be bounded. But one notes from (5.3) that a_{ij} and a_{ji} occur in $\operatorname{tr} B(x)$ in the form

$$q_{ji} a_{ij} + q_{ij} a_{ji}. \tag{5.5}$$

The symmetry of Q^{-1} tells us that $q_{ji} = q_{ij}$; thus if $\operatorname{tr} B(x)$ is unbounded, it is unbounded because the sum $a_{ij} + a_{ji}$ is unbounded. Hence for this last case one takes ξ to be the unit vector with zeros in all components except the ith and jth, where $1/\sqrt{2}$ appears, and obtains (5.4) again, this time with $a_{ij} + a_{ji}$ becoming unbounded and all other terms on the right-hand side remaining bounded. This proves Lemma 1.

LEMMA 2. *Suppose*

(1) *$T(x)$ is a solution of the matrix differential equation $T'(x) = F(x, T(x))$ for $0 < a \leqslant x \leqslant b$, where the elements of the matrix F are to be continuous functions of x,*

(2) *there exists a matrix $\Phi(x)$ whose elements are of class C^1 for $a \leqslant x \leqslant b$ such that $\Phi'(x) > F(x, \Phi(x))$ for $a \leqslant x \leqslant b$ with $\Phi(a) \geqslant T(a)$.*

Then $\Phi(x) > T(x)$ for $a < x \leqslant b$.

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Proof. We first show that there is an interval to the right of a where $\Phi(x) > T(x)$. This is trivial if $\Phi(a) > T(a)$. If $\Phi(a) = T(a)$, then

$$\Phi'(a) > F(a, \Phi(a)) = F(a, T(a)) = T'(a)$$

and the existence of such an interval follows again. Suppose now that the inequality $\Phi(x) > T(x)$ cannot be continued over the whole interval $a < x \leqslant b$. Then there exists a number c $(a < c \leqslant b)$ such that $\Phi(c) = T(c)$ and $\Phi(x) > T(x)$, $a < x < c$. But this implies

$$\Phi'(c) \leqslant T'(c). \tag{5.6}$$

For suppose this last matrix inequality fails to hold. Then there exists a vector $\xi \neq 0$ such that

$$\xi^*(\Phi'(c) - T'(c))\xi > 0. \tag{5.7}$$

By continuity of the elements of the Φ' and T' matrices, there must exist an interval, I say, to the left of c (and including c as an interior point or the right end point) such that for $x \in I$, $\xi^*(\Phi'(x) - T'(x))\xi > 0$. Integrating from x in I to c, $x < c$, we obtain

$$\xi^*(-\Phi(x) + \Phi(c) + T(x) - T(c))\xi > 0. \tag{5.8}$$

Thus

$$\xi^*(T(x) - \Phi(x))\xi > 0, \tag{5.9}$$

(since $\Phi(c) = T(c)$), a contradiction, since $\Phi > T$ in (a, c). Now using (5.6), we have

$$\Phi'(c) > F(c, \Phi(c)) = F(c, T(c)) = T'(c), \tag{5.10}$$

in contradiction with (5.6). This proves Lemma 2. We now have the following non-oscillation theorem.

THEOREM 6. *Suppose*

(1) $Y(x)$ *is a solution of equation* (3.1) *with* $|Y(a)| \neq 0$, $a > 0$,

(2) $G(x)$ *is a scalar matrix with a positive diagonal element* $g(x) \in C^1$ *for* $0 < x < \infty$,

(3) *for every* $b > a$,

$$L(x) = \int_a^x \{G(t)P(t) - \tfrac{1}{4}Q(t)(G'(t))^2G^{-1}(t)\}\,dt + \tfrac{1}{2}Q(x)G'(x),$$

where $a \leqslant x \leqslant b$, *can be appraised so that one can demonstrate the existence of matrices* Φ *and* Ψ *with elements of class* C^1 *and* C *for* $a \leqslant x \leqslant b$, *respectively, with the properties that for* $a \leqslant x \leqslant b$,

$$\Psi^2(x) \geqslant (L(x) + \Phi(x))Q^{-1}(x)(L(x) + \Phi(x)), \qquad \Phi'(x) > G^{-1}(x)\Psi^2(x)$$

and $\Phi(a) = c\,I$, *c an arbitrarily large positive number.*
Then $|Y(x)| \neq 0$ *for* $x > a$, *that is* Y *is a non-oscillatory solution of* (3.7).

Proof. The proof is by contradiction. Suppose $Y(x)$ is an oscillatory solution. Then there exists a first zero of the equation $|Y(x)| = 0$ after the point a mentioned in Hypothesis 1, at $x = b$, say. Thus for x in $a \leqslant x < b$ we can transform (3.1) into (3.10) as in the proof of Theorem 3. By Lemma 1,

$$|\xi^* Q(x) Y'(x) Y^{-1}(x) \xi|$$

must assume arbitrarily large positive values in $[a, b)$, for at least one properly chosen unit vector ξ, say for ξ_0. The same must be true of $|\xi_0^* T \xi_0|$ by an inspection of the transformations used to obtain (3.10) from (3.1) and our knowledge, by use of Hypothesis 3, that $|\xi_0^* L \xi_0|$ is bounded in $[a, b]$. But from Hypothesis 3, we have the existence of a matrix Φ such that

$$\Phi' > G^{-4}\Psi^2 \geqslant (L + \Phi)(QG)^{-1}(L + \Phi)$$

with $\Phi(a) = cI \geqslant T(a)$ since c is an arbitrarily large positive number. Hence, by Lemma 2, $\Phi > T$ for $a < x \leqslant b$. But $|\xi_0^* T \xi_0|$ assures arbitrarily large values in (a, b), an impossibility since $\Phi \in C^1$ in $[a, b]$. This contradiction proves the theorem.

It may be noted that there are no hypotheses concerning the symmetry or definiteness of the coefficient matrix P or the symmetry of the solution matrix Y.

6. Non-linear matrix equations. In this section we shall use the same basic techniques and ideas of previous sections to obtain some oscillation criteria for the equation

$$Y''(x) + P(x) f(Y(x)) = 0. \tag{6.1}$$

It is assumed here, in the interests of simplicity, that f is a real-valued function with a power series expansion in powers of x with an infinite radius of convergence, so that

$$f(Y(x)) = \sum_0^\infty a_i Y^i(x),$$

$Y^0(x) \equiv I$, a_i real scalars, is well defined. There are, of course, other ways of defining $f(Y(x))$, Y a matrix, and the reader is referred to **(15)** for further details on the subject.

We have the following theorems.

THEOREM 7. *Suppose*

(1) *Y is a symmetric solution of* (6.1) *existing for* $0 < x < \infty$ *such that Y commutes with Y',*

(2) *Whenever Y^{-1} exists, $f(Y) Y^{-1}$ is positive definite and*

$$R(x) \equiv f(Y(x)) Y^{-1}(x) \geqslant g(x) I,$$

$g(x)$ a non-negative scalar function of class C for $0 < x < \infty$,

(3) *$P(x)$ is symmetric and positive semi-definite for* $0 < x < \infty$,

(4) $K(x) \equiv \int_{x_1}^{x} g(t)\, P(t)\, dt$

has property D ($x_1 > 0$ *a convenient lower limit of integration*).
Then Y is an oscillatory solution of (6.1).

Proof. The proof is by contradiction, once again. If $|Y(x)| = 0$ has a last root before $x_2 > 0$, then $S(x) = -Y'(x)Y^{-1}(x)$ is well defined for $x > x_2$ and (6.1) becomes

$$S'(x) = S^2(x) + P(x)f(Y(x))Y^{-1}(x). \tag{6.2}$$

Letting

$$S(x) = T(x) + \int_{x_1}^{x} P(t)R(t)\, dt$$

we obtain

$$T'(x) = \left(T(x) + \int_{x_1}^{x} P(t)R(t)\, dt\right)^2, \qquad x \geqslant x_2. \tag{6.3}$$

The assumption of the commutativity of Y and Y', coupled with the symmetry of Y and P, by use of (6.1), leads to

$$Pf(Y)Y^{-1} = Y^{-1}(f(Y))^*P = Y^{-1}f(Y)P = f(Y)Y^{-1}P \qquad \text{for } x \geqslant x_2. \tag{6.4}$$

Hence P and R are commutative matrices, and since P and R are both symmetric, PR is also symmetric. But using $YY' = Y'Y$ again and the symmetry of Y, we conclude that S is symmetric. Hence T is also symmetric and positive semi-definite since its eigenvalues are non-negative.

Now note that

$$P(x)R(x) \geqslant g(x)P(x), \qquad x \geqslant x_2. \tag{6.5}$$

Indeed this last inequality is equivalent to

$$P(x)(R(x) - g(x)I) \geqslant 0. \tag{6.6}$$

But P is positive semi-definite and symmetric, so its eigenvalues are non-negative real numbers; $R(x) - g(x)I$ is also positive semi-definite and symmetric, so its eigenvalues are non-negative real numbers. Since P and R commute a theorem of Frobenius (**5**, p. 100, Problem 11) implies that the eigenvalues of the left member of (6.6) are, with proper ordering, just the products of the eigenvalues of P and $R - gI$. Thus the inequality in (6.6) must hold. Hence, using Hypothesis 4, we conclude that

$$\int_{x_1}^{x} PR\, dt$$

possesses property D and, as in previous theorems, $T + K$ becomes and remains positive definite as $x \to \infty$. Thus

$$(T(x) + K(x))^{-1}(T'(x) + K'(x))(T(x) + K(x))^{-1} \geqslant I \tag{6.7}$$

as $x \to \infty$ and a contradiction is reached as in previous theorems, by integrating. This proves the theorem.

Another oscillation theorem for (6.1) is the following:

THEOREM 8. *Suppose*

(1) *Y is a symmetric solution of* (6.1) *existing for* $0 < x < \infty$ *such that* $Y'Y = YY'$;

(2) *$f(Y)$ is singular if and only if Y is singular, and*

$$f'(Y) \equiv df(Y)/dY \geqslant g(x)I,$$

$g(x)$ a positive scalar function of class C for $0 < x < \infty$ *such that*

$$\int_{x_1}^{x} (1/g(t))\,dt \to \infty \qquad as\ x \to \infty;$$

(3) *P is symmetric and*

$$K(x) \equiv \int_{x_1}^{x} P(t)\,dt$$

has property D.

Then Y is an oscillatory solution of (6.1).

Proof. As before, if the equation $|Y(x)| = 0$ has a last zero before $x_2 > 0$, we make the substitution

$$S(x) = -Y'(x)(f(Y(x)))^{-1}, \qquad x \geqslant x_2, \tag{6.8}$$

and obtain

$$S'(x) = P(x) + S(x)f'(Y(x))S(x). \tag{6.9}$$

Letting

$$S(x) = T(x) + \int_{x_1}^{x} P(t)\,dt$$

we have

$$T'(x) = (T(x) + K(x))f'(Y(x))(T(x) + K(x)). \tag{6.10}$$

But the commutativity of Y' and Y and our assumption about the form of $f(Y)$ mean that $Y'f(Y) = f(Y)Y'$; from this and the symmetry of Y we conclude that $S(x)$ is symmetric for $x \geqslant x_2$. Hence T is also symmetric. By Hypothesis 3, $T + K$ becomes and remains positive definite for $x \to \infty$. Thus

$$\begin{aligned}(T(x) + K(x))^{-1}(T'(x) + K'(x))(T(x) + K(x))^{-1} &\geqslant f'(Y(x)) \\ &\geqslant g(x)I \qquad \text{as } x \to \infty.\end{aligned} \tag{6.11}$$

A contradiction then follows easily from this last line. This proves the theorem.

7. An application. In a recent paper R. W. Hunt **(8)** has given conditions for the existence of a non-trivial solution of the differential equation

$$(r(x)y^{(n)}(x))^{(n)} + (-1)^{n+1}p(x)y = 0 \tag{7.1}$$

(where $r(x) > 0$, $p(x) > 0$, $r, p \in C[0, \infty)$) satisfying one of the following sets of two-point boundary conditions:

$$(7.2) \quad y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0 = y(b) = y'(b) = \ldots = y^{(n-1)}(b),$$

$$(7.3) \quad y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0 = y_1(b) = y'_1(b) = \ldots = y_1^{(n-1)}(b),$$

where $y_1(x) \equiv r(x)y^{(n)}(x)$ and $b > 0$. W. T. Reid **(14)** has also discussed problems of this sort for equations which include (7.1), by variational methods, obtaining oscillation criteria for general quasi-differential equations.

In particular Hunt has shown that one may represent (7.1) as a system (in Hunt's notation)

$$(7.4) \qquad Y' = EZ, \qquad Z' = -FZ,$$

with

$$E = (e_{ij}), \qquad e_{ij} = (-)^{i+j}x^{2n-i-j}/(n-i)!\,(n-j)!\,r(x),$$
$$F = (f_{ij}), \qquad f_{ij} = x^{i+j-2}p(x)/(i-1)!\,(j-1)!$$

for $i \leqslant j$, $f_{ij} = f_{ji}$; E and F are symmetric with $E \geqslant 0$. Moreover the system is so constructed that

$$Y(x) = D^{-1}(x)W(u_1(x), \ldots, u_n(x))$$

where $|D(x)| \equiv 1$ and W stands for the Wronskian matrix formed from the $u_i(x)$, the u_i being linearly independent solutions of (7.1) satisfying

$$u_i^{(j-1)}(0) = 0, \qquad i, j = 1, 2, \ldots, n.$$

Thus $Y(0) = 0$ and a solution of (7.1) which is a linear combination of the u_i has an nth order zero at $b > 0$ if and only if $|Y(b)| = 0$.

Returning to Theorem 5, considering the special case $A \equiv 0$, $G \equiv I$, making the identifications $U \equiv Y$, $V \equiv Z$, $B \equiv E$, $C \equiv -F$, and noting that (4.2) holds, we see that the assumption $B > 0$ may be relaxed to $B \geqslant 0$ and that we then have the following theorem concerning equation (7.1) and boundary conditions (7.2).

THEOREM 9. *Suppose*
(1) *the matrix*

$$\int_{x_1}^{x} F(t)\,dt$$

has property D,
(2) $E(t) \geqslant e(t)I$, $e \geqslant 0$, *with*

$$\int_{x_1}^{x} e(t)\,dt \to \infty$$

with x.
Then for any positive number b_1, *there exists a solution of* (7.1) *possessing* nth *order zeros at the points* 0 *and* b, *where* $b \geqslant b_1$.

It may be noted that there is no restriction imposed by this theorem on the sign of $p(x)$.

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