MAGIC GRAPHS

B. M. STEWART

Introduction. In this paper we use number-theoretic properties to classify ordinary graphs that are finite and have no isolated vertices. The classification depends on whether there is an assignment of real values, usually rational integer values, to the edges of the graph, such that the set of assigned values and the set of vertex sums of these values, summed at each vertex over all the edges incident to the vertex, will be a pair of sets with prescribed properties. Then we seek corresponding graph-theoretic properties.

It is possible to describe the problem in terms of a symmetric matrix having specified properties for its row sums, but in this paper we make no use of this interpretation; however, see **(3)**.

It is possible also to interpret the problem for any graph G by passing to its plane projective dual $P(G)$ in which vertex-incident edges in G are replaced by collinear vertices in $P(G)$. This idea is a bit foreign for graph theory, which does not ordinarily concern itself with collinearity; and also a bit foreign for projective geometry, since the ordinary restriction, that a pair of vertices in G be joined by at most one edge, results in $P(G)$ having exactly two lines through each vertex. But the edge-assignment, vertex-sum concepts in G become rather natural vertex-assignment, line-sum concepts in $P(G)$. In particular, for the regular complete bipartite graph $K(n, n)$, the corresponding $P(K(n, n))$ is a square grid; and under certain conditions for assignments and sums (see Example 1, §7), we find that our results for $K(n, n)$ correspond in $P(K(n, n))$ to the magic squares of number theory. We borrow the language of the past and ascribe to the original graphs various degrees of supernatural power, such as trivially-magic, zero-magic, semi-magic, pseudo-magic, magic, super-magic, and prime-magic.

Our interest in this topic originated in a problem proposed by J. Sedlacek in **(1)**. In particular, our use of the words "magic" and "prime-magic" agrees with Sedlacek, but we find it useful to introduce the five other classes.

1. The edge space of a graph. Consider a graph G with vertices $v_1, v_2, \ldots, v_n$ and undirected edges $e_1, e_2, \ldots, e_E$, having no multiple edges, no loops, and no isolated vertices (hence $n \geqslant 2$, $E \geqslant n/2$). Let $\alpha(e)$, or α, indicate a function defined on the edges of G with values in the real field R. The set $A(G)$ of all α is a vector space over R under the rule

$$(x_1 \alpha_1 + x_2 \alpha_2)(e) = x_1 \alpha_1(e) + x_2 \alpha_2(e) \tag{1}$$

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for all x_1, x_2 in R, all α_1, α_2 in $A(G)$, and all e in G. We shall call $A(G)$ the edge space of G.

In particular, let ϵ_i indicate the function $\epsilon_i(e_j) = d_{ij}$, $i, j = 1, 2, \ldots, E$, where $d_{ij} = 0$ if $i \neq j$ and $d_{ii} = 1$. The set $\{\epsilon_i\}$ is a basis for $A(G)$. For if α is any function in $A(G)$, say with $\alpha(e_i) = a_i$, we can represent α in the form $\alpha = a_1\epsilon_1 + a_2\epsilon_2 + \ldots + a_E\epsilon_E$; and the representation is unique, for if ζ is the zero of the vector space with $\zeta(e) = 0$ for every e in G, then $\alpha = \zeta$ if and only if every $a_i = 0$. Hence in dimension we have $\dim A(G) = E$.

Each function α in $A(G)$ creates a partitioning $P(\alpha)$ of the edges of G into classes, where e_i and e_j are in the same class if and only if $\alpha(e_i) = \alpha(e_j)$. Let us use a double subscript notation e_{uv} in which u indicates the class and v indicates an (arbitrary) enumeration within the class. Suppose there are w classes and that in the u class there are t_u edges. Since R is an ordered field, if $w > 1$, we can order the classes and describe α as follows:

$$\alpha(e_{uv}) = a_u, \qquad 1 \leqslant u \leqslant w,\ 1 \leqslant v \leqslant t_u,\ a_1 < a_2 < \ldots < a_w,$$
$$t_1 + t_2 + \ldots + t_w = E.$$

As usual we speak of a partitioning P' as a proper refinement of a partitioning P if every class in P' is a subclass of P and the number of classes in P' exceeds the number of classes in P.

THEOREM 1. *If the partitioning $P(\alpha)$ has $t_u > 1$, if α' has its values in the rational domain and $\alpha'(e_{ut_u}) \neq \alpha'(e_{u1})$, then $P(\alpha + x\alpha')$ is a proper refinement of $P(\alpha)$ when $x > a_w - a_1$.*

Proof. If $w > 1$, let e_{ps} amd e_{qt} be representatives of classes in $P(\alpha)$. Suppose e_{ps} and e_{qt} belong to the same class in $P(\alpha + x\alpha')$ so that

$$(\alpha + x\alpha')(e_{ps}) = (\alpha + x\alpha')(e_{qt}).$$

Then by (1) we have

$$x(\alpha'(e_{ps}) - \alpha'(e_{qt})) = \alpha(e_{qt}) - \alpha(e_{ps}).$$

If we assume that $\alpha'(e_{ps}) > \alpha'(e_{qt})$, since the values of α' are rational integers it follows that

$$x(\alpha'(e_{ps}) - \alpha'(e_{qt})) \geqslant x > a_w - a_1 \geqslant \alpha(e_{qt}) - \alpha(e_{ps}),$$

a contradiction. There is a similar contradiction if we assume that

$$\alpha'(e_{ps}) < \alpha'(e_{qt}).$$

Hence $\alpha'(e_{ps}) = \alpha'(e_{qt})$. Since $x \neq 0$, this implies that $\alpha(e_{qt}) = \alpha(e_{ps})$ which in turn implies that $q = p$. In other words, $P(\alpha + x\alpha')$ is a refinement of $P(\alpha)$. Furthermore, since $x \neq 0$, we have $a_u + x\alpha'(e_{ut_u}) \neq a_u + x\alpha'(e_{u1})$ so that $(\alpha + x\alpha')(e_{ut_u}) \neq (\alpha + x\alpha')(e_{u1})$. Hence $P(\alpha + x\alpha')$ is a proper refinement of $P(\alpha)$.

If $w = 1$, only the matter of a proper refinement is in question, and the argument in the preceding two sentences is adequate with $u = 1, t_1 > 1, x > 0$.

In the usual applications of Theorem 1, the original α, like α', will have all its values in the rational domain and x will be a rational integer; then $\alpha + x\alpha'$ also will have its values in the rational domain, and $P(\alpha + x\alpha')$ will be a proper refinement of $P(\alpha)$ when $x \geqslant a_w - a_1 + 1$.

2. The semi-magic space of a graph. For each vertex v_i in G we define the *vertex sum* $\sigma^\alpha(v_i) = \sum^i \alpha(e)$ where the sum $\sum^i$ extends over all edges of G which have v_i as an end point. The vertex sum is defined for each v_i in G since we assumed G has no isolated vertices. The vertex sum is a covariant of G under $A(G)$ since

$$\sigma^{x_1\alpha_1+x_2\alpha_2}(v_i) = x_1\,\sigma^{\alpha_1}(v_i) + x_2\,\sigma^{\alpha_2}(v_i). \tag{2}$$

Let $S(G)$ be the set of all α in $A(G)$ which satisfy the following *semi-magic condition:*

$$\sigma^\alpha(v_i) = \sigma(\alpha), \qquad 1 \leqslant i \leqslant n; \tag{M.1}$$

i.e., there is a "constant vertex sum, $\sigma(\alpha)$" for all the vertices of G.

The set $S(G)$ is not empty, since ζ satisfies (M.1) with $\sigma(\zeta) = 0$. In fact, $S(G)$ is a subspace of $A(G)$ which we shall call the *semi-magic space of G.* The proof follows readily from (2), for if α_1 and α_2 are in $S(G)$ with

$$\sigma^{\alpha_1}(v_i) = \sigma(\alpha_1), \qquad \sigma^{\alpha_2}(v_i) = \sigma(\alpha_2), \qquad 1 \leqslant i \leqslant n,$$

then

$$\sigma^{x_1\alpha_1+x_2\alpha_2}(v_i) = x_1\,\sigma(\alpha_1) + x_2\,\sigma(\alpha_2), \qquad 1 \leqslant i \leqslant n.$$

Thus $x_1\alpha_1 + x_2\alpha_2$ is in $S(G)$ for all x_1, x_2 in R. Furthermore,

$$\sigma(x_1\alpha_1 + x_2\alpha_2) = x_1\,\sigma(\alpha_1) + x_2\,\sigma(\alpha_2) \tag{3}$$

so that the constant vertex sum is a covariant of G under $S(G)$.

THEOREM 2. *If G is connected, then*

$$E - n + 1 \leqslant \dim S(G) \leqslant E - n + 2. \tag{4}$$

Proof. Note that $G_1 = K(1, 2)$, the arc of length 2, has

$$\dim S(G_1) = 0 = 2 - 3 + 1 = E - n + 1$$

and that $G_2 = K(1, 1) = K_2$, the arc of length 1, has

$$\dim S(G_2) = 1 = 1 - 2 + 2 = E - n + 2.$$

Thus the limits in (4) are the best possible. Note that G connected implies $E \geqslant n - 1$; hence the lower bound in (4) is non-negative.

We can eliminate $\sigma(\alpha)$ from pairs of equations in (M.1) and obtain an equivalent system in which the last $n - 1$ equations form a subsystem M, linear and homogeneous in the E coordinates a_i of α. For definiteness we take for M the equations:

$$\sigma^{\alpha}(v_i) - \sigma^{\alpha}(v_{i+1}) = 0, \qquad i = 1, 2, \ldots, n - 1.$$

(A) If G is connected, then $E \geqslant n - 1$, so the system M has a rank r satisfying $r \leqslant \min(E, n - 1) = n - 1$. Hence r of the variables a_i may be expressed in terms of the remaining $E - r$ variables or parameters. In other words, $E - n + 1 \leqslant E - r = \dim S(G)$. It is worth noting that the coefficients in M are restricted to $-1, 0, +1$; hence the solutions are linear combinations of the parameters with rational coefficients. By a proper selection of the parameters we can obtain solutions in which all the a_i are rational integers, in agreement with the remarks in the Introduction.

(B) If G is connected, we can establish the other part of (4) by an edge-completion argument, holding n fixed and making a finite induction on E. We assume the E for G satisfies $n - 1 \leqslant E < \binom{n}{2}$ and that

$$\dim S(G) \leqslant E - n + 2, \qquad n \geqslant 3.$$

Let G' be formed from G by inserting any additional edge e'. The vertices may be labelled so that $e' = v_{n-1} v_n$. The system M' for G', although involving $E' = E + 1$ coordinates, differs from the system M for G only in having the next-to-last equation contain an extra term $-a_{E'}$. Consequently the rank r' of M' and the rank r of M satisfy $r \leqslant r' \leqslant r + 1$. Then

$$E - r \leqslant E + 1 - r' = E' - r' \leqslant 1 + E - r;$$

hence

$$\dim S(G) \leqslant \dim S(G') \leqslant 1 + \dim S(G). \tag{5}$$

Applying the induction hypothesis, we find that

$$\dim S(G') \leqslant 1 + \dim S(G) \leqslant 1 + (E - n + 2) = E' - n + 2.$$

As a basis for the induction we note that a connected G with $E = n - 1$ is a tree T. The condition (M.1) dictates that $\alpha(e)$ must be the same for all terminal edges of the tree. This completely determines the assignment for all interior edges. Hence

$$0 \leqslant \dim S(T) \leqslant 1 = (n - 1) - n + 2 = E - n + 2.$$

We can pursue the completion argument in (B) and obtain a number of corollaries to Theorem 2. Some preliminary notes and lemmas will be helpful.

When G is connected, we note that $E - n + 1$ is the circuit rank, $C(G)$, with a variety of interpretations.

LEMMA 1. *If G is connected, but not complete, if G' is obtained from G by edge completion, and if* $\dim S(G) = C(G)$, *then* $\dim S(G') = C(G')$.

Proof. We note that $C(G') = 1 + C(G)$. Then we apply the hypothesis $\dim S(G) = C(G)$ and the relations (4) and (5) to obtain

$$1 + \dim S(G) = 1 + C(G) = C(G') \leqslant \dim S(G') \leqslant 1 + \dim S(G).$$

Hence $\dim S(G') = C(G')$.

With edge completion in mind, we know the basic connected graph is a tree. We may fix attention on one terminal end of a tree by calling this vertex a root. By Euler's relation, the tree has at least one other terminal end, say v', where the requirement $\sigma^\alpha(v') = a \neq 0$, in the following lemma, can be realized by taking $\alpha(e') = a$ for the terminal edge e' incident with v'.

LEMMA 2. *If α is in the edge space of a tree T with root q, if $\sigma^\alpha(v_i) = a \neq 0$, for all vertices of T, except possibly the root q, and if $\sigma^\alpha(q) = ka$, then k is an integer and $n \equiv 1 + k$* (mod 2).

Proof. The proof is by induction on $n \geqslant 2$. If $n = 2$, the single edge must have the assignment a, so $k = 1$ and $n \equiv 1 + k$ (mod 2). For $n \geqslant 3$ assume the lemma is correct for all trees when the number of vertices is less than n. Let qv be the root edge assigned the value ka. At the stem point v, there are (in addition to the root edge), say, t trees $T_1, T_2, \ldots, T_t$, disjoint except for having the common root v, and having $n_1, n_2, \ldots, n_t$ vertices, respectively, where every $n_i < n$. Since $n \geqslant 3$, we have $t \geqslant 1$. The induction hypothesis implies that each tree T_i has a root edge, say vv_i, which has an edge value $k_i a$, where k_i is an integer and $n_i \equiv 1 + k_i$ (mod 2). Since the vertex sum at v is a, we have $(k + k_1 + \ldots + k_t)a = a$. Since $a \neq 0$, $k + k_1 + \ldots + k_t = 1$. Hence k is an integer and

$$n = 2 + (n_1 - 1) + \ldots + (n_t - 1) \equiv k_1 + \ldots + k_t \equiv 1 + k \pmod 2.$$

LEMMA 3. *If T is a tree with n vertices and if n is odd, $n \geqslant 3$, then* $\dim S(T) = 0$.

Proof. A tree has $E = n - 1$; hence by (4) we have $0 \leqslant \dim S(T) \leqslant 1$. If $\dim S(T) = 1$, there is at least one α in $S(T)$ with $\sigma(\alpha) = a \neq 0$. For a star T^* the condition $\sigma(\alpha) = a$, when enforced at the centre, requires $(n-1)a = a$; so if $n \geqslant 3$ (odd or even), we find $a = 0$, hence $\dim S(T^*) = 0$. If $n = 3$, the only tree is a star.

If T is a tree which is not a star (hence $n \geqslant 4$), then there is an interior edge e with vertices q_1 and q_2. There are subgraphs T_1 and T_2 of T determined by e as follows: there is a tree T_1 with q_1 as a root and q_2 as a stem point with, say, $1 + n_1$ vertices, $n_1 \geqslant 2$; and there is a tree T_2 with q_2 as a root and q_1 as a stem point with, say, $1 + n_2$ vertices, $n_2 \geqslant 2$; furthermore, $n = n_1 + n_2$. Suppose $\alpha(e) = ka$. By Lemma 2, the condition $\sigma(\alpha) = a \not\equiv 0$ implies that k is an integer with $k \equiv n_1$ and $k \equiv n_2 \pmod 2$. Hence $n = n_1 + n_2 \equiv 2k \equiv 0 \pmod 2$. Thus the case $\dim S(T) = 1$ can arise only if n is even.

COROLLARY 2.1. *If G is connected and n is odd, $n \geqslant 3$, then* $\dim S(G) = C(G)$.

Proof. Regard G as obtained from a tree T having n vertices by repeated applications of the edge-completion process. Since n is odd, $n \geqslant 3$, according to Lemma 3 we have $\dim S(T) = 0 = C(T)$. Repeated application of Lemma 1 establishes Corollary 2.1.

COROLLARY 2.2. *If G has a vertex of degree $n - 1$, $n \geqslant 3$, then*

$$\dim S(G) = C(G).$$

Proof. The hypothesis $\rho(v) = n - 1$ for some vertex v of G implies that G is connected and contains a star graph T^* with n vertices from which G can be obtained by repeated applications of the edge-completion process. Since $n \geqslant 3$, $\dim S(T^*) = 0 = C(T^*)$; then repeated application of Lemma 1 shows $\dim S(G) = C(G)$.

COROLLARY 2.3. *For a complete graph K_n, $n \geqslant 3$,* $\dim S(K_n) = \binom{n-1}{2}$.

Proof. This is a consequence of Corollary 2.2 since K_n does contain a vertex of degree $n - 1$ and

$$C(K_n) = \binom{n}{2} - n + 1 = \frac{(n-1)(n-2)}{2} = \binom{n-1}{2}.$$

For the discussion of a graph that may not be connected, let $\tau(G)$ indicate the number of components of G. Let a typical component be G_i of order n_i with E_i edges (of course, since G_i is connected, $E_i \geqslant n_i - 1$). Let $\tau'(G)$ indicate the number of components of G for which $\dim S(G_i) = 0$. Let $\sum$ indicate summation with an index $i = 1, 2, \ldots, \tau(G)$.

THEOREM 2′. (A) *If $\tau'(G) = 0$, then* $\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i)$ *and*

$$\max(0, E - n + 1) \leqslant \dim S(G) \leqslant E - n + 1 + \tau(G). \tag{4'}$$

(B) *If $\tau'(G) > 0$, then* $\dim S(G) = \tau'(G) - \tau(G) + \sum \dim S(G_i)$ *and*

$$\max(0, E - n + \tau'(G)) \leqslant \dim S(G) \leqslant E - n + \tau(G). \tag{4''}$$

Proof. With each G_i the condition (M.1) associates a system of n_i equations involving a common vertex sum σ_i. Actually (M.1) applies to all of G and demands $\sigma_1 = \sigma_2 = \ldots = \sigma_{\tau(G)}$, but we hold this in abeyance.

(A) If $\tau'(G) = 0$, the systems of equations for the components G_i do not force any $\sigma_i = 0$; hence to satisfy (M.1) for all of G, there are exactly $\tau(G) - 1$ added conditions on the edge assignments, resulting, say, from setting $\sigma_i = \sigma_1$, $i = 2, 3, \ldots, \tau(G)$. Consequently, $\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i)$. Applying Theorem 2 to each component, we have

$$E_i - n_i + 1 \leqslant \dim S(G_i) \leqslant E_i - n_i + 2.$$

Using $E = \sum E_i$ and $n = \sum n_i$, we obtain (4′).

(B) If $\tau'(G) > 0$, suppose $\dim S(G_i) = 0$ for $i = 1, 2, \ldots, \tau'(G)$ and $\dim S(G_j) > 0$ for $\tau'(G) \leqslant j \leqslant \tau(G)$. Of course, each $\sigma_i = 0$, $1 \leqslant i \leqslant \tau'(G)$; furthermore, the edge assignments for these components are necessarily all 0. To satisfy (M.1) for all of G, there are exactly $\tau(G) - \tau'(G)$ added conditions on the edge assignments, resulting from setting $\sigma_j = 0$, $\tau'(G) < j \leqslant \tau(G)$. Consequently,

$$\dim S(G) = \tau'(G) - \tau(G) + \sum \dim S(G_i).$$

We obtain the first inequality in (4″) by using

$$E_i - n_i + 1 \leqslant \dim S(G_i), \qquad 1 \leqslant i \leqslant \tau(G).$$

To obtain the second inequality in (4″) we note from (4) that $\dim S(G_i) = 0$ implies $E_i - n_i + 1 = 0$. We let $\sum'$ indicate summation over $1 \leqslant i \leqslant \tau'(G)$ and $\sum''$ indicate summation over $1 + \tau'(G) \leqslant j \leqslant \tau(G)$. Then

$$\begin{aligned}\dim S(G) &= \tau'(G) - \tau(G) + \textstyle\sum'(E_i - n_i + 1) + \sum'' \dim S(G_j)\\ &\leqslant \tau'(G) - \tau(G) + \textstyle\sum'(E_i - n_i + 1) + \sum''(E_j - n_j + 2)\\ &= \tau'(G) - \tau(G) + E - n + \tau'(G) + 2\tau(G) - 2\tau'(G)\\ &\qquad\qquad\qquad\qquad\qquad\qquad = E - n + \tau(G).\end{aligned}$$

To show that the limits in (4′) and (4″) are the best possible consider the following graphs T, H, Q as possible components of G (see Figure 1). By Corollary 2.2 the star graph $T = K(1, 3)$ has $\dim S(T) = C(T) = 3 - 4 + 1 = 0$. By Corollary 2.2 the graph H consisting of a triangle with a tail of length 1 has

FIGURE 1.

$\dim S(H) = C(H) = 4 - 4 + 1 = 1$. By Theorem 2 the quadrilateral Q has $4 - 4 + 1 = 1 \leqslant \dim S(Q) \leqslant 2$; but it is easy to check that $\dim S(Q) = 2$, for one pair of opposite edges may be assigned a common value a, and the other pair a common value b, independent of a, and the assignment α has $\sigma(\alpha) = a + b$ at every vertex.

Let G contain t_1 copies of T, t_2 copies of H, and t_3 copies of Q so that $\tau(G) = t_1 + t_2 + t_3$ and $\tau'(G) = t_1$.

(A) If $t_1 = 0$, then by Theorem 2′,

$$\dim S(G) = 1 - (t_2 + t_3) + t_2 + 2t_3 = 1 + t_3.$$

Note that

$$E - n + 1 = 4(t_2 + t_3) - 4(t_2 + t_3) + 1 = 1.$$

Note that $t_2 = \tau(G) - t_3$, so that for any desired $\tau(G) \geqslant 1$, the value of t_2 is determined by the value of t_3. Since $\dim S(G) = 1 + t_3$ can be made to take every value between $E - n + 1 = 1$ and $E - n + \tau(G) = 1 + \tau(G)$, by taking $t_3 = 0, 1, 2, \ldots, \tau(G)$, the limits in (4′) are the best possible.

(B) If $t_1 > 0$, then by Theorem 2′,

$$\dim S(G) = t_1 - (t_1 + t_2 + t_3) + t_2 + 2t_3 = t_3.$$

Note that

$$E - n + \tau'(G) = 3t_1 + 4t_2 + 4t_3 - 4(t_1 + t_2 + t_3) + t_1 = 0.$$

Note that $t_2 = \tau(G) - t_1 - t_3$, so that for any desired $\tau(G) \geqslant 1$ and any desired $\tau'(G) = t_1$, $1 \leqslant t_1 \leqslant \tau(G)$, the value of t_2 is determined by t_3. Since $\dim S(G) = t_3$ can be made to take every value between $E - n + \tau'(G) = 0$ and $E - n + \tau(G) = \tau(G) - t_1$, by taking $t_3 = 0, 1, 2, \ldots, \tau(G) - t_1$, the limits in (4″) are the best possible.

It is worth checking that the results in Theorem 2′ agree with those in Theorem 2 when $\tau(G) = 1$ in both the cases $\tau'(G) = 0$ and $\tau'(G) = 1$. We note that $E - n + \tau(G) = C(G)$, the circuit rank of G. If we define $D(G) = E - n + \tau'(G)$, then we may combine (4), (4′), (4″) into

$$\max(0, D(G) + d) \leqslant \dim S(G) \leqslant C(G) + d,$$

where $d = 1$ or $d = 0$ according as $\tau'(G) = 0$ or $\tau'(G) > 0$.

3. The zero-magic space of a graph. Let $Z(G)$ be the set of all α in $S(G)$ which satisfy the condition

$$\text{(M.2)} \qquad \sigma(\alpha) = 0.$$

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The set $Z(G)$ is a subspace of $S(G)$ which we shall call the *zero-magic space of G*. The proof follows from (3) since $\sigma(\alpha_1) = 0$ and $\sigma(\alpha_2) = 0$ imply that

$$\sigma(x_1\alpha_1 + x_2\alpha_2) = x_1\sigma(\alpha_1) + x_2\sigma(\alpha_2) = 0 \qquad \text{for all } x_1, x_2 \text{ in } R.$$

To the condition (M.1) we add the condition (M.2) and obtain a homogeneous system M'' equivalent to using the equation $\sigma^\alpha(v_1) = 0$ and the previously defined system M. Hence the rank r'' of M'' and the rank r of M have the relation $r \leqslant r'' \leqslant r + 1$. Since $\dim S(G) = E - r$ and $\dim Z(G) = E - r''$, we obtain

$$\dim Z(G) \leqslant \dim S(G) \leqslant 1 + \dim Z(G). \tag{6}$$

Relation (6) suggests that we define a semi-canonical basis for $S(G)$ in which either (Case 1) every basis element is in $Z(G)$ or (Case 2) every basis element, except one, is in $Z(G)$. For examples, see §5.

4. Trivially-magic, zero-magic, and semi-magic graphs. On the basis of the preceding discussion we describe G as being

(G.1)	*trivially-magic*	if and only if $\dim S(G) = 0$;
(G.2)	*zero-magic*	if and only if $\dim S(G) = \dim Z(G) > 0$;
(G.3)	*semi-magic*	if and only if $\dim S(G) > \dim Z(G)$.

We illustrate these concepts with some infinite families of appropriate graphs. Other examples will appear in later sections.

We showed in Lemma 3 that all trees of odd order have $\dim S(T) = 0$; hence these are trivially-magic.

We shall show that the complete bipartite graph $G = K(2, t)$ is zero-magic if $t \geqslant 3$. For if the vertices are $v_1, v_2, w_1, w_2, \ldots, w_t$, we can use the edge-completion process (repeated) to obtain G from the tree T having the edges $v_1w_1, v_1w_2, \ldots, v_1w_t$ and the edge w_1v_2 (see Figure 2). In an attempt to

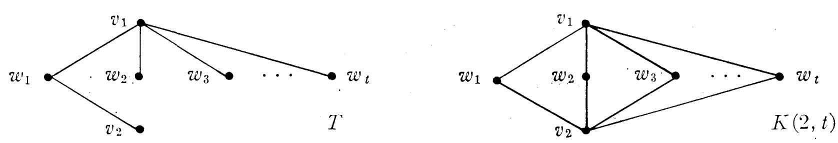

FIGURE 2.

satisfy (M.1) for T, we must assign to all the terminal edges the same value a. The vertex sum condition $\sigma^\alpha(w_1) = a$ requires $\alpha(v_1w_1) = 0$. Then the condition $\sigma^\alpha(v_1) = a$ has the form $(t - 1)a = a$. Since $t \geqslant 3$, it follows that $a = 0$. Hence $\dim S(T) = 0$. It follows from Lemma 1 that $\dim S(G) = C(G) = t - 1$. A set of $t - 1$ independent basis elements for $S(G)$ is given by

$$\{\alpha_i = \epsilon_{1i} - \epsilon_{2i} + \epsilon_{2,i+1} - \epsilon_{1,i+1}, i = 1, 2, \ldots, t - 1\},$$

where ϵ_{1i} and ϵ_{2i} are the unit vectors associated with the edges $v_1 w_i$ and $v_2 w_i$, respectively. Since every α_i is in $Z(G)$, it follows that $S(G) = Z(G)$. Hence $G = K(2, t)$ is zero-magic if $t \geqslant 3$.

The previous example shows there are zero-magic graphs of every dimension $d \geqslant 2$. For an example of a zero-magic graph of dimension 1, we can use a *kite* consisting of a quadrilateral with a tail, where the tail is an arc of odd length (for an outline of the argument see Figure 3).

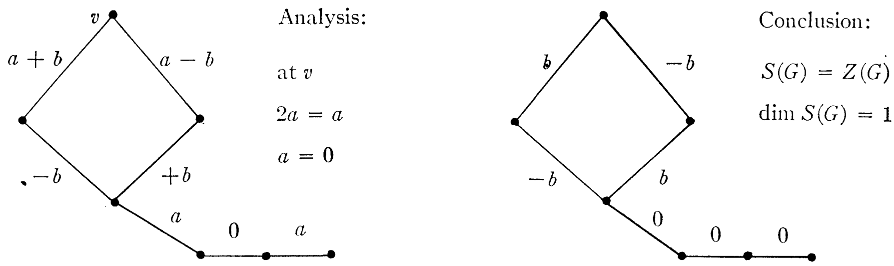

FIGURE 3.

It is clear that (G.3) can hold if and only if there is an α in $S(G)$ for which $\sigma(\alpha) \neq 0$. In fact we may change this condition to $\sigma(\alpha) > 0$, for from (3) we have $\sigma(-\alpha) = -\sigma(\alpha)$.

We can apply this test to conclude that any regular graph is semi-magic; for if we use the α that assigns to every edge the value $a > 0$, we find that (M.1) is satisfied with $\sigma(\alpha) = \rho a > 0$, where ρ is the common vertex degree. For example, a triangle K_3 is semi-magic with $\dim S(K_3) = 1$; and the graph G consisting of t_3 copies of the quadrilateral Q (see the example following Theorem 2′) is semi-magic with $\dim S(G) = 1 + t_3$.

Let us turn from examples to a partial characterization of the trivially-magic graphs. Theorem 2 shows that a necessary condition for a connected graph to be trivially-magic is that the graph be a tree. For if G is connected, $n - 1 \leqslant E$; and if $\dim S(G) = 0$, then (4) demands that $E - n + 1 \leqslant 0$, so that $E \leqslant n - 1$. Hence $E = n - 1$ and T is a tree.

Lemma 3 shows that a sufficient condition for a tree to be trivially-magic is that the tree be of order n where n is odd, $n \geqslant 3$.

If T is a tree of even order, then T is either trivially-magic with $\dim S(T) = 0$, as in the case of stars, or T is semi-magic with $\dim S(T) = 1 > \dim Z(T)$, as in the case of arcs. For T has $E - n + 1 = 0$, hence (4) limits $\dim S(T)$ to the two values 0 or 1. If $\dim S(T) = 1$, then $\dim Z(T) = 0$; for if α is in $S(T)$ with $\sigma(\alpha) = a$, then the terminal edges of T must have the assignment a; hence it is possible to satisfy (M.2) only by taking $a = 0$. We have not been able to find a neat graph-theoretic distinction between these cases. As an example, the 23 trees of order 8 are separated into 14 trivially-magic and 9 semi-magic.

If G has more than one component, we can use the notation of Theorem 2′ and show that G is trivially-magic if and only if $\tau'(G) > 0$ and $\dim S(G_j) = 1$, $\tau'(G) < j \leqslant \tau(G)$. For if $\tau'(G) = 0$, every component G_i of G has $\dim S(G_i) \geqslant 1$, and part (A) of Theorem 2′ shows that

$$\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i) \geqslant 1.$$

If $\tau'(G) > 0$, of course $\dim S(G_j) \geqslant 1$, but if any G_j has $\dim S(G_j) \geqslant 2$, then part (B) of Theorem 2′ shows that

$$\dim S(G) = \tau'(G) - \tau(G) + \sum'' \dim S(G_j) \geqslant 1.$$

In both cases (G.1) is not satisfied. Thus if G is to be trivially-magic, the conditions stated are necessary. Conversely, if the conditions are satisfied, it is easy to use the formula for $\dim S(G)$ in (B) to check that $\dim S(G) = 0$, so that G is trivially-magic.

THEOREM 3. *A graph is semi-magic if and only if each component is semi-magic.*

Proof. (A) If G has components $G_1, G_2, \ldots, G_\tau$, then the corresponding vector space $A(G)$ has disjoint components $A_1, A_2, \ldots, A_\tau$. If one component, say G_i, is not semi-magic, then G is not semi-magic. For condition (M.1) on G can only be satisfied by having (M.1) hold on G_i. Since G_i is not semi-magic, G_i must be trivially-magic or zero-magic. In either case if α is in $S(G)$, then $\sigma^\alpha(v) = 0$ on the vertices of G_i, hence $\sigma^\alpha(v) = 0$ for all the vertices of G. Thus $S(G) = Z(G)$ and G is not semi-magic.

(B) If each component G_i is semi-magic, there exists an associated α^i in $S(G_i)$ for which $\sigma(\alpha^i) > 0$, $1 \leqslant i \leqslant \tau$. Let M be the product of all the $\sigma(\alpha^i)$, or if the $\sigma(\alpha^i)$ are all rational integers, let M be the least common multiple of $\sigma(\alpha^1), \sigma(\alpha^2), \ldots, \sigma(\alpha^\tau)$. Define $M_i = M/\sigma(\alpha^i)$. Define α_i for G by $\alpha_i(e) = \alpha^i(e)$ for all e in G_i, and $\alpha_i(e) = 0$ for all e in G, but not in G_i. Finally, define

$$\alpha = M_1\alpha_1 + M_2\alpha_2 + \ldots + M_\tau\alpha_\tau. \tag{7}$$

Note that each edge and each vertex of G appears in exactly one component. Hence if v_k appears in the component G_i we find from (2) that

$$\sigma^\alpha(v_k) = \sum{}^k\alpha(e) = M_i \sum{}^k\alpha_i(e) = M_i \sum{}^k\alpha^i(e) = M_i\,\sigma(\alpha^i) = M.$$

Since this result is independent of the choice of v_k, we know that α is in $S(G)$. Since $\sigma(\alpha) = M > 0$, it follows that G is semi-magic.

We say that U is a *skeleton* for G if U is a proper subgraph of G which includes every vertex of G and the components of U are not isolated vertices.

THEOREM 4. *A sufficient condition that G be semi-magic is that G has a skeleton U all of whose components are semi-magic.*

Proof. Part (B) of Theorem 3 shows that U is semi-magic under an α^U of the type defined in (7). Since the edges of U are edges of G, it is possible to define an α for G by using $\alpha(e) = \alpha^U(e)$ if e is in U, and $\alpha(e) = 0$ if e is in G, but not in U. Since every vertex of G is in U and since U is semi-magic under α^U, it follows easily that G is semi-magic under α with $\sigma(\alpha) = \sigma(\alpha^U) > 0$.

COROLLARY 4.1. *A sufficient condition that G be semi-magic is that G has a skeleton U all of whose components are regular.*

Proof. A regular graph U_i of degree ρ_i is semi-magic under an assignment $\alpha^i(e) = 1$ for each e of U_i, for this implies $\sigma(\alpha^i) = \rho_i > 0$. The method of (7) and Theorem 4 now applies.

For example, consider the graph G with skeleton U in Figure 4. Here U has

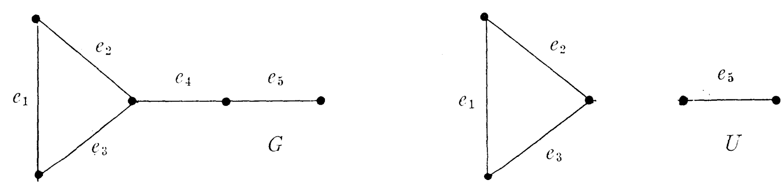

FIGURE 4.

two regular components: U_1 with $\rho_1 = 2$ and U_2 with $\rho_2 = 1$. With $M = 2$, $M_1 = 1$, $M_2 = 2$, we apply (7) to find $\alpha = (\epsilon_1 + \epsilon_2 + \epsilon_3) + 2\epsilon_5$ with $\sigma(\alpha) = 2$.

The condition in Theorem 4 is not a necessary one for G to be semi-magic. For consider the graph G in Figure 5, which has several proper skeletons, as shown. However, no one of these is semi-magic. Nevertheless, G itself is

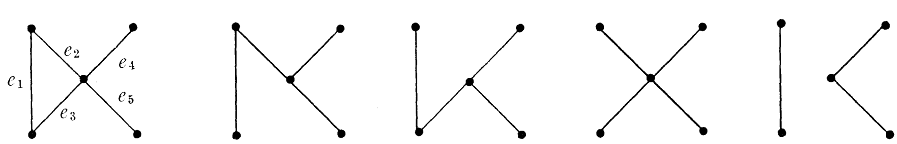

FIGURE 5.

semi-magic. By Corollary 2.1, dim $S(G) = C(G) = 1$. We find $S(G)$ generated by $\alpha = 3\epsilon_1 - \epsilon_2 - \epsilon_3 + 2\epsilon_4 + 2\epsilon_5$ for which $\sigma(\alpha) = 2 > 0$; hence G is semi-magic.

5. Pseudo-magic graphs. We describe a graph G (with $E \geqslant 2$) as being *pseudo-magic* if and only if there exists an α in S(G) which has the property of *distinctness:*

(M.3) *for all pairs of edges in* G, $e_i \neq e_j$ *implies* $\alpha(e_i) \neq \alpha(e_j)$.

We agree that K_2 with $E = 1$ is also pseudo-magic.

THEOREM 5. *A necessary and sufficient condition that* G *(with* $E \geqslant 2$*) be pseudo-magic is that*

(C.1) *for each pair of edges in* G, $e_i \neq e_j$, *there exists an* α_{ij} *in* $S(G)$, *with values in the rational integers, such that* $\alpha_{ij}(e_i) \neq \alpha_{ij}(e_j)$.

Proof. We use the notation of Theorem 1.

(A) Suppose (C.1) is satisfied. Then there exists $\alpha = \alpha_{12}$ in $S(G)$ with $\alpha(e_1) \neq \alpha(e_2)$; hence $P(\alpha)$ has $2 \leqslant w \leqslant E$. If $w < E$, then there exists u with $t_u > 1$ and $\alpha(e_{u1}) = \alpha(e_{ut_u})$. If we take $e_i = e_{u1}$ and $e_j = e_{ut_u}$, then (C.1) implies the existence of $\alpha' = \alpha_{ij}$ with values in the rational domain such that $\alpha'(e_{u1}) \neq \alpha'(e_{ut_u})$. According to Theorem 1 the partitioning $P(\alpha + x\alpha')$ is a proper refinement of $P(\alpha)$ if we choose $x \geqslant a_w - a_1 + 1$. Since $S(G)$ is a subspace of $A(G)$, it follows that $\alpha^* = \alpha + x\alpha'$ is in $S(G)$, and with $x \geqslant a_w - a_1 + 1$ we have $w < w^*$. If $w^* < E$, we can repeat this procedure. In a finite number of steps we can reach α^{**} in $S(G)$ with $w^{**} = E$. But this is equivalent to the condition (M.3); hence G is pseudo-magic.

(B) Suppose (C.1) is not satisfied. If there exists a pair of edges $e_i \neq e_j$ in G such that every α in $S(G)$ with values in the rational domain has $\alpha(e_i) = \alpha(e_j)$, we can show that $\alpha(e_i) = \alpha(e_j)$ for every real-valued α in $S(G)$. Hence (M.3) fails and G is not pseudo-magic. If $\dim S(G) = 0$, there is no problem, for then ζ is the only function in $S(G)$. If $\dim S(G) > 0$, the α in $S(G)$ are determined by solving (M.1) for the coordinates a_k of α. As remarked in part (A) of the proof of Theorem 2, if the independent parameters are a'_s, $s = 1, 2, \ldots,$ $E - r = \dim S(G)$, then the dependent coordinates are expressed as $a_k = \sum d_{ks} a'_s$, $k = 1, 2, \ldots, r$, where the d_{ks} are rational. Let D be the least common denominator of the set of d_{ks}. Among the α in $S(G)$ whose values are in the rational domain are the α'' determined by taking every a'_s to be an integer multiple of D. If $\alpha(e_i) = \alpha(e_j)$ for all α with values in the rational domain, it cannot be that both a_i and a_j are independent parameters, for then there are functions of the type α'' which have $\alpha''(e_i) \neq \alpha''(e_j)$. If a_i, say, is dependent and a_j independent, or if both a_i and a_j are dependent, the hypothesis that $\alpha(e_i) = \alpha(e_j)$ for all α with values in the rational domain includes the infinitely many α''. Hence, in both cases, a_i and a_j must be identical linear expressions in the independent parameters. Therefore $\alpha(e_i) = \alpha(e_j)$ for all real-valued α in $S(G)$.

For example, K_3 is semi-magic, but not pseudo-magic. Since n is odd, $\dim S(K_3) = C(K_3) = 1$, and $S(K_3)$ is generated by $\alpha_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$. Thus for every $\alpha = x\alpha_1$ in $S(K_3)$ we find that $\alpha(e_1) = \alpha(e_2) = \alpha(e_3) = x$. Hence

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(M.3) fails. However, $\sigma(\alpha) = 2x > 0$ when $x > 0$; hence (G.3) holds (see Figure 6).

Similarly, K_4 is semi-magic, but not pseudo-magic. By Corollary 2.3, $\dim S(K_4) = 3$. If we let ϵ_{ij} indicate the unit vector associated with the edge $e_{ij} = v_i v_j$, then a basis for $S(K_4)$ is given by

$$\alpha_1 = \epsilon_{12} + \epsilon_{34}, \qquad \alpha_2 = \epsilon_{13} + \epsilon_{24}, \qquad \alpha_3 = \epsilon_{14} + \epsilon_{23}.$$

Thus for every $\alpha = a\epsilon_1 + b\epsilon_2 + c\epsilon_3$ in $S(K_4)$ we find $\alpha(e_{12}) = a = \alpha(e_{34})$. Hence (M.3) fails. However, $\sigma(\alpha) = a + b + c > 0$ when, say, $a > 0$, $b = c = 0$; hence (G.3) holds (see Figure 6).

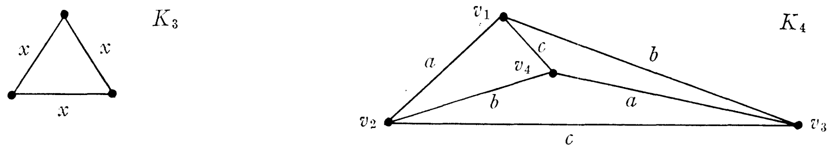

FIGURE 6.

In a later section we shall show that K_n is a special kind of semi-magic and pseudo-magic graph when $n \geqslant 5$.

Graphs may be pseudo-magic and not semi-magic. We showed that the complete bipartite graph $K(2, t)$ is zero-magic when $t \geqslant 3$. We can now show that $K(2, t)$ is pseudo-magic for $t \geqslant 3$. For a direct demonstration we define

$$\alpha(v_1 w_i) = b_i = -\alpha(v_2 w_i), \qquad i = 1, 2, \ldots, t-1,$$

where $0 < b_1 < b_2 < \ldots < b_{t-1}$, and

$$\alpha(v_2 w_t) = b_1 + b_2 + \ldots + b_{t-1} = -\alpha(v_1 w_t).$$

Then we readily check that (M.2) and (M.3) are satisfied.

It is clear that the selection of the α_{ij} in (C.1) may be made easier by the use of the skeletons described in Theorem 4. For example, to show that the "fan" F_4 (see Figure 7) is pseudo-magic, we use $\alpha_1 = \epsilon_1 + \epsilon_3 + \epsilon_8 + \epsilon_9 + 2\epsilon_6$, $\alpha_2 = \epsilon_1 + \epsilon_6 + \epsilon_8$, $\alpha_3 = \epsilon_5 + \epsilon_7 + \epsilon_9$, which are based on skeletons, and $\alpha_4 = \epsilon_2 - \epsilon_4 + \epsilon_7 - \epsilon_8$, which is in $Z(F_4)$, and hence in $S(F_4)$. Since the set $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is extensive enough to guarantee (C.1), it follows that F_4 is pseudo-magic. We find that $\alpha = 2\alpha_1 + 6\alpha_2 + 4\alpha_3 + \alpha_4$ has property (M.1) with $\sigma(\alpha) = 14$ and property (M.3) with $w = E = 9$ distinct edge values.

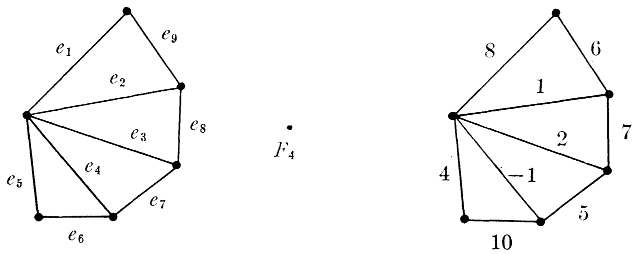

FIGURE 7.

To motivate the next section we note that the most general α in $S(F_4)$, namely, $\alpha = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4$ (for by Corollary 2.2,

$$\dim S(F_4) = C(F_4) = 4$$

and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are independent), has the property $\alpha(e_2) = x_4$, $\alpha(e_4) = -x_4$. Hence it is impossible to find an α in $S(F_4)$ for which every edge has a positive assignment.

6. Magic graphs. We describe G (with $E \geqslant 2$) as being *magic* if and only if there exists an α in $S(G)$ which has the property (M.3) of distinctness and the property of "positiveness":

(M.4) *for every edge e in G, $\alpha(e) > 0$.*

We agree that K_2 with $E = 1$ is also magic.

THEOREM 6. *If G is pseudo-magic, a sufficient condition that G be magic is that*

(C.2) *for each edge e in G there exists a skeleton R that contains e and whose components are semi-magic and have property* (M.4) *under functions that have values in the rational domain.*

Proof. Since G is assumed pseudo-magic, there exists an α in $S(G)$ with $w = E$ and $a_1 < a_2 < \ldots < a_E$. We may assume that $0 < a_E$; otherwise G is already known to be magic under $-\alpha$. If $0 < a_1$, then G is already magic under α. If $a_1 \leqslant 0$ and $\alpha(e_1) = a_1$, let R be the skeleton, provided by the hypothesis (C.2), which contains e_1. Using the components of R, construct as in (7) the function $\alpha^R = M_1\alpha_1^R + \ldots + M_t\alpha_t^R$. From (C.2) each α_i^R has values that are positive integers, each M_i is a positive integer, and hence $\alpha^R(e)$ is a positive integer for each e in R. Extend α^R to an α' for G by the usual plan: $\alpha'(e) = \alpha^R(e)$ if e is in R; and $\alpha'(e) = 0$ if e is in G, but not in R. Consider the function $\alpha^* = \alpha + x\alpha'$ with $x \geqslant a_E - a_1 + 1$. According to Theorem 1, $w^* = w = E$. According to Theorem 4, G is semi-magic under α'; since $\sigma(\alpha) \geqslant 0$ and

$$\sigma(\alpha^*) = \sigma(\alpha) + x\sigma(\alpha') \geqslant \sigma(\alpha') > 0,$$

it follows that G is semi-magic under α^*. Thus G is pseudo-magic under α^*.

If $i > 1$, then $\alpha^*(e_i) = \alpha(e_i) + x\alpha'(e_i) \geqslant \alpha(e_i) = a_i$. If e_1 belongs to the jth component of R, $\alpha^*(e_1) = \alpha(e_1) + xM_j\alpha_j^R(e_1)$; hence $\alpha^*(e_1) \geqslant a_1 + 1$. Thus the minimum a_1^* for α^* has the property $a_1^* \geqslant \min(a_2, a_1 + 1)$. In a finite number of steps, we can find an α^{**} under which G is pseudo-magic with $w^{**} = E$ and for which the minimum $a_1^{**} > 0$; hence G is magic under α^{**}.

Let U^* indicate the set of all the skeletons U of G which have the property that each component of U is regular. If U is in U^*, let β be the corresponding function described in the proof of Corollary 4.1.

COROLLARY 6.1. *Sufficient conditions that G be magic are that both*

(U.1) *for each pair of edges in G, $e_i \neq e_j$, there exists a skeleton U_{ij} in U^* such that $\beta_{ij}(e_i) \neq \beta_{ij}(e_j)$; and*

(U.2) *for each edge e_i in G there exists a skeleton U_i in U^* such that U_i contains e_i.*

Proof. From Corollary 4.1 we see that (U.1) corresponds to (C.1) and we can apply Theorem 5 to see that G is pseudo-magic. From Corollary 4.1 we see that (U.2) implies that the components of U_i have the property (M.4) under the function β_i; hence (U.2) corresponds to (C.2) and we can apply Theorem 6 to see that G is magic.

7. Applications of Theorem 6.

Example 1. A complete bipartite graph $K(n, n)$ is magic for $n \geqslant 3$.

Proof. The graph $K(n, n)$ consists of two sets of vertices $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ with every v joined to every w, but no v joined to another v and no w joined to another w. Because of the great symmetry of the graph, the condition (U.1) of Corollary 6.1 can be checked by examination of only two cases.

Case 1. We produce a skeleton U in U^* containing $v_1 w_1$ but not containing $v_2 w_2$; hence $\beta(v_2 w_2) = 0 < \beta(v_1 w_1)$. We use

$$U = (v_1 w_1) \oplus (v_2 w_3) \oplus (v_3 w_2) \oplus (v_4 w_4) \oplus \ldots \oplus (v_n w_n)$$

which is possible since $n \geqslant 3$.

Case 2. We produce a skeleton U in U^* containing $v_1 w_1$, but not containing $v_1 w_2$, by using

$$U = (v_1 w_1) \oplus (v_2 w_2) \oplus (v_3 w_3) \oplus \ldots \oplus (v_n w_n).$$

To satisfy condition (U.2) we may use the last-described skeleton, for $v_1 w_1$ is a typical edge.

By Corollary 6.1 the proof for Example 1 is now complete.

If we interpret $\alpha(v_i w_j)$ to be the entry in the ith row and jth column of a square table, then $\sigma^\alpha(v_i)$ is the row sum in the ith row of the table and $\sigma^\alpha(w_j)$ is the column sum in the jth column of the table. Hence if $K(n, n)$ is magic under α, then the table is one of the weakly-magic squares of number theory and Example 1 shows (by graph-theoretic methods!) the existence of weakly-magic squares of all orders $n \geqslant 3$. However, the magic squares of number theory require the use of consecutive integers, a feature which we shall consider in §9.

Exercise 1. Show that the complete bipartite graph $K(n, m)$ with $n \geqslant m \geqslant 1$ is semi-magic if and only if $n = m$.

Example 2. A complete graph K_n is magic for $n = 2$ and all $n \geqslant 5$.

Proof. We have already discussed the cases $n = 2, 3, 4$, discovering that K_2 is magic, while K_3 and K_4 are not pseudo-magic, and hence not magic. Consider K_n with $n \geqslant 5$. To satisfy (U.1) consider two cases.

Case 1. Suppose p, q, u, v are four distinct vertices fo K_n. We can produce a skeleton U in U^* which includes the edge pq, but does not include the edge uv; hence $\beta(uv) = 0 < \beta(pq)$. Select a fifth distinct vertex t, which is possible since $n \geqslant 5$. In K_n there is an arc from v to t, direct if $n = 5$, and passing through all the other $n - 5$ vertcies if $n > 5$. For U we can use the circuit $((tupqv \ldots t))$ containing pq and not containing uv.

Case 2. Suppose p, q, v are three distinct vertices in K_n. We can produce a skeleton U in U^* which includes the edge pq, but does not include the edge qv. Select a fourth distinct vertex t. In K_n there is an arc from v to t, passing through the other $n - 4$ vertices. For U we can use the circuit $((tqpv \ldots t))$ containing pq and not containing qv.

To satisfy (U.2) we note that K_{n-2} is regular and is not an isolated vertex since $n \geqslant 5$. As a typical case we consider $e = pq$ and use the skeleton $U = (pq) \oplus K_{n-2}$ where K_{n-2} is based on the other $n - 2$ vertices.

It follows from Corollary 6.1 that K_n is magic for all $n \geqslant 5$.

If each vertex $v_0, v_1, \ldots, v_k$ of an arc $(k \geqslant 1)$ is connected to another vertex v, we describe the resulting graph as a *fan with k blades*, denoted by F_k (see Figure 7 for F_4). We note that F_k has $n = k + 2$ and $E = 2k + 1$. By Corollary 2.2, $\dim S(F_k) = C(F_k) = k$.

Example 3. A fan F_k is magic if and only if k is odd and $k \geqslant 3$.

Proof. (A) We note that F_1 is the triangle K_3, which is not magic. We set $k = 2K + 1$ and give two proofs that F_k is magic for $K \geqslant 1$.

(A.1) For an existence proof we use Corollary 6.1. Consider the following series of skeletons, each in U^* (see Figure 8):

$$U_i = (v_0 v_1) \oplus \ldots \oplus (v_{2i-2} v_{2i-1}) \oplus ((vv_{2i} v_{2i+1} v)) \oplus (v_{2i+2} v_{2i+3}) \oplus \ldots \oplus (v_{2K} v_{2K+1}), \qquad 0 \leqslant i \leqslant K,\ K \geqslant 1;$$

$$U'_i = (v_0 v_1) \oplus \ldots \oplus (v_{2i-2} v_{2i-1}) \oplus ((vv_{2i} v_{2i+1} v_{2i+2} v_{2i+3} v)) \oplus (v_{2i+4} v_{2i+5}) \oplus \ldots \oplus (v_{2K}v_{2K+1}), \qquad 0 \leqslant i \leqslant K - 1, K \geqslant 1.$$

It is not difficult to check that this set of skeletons is extensive enough to guarantee (U.1) and (U.2). In fact, the associated sets $\{\beta_i\}$ and $\{\beta'_i\}$ form a basis for $S(F_k)$.

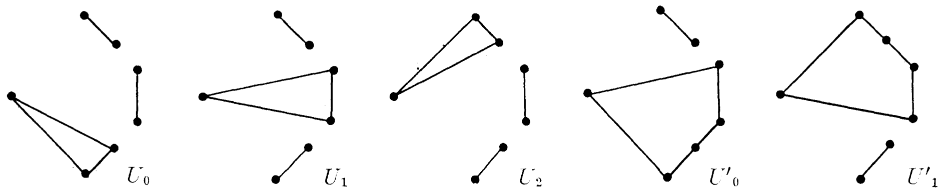

FIGURE 8.

(A.2) For a constructive proof we describe a specific α (see Figure 9) in which the edge assignments are positive (M.4) and readily checked to be distinct (M.3):

$$\alpha(vv_i) = i, \quad 1 \leqslant i \leqslant 2K; \qquad \alpha(vv_0) = 3K + 1; \qquad \alpha(vv_{2K+1}) = 4K + 1;$$

$$\alpha(v_{2i}\, v_{2i-1}) = 2K + (K - i) + 1, \quad K \geqslant i \geqslant 1;$$

$$\alpha(v_{2i+1}\, v_{2i}) = 2K^2 + 4K + 1 + (K - i), \quad K \geqslant i \geqslant 0.$$

The first set runs consecutively from 1 to $2K$; the second, from $2K + 1$ to $3K$; the third, from $2K^2 + 4K + 1$ to $2K^2 + 5\mathrm{K} + 1$. It is obvious that the other two values are distinct from these. It remains to check (M.1) by verifying that the vertex sums are all rearrangements of $2K^2 + 8K + 2$, viz.:

$$\sigma^\alpha(v_0) = (3K + 1) + (2K^2 + 5K + 1) = (4K + 1) + (2K^2 + 4K + 1) = \sigma^\alpha(v_{2K+1});$$

$$\sigma^\alpha(v_{2i}) = (3K - i + 1) + (2i) + (2K^2 + 5K + 1 - i), \qquad 1 \leqslant i \leqslant K;$$

$$\sigma^\alpha(v_{2i+1}) = (2K^2 + 5K + 1 - i) + (2i + 1) + (3K - i), \qquad 0 \leqslant i \leqslant K - 1;$$

$$\sigma^\alpha(v) = (3K + 1) + (4K + 1) + (1 + 2 + \ldots + 2K) = 7K + 2 + K(2K + 1) = 2K^2 + 8K + 2.$$

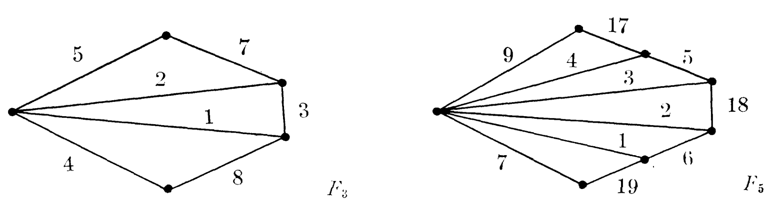

FIGURE 9.

(B) There is a hint that F_{2K} is not magic for $K \geqslant 1$, since there is no skeleton U which includes vv_{2i-1}, $1 \leqslant i \leqslant K$; hence (U.1) and (U.2) cannot be satisfied. However, we recall that the conditions in Corollary 6.1 are sufficient, not necessary, so we resort to an algebraic proof (underneath whose thin disguise we are really finding a basis for $S(F_{2K})$).

Let α be any function in $S(F_{2K})$; hence α satisfies (M.1).

To certain selected edges we make the arbitrary assignments:

$$\alpha(vv_0) = a, \qquad \alpha(v_0v_1) = b, \qquad \alpha(vv_i) = x_i, \quad 1 \leqslant i \leqslant 2K-1;$$

for we can show that all the other assignments are determined by the condition (M.1) which requires that every vertex sum be equal to $\sigma^\alpha(v_0) = a + b$. For example, $\sigma^\alpha(v) = a + b$ requires

$$\alpha(vv_{2K}) = b - \sum_1^{2K-1} x_s.$$

By (limited) induction we can show that we must have

$$\alpha(v_{2i-1}v_{2i}) = a + \sum_1^{2i-1} (-1)^s x_s, \qquad 1 \leqslant i \leqslant K. \tag{8}$$

For when $i = 1$, we require

$$\sigma^\alpha(v_1) = a + b = \alpha(v_0v_1) + \alpha(vv_1) + \alpha(v_1v_2) = b + x_1 + \alpha(v_1v_2);$$

hence $\alpha(v_1v_2) = a - x_1$, in agreement with the formula (8). From the requirements $\sigma^\alpha(v_{2i}) = \sigma^\alpha(v_{2i+1})$ we find that

$$\alpha(v_{2i-1}v_{2i}) + \alpha(vv_{2i}) + \alpha(v_{2i}v_{2i+1}) = \alpha(v_{2i}v_{2i+1}) + \alpha(vv_{2i+1}) + \alpha(v_{2i+1}v_{2i+2})$$

hence $\alpha(v_{2i+1}v_{2i+2}) = \alpha(v_{2i-1}v_{2i}) + x_{2i} - x_{2i+1}$. If we use the induction hypothesis (8) for $1 \leqslant i < K$, we find that

$$\alpha(v_{2i+1}v_{2i+2}) = a + \sum_1^{2i+1} (-1)^s x_s,$$

which is the correct form for (8) in the case $i + 1$.

In particular, from (8) we have

$$\alpha(v_{2K-1}v_{2K}) = a + \sum_1^{2K-1} (-1)^s x_s.$$

Hence

$$\sigma^\alpha(v_{2K}) = \alpha(v_{2K-1}v_{2K}) + \alpha(vv_{2K}) = a + b - 2\sum_1^K x_{2S-1}.$$

Requiring $\sigma^\alpha(v_{2K}) = a + b$ demands

$$\sum_1^K x_{2S-1} = 0.$$

Since this is impossible using only positive real values for the x_{2S-1}, there is no α in $S(F_{2K})$ which satisfies (M.4). Hence F_{2K} is not magic for all $K \geqslant 1$.

Exercise 2. Consider a rectangular lattice $L(s, t)$ with $n = (s+1)(t+1)$ and $E = s(t+1) + t(s+1)$. Note that $C(L) = st$ and that st linearly independent α in $Z(L)$ are suggested by the st cells of L, of the type $\alpha = \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4$.

(1) Use the method of Corollary 4.1 to show that $L(1, b)$ is semi-magic, but not pseudo-magic, for all $b \geqslant 1$; and that $\dim S(L) = 1 + C(L)$.

(2) Use Lemma 1 and Theorem 5 to show that $L(2a, 2b)$ is zero-magic and pseudo-magic for all $a \geqslant 1$, $b \geqslant 1$; and that $\dim S(L) = C(L)$.

(3) Use Corollary 6.1 to show that $L(a, 2b+1)$ is magic for all $a \geqslant 2$, $b \geqslant 1$; and that $\dim S(L) = 1 + C(L)$.

If each vertex $v_1, v_2, \ldots, v_k$ of a circuit ($k \geqslant 3$) is connected to another vertex v, we call the resulting graph a *wheel with k spokes*, denoted by W_k. If one spoke is removed from W_k, say v and v_k are not joined, we call the resulting graph a *basket*, denoted by B_k.

Example 4. A *basket* B_k is magic for $k = 4$ and all $k \geqslant 6$.

Proof. (A) If $k = 2K+1$, $K \geqslant 3$, define α as follows (see Figure 10):

$$\alpha(vv_i) = i, \qquad 1 \leqslant i \leqslant 2K-2;$$

$$\alpha(vv_{2K}) = 2K-1; \qquad \alpha(vv_{2K-1}) = 4K-2; \qquad \alpha(v_{2K}\, v_{2K-1}) = K^2 - K;$$

$$\alpha(v_{2i+1}\, v_{2i}) = K^2 + K - 1 - i, \quad K-1 \geqslant i \geqslant 1;$$

$$\alpha(v_1\, v_{2K+1}) = K^2 + K - 1;$$

$$\alpha(v_{2i}\, v_{2i-1}) = K^2 + 2K + 1 - i, \quad K-1 \geqslant i \geqslant 1;$$

$$\alpha(v_{2K+1}\, v_{2K}) = K^2 + 2K - 1.$$

It is easy to check that (M.3) holds for $K \geqslant 5$ with the listed ordering; luckily (M.3) holds for $K = 3$ and $K = 4$ with a reordering of the second, third, and fourth items. Finally, (M.1) holds with $\sigma(\alpha) = 2K^2 + 3K - 2$.

(B) If $k = 2K$, $K \geqslant 4$, define α as follows (see Figure 10):

$$\alpha(vv_i) = i, \quad 1 \leqslant i \leqslant 2K-3;$$

$$\alpha(vv_{2K-1}) = 2K-2; \qquad \alpha(v_{2K-1}\, v_{2K-2}) = 2K-1; \qquad \alpha(vv_{2K-2}) = 3K-3;$$

$$\alpha(v_{2i+1}\, v_{2i}) = 4K-3-i, \quad K-2 \geqslant i \geqslant 1; \qquad \alpha(v_1\, v_{2K}) = 4K-3;$$

$$\alpha(v_{2i}\, v_{2i-1}) = 2K^2 - 4K + 1 - i, \quad K-1 \geqslant i \geqslant 1;$$

$$\alpha(v_{2K}\, v_{2K-1}) = 2K^2 - 4K + 1.$$

It is easy to check that (M.3) and (M.4) are satisfied because $K \geqslant 4$; and (M.1) is satisfied with $\sigma(\alpha) = 2K^2 - 2$.

(C) For the cases B_4 and B_6 we produce individual solutions (see Figure 10).

(D) It is easy to show that B_3 and B_5 are not magic by finding bases for $S(B_3)$ and $S(B_5)$ (see Figure 10).

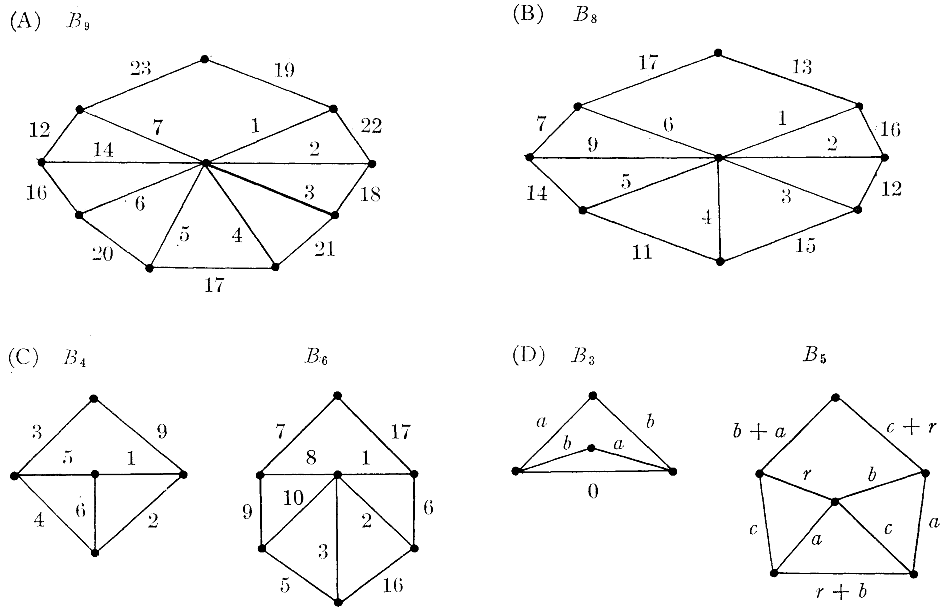

FIGURE 10.

8. Completion. If G is not complete, let G^* be formed from G by inserting one new edge e^*. Thus G^* has $n^* = n$ and $E^* = E + 1$.

THEOREM 7. *If G is magic and if G^* has a skeleton R^* which contains e^* and the components of R^* are semi-magic and have property* (M.4) *under functions which have values in the rational domain, then G^* is magic.*

Proof. If α is in $A(G)$, let α^* be defined in $A(G^*)$ by $\alpha^*(e^*) = 0$ and $\alpha^*(e) = \alpha(e)$ when e is in G, but $e \neq e^*$. Given that G is magic under α, certainly G^* is pseudo-magic under α^* for the vertex sums are the same as for α, and the edge assignments are distinct with $w^* = E + 1 = E^*$. The only discrepancy, keeping G^* from being magic under α^*, is the assignment $\alpha^*(e^*) = 0$. The assumptions on R^* and e^* are almost like (C.2) in Theorem 6. The same argument applies, but only one step is required to arrive at an appropriate $\alpha^{**} = \alpha^* + x\alpha'$ under which G^* is magic. Since the minimum edge assignment for α^* is 0 and the maximum edge assignment is a_E (the same as for α), a suitable condition on x is $x \geqslant a_E - 0 + 1 = a_E + 1$.

COROLLARY 7.1. *If G is magic and if G^* has a skeleton R^* which contains e^* and the components of R^* are regular, then G^* is magic.*

Proof. Combine Corollary 4.1 and Theorem 7.

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Example 5. A wheel W_t is magic for $t \geqslant 4$.

Proof. The wheel W_t can be obtained from the basket B_t by edge completion, restoring the edge $e^* = v_t v$. The Hamilton circuit $((v_t v v_1 v_2 \ldots v_t))$ is a regular skeleton of W_t which includes the edge e^*. By Corollary 7.1 and Example 4 the proof is complete for $t = 4$ and all $t \geqslant 6$. For the case $t = 5$ we use the scheme in Figure 11.

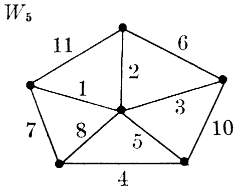


FIGURE 11.

If a graph is magic, it is sometimes amusing to seek an α, with each $\alpha(e)$ a positive integer, satisfying (M.1) and (M.3), and having a minimal value for $\sigma(\alpha)$. The examples given for B_4, B_6, and W_5 result from such a search.

A proper skeleton F of K_n, $n \geqslant 5$, will be called a *magic base* for K_n if every skeleton G of K_n which contains F is magic. We say that F, a magic base for K_n, is *irreducible* if each skeleton of F, obtained by deleting an edge, is not magic. The following Theorem 8 demonstrates the existence of magic bases, chosen for ease of description and irreducibility, but not unique.

THEOREM 8. *For $n = 5$ and all $n \geqslant 7$, the basket B_{n-1} is an irreducible magic base for K_n. The wheel W_5 is an irreducible magic base for K_6.*

Proof. (A) Example 4 shows that B_{n-1} is magic for $n = 5$ and all $n \geqslant 7$. Let H^* be a graph obtained from B_{n-1} by the addition of an edge e^*. We claim that H^* has a skeleton R^* which contains e^* and whose components are regular (see the cases listed below). By Corollary 7.1 the graph H^* is magic. If G is any skeleton of K_n which includes B_{n-1}, but does not include e^*, and if G^* is formed from G by adding e^*, then G^*, like H^*, has the same skeleton R^* containing e^*. By finite induction, we can add edge after edge and make repeated applications of Corollary 7.1 to conclude that any skeleton of K_n which contains B_{n-1} is magic. Hence B_{n-1} is a magic base for K_n for $n = 5$ and all $n \geqslant 7$.

We use the previously established notation in Example 4 in which B_{n-1} has the centre v, the circuit $((v_1 v_2 \ldots v_{n-1} v_1))$, and all the spokes vv_i except vv_{n-1}.

Case 1. $e^* = vv_{n-1}$, $R^* = ((vv_1 v_2 \ldots v_{n-2} v_{n-1} v))$.

Case 2. $e^* = v_{n-1} v_i$, $i \neq 1$, $i \neq n-2$,

$$R^* = ((v_i v_{i+1} \ldots v_{n-2} v_{n-1} v_i)) \oplus ((v_1 v_2 \ldots v_{i-1} v v_1)).$$

Case 3. $e^* = v_j v_i$, $1 < i < j - 1 < n - 3$,

$$R^* = ((v_i v_{i+1} \ldots v_{j-1} v_j v_i)) \oplus ((v_1 v_2 \ldots v_{i-1} v v_{j+1} \ldots v_{n-1} v_1)).$$

Case 4. $e^* = v_1 v_i$, $i \neq 2$, $i \neq n - 1$,

$$R^* = ((v_1 v_i v_{i+1} \ldots v_{n-2} v_{n-1} v_1)) \oplus ((v v_2 v_3 \ldots v_{i-2} v_{i-1} v)).$$

Case 5. $e^* = v_i v_{n-2}$, $i \neq n - 3$, $i \neq n - 1$, $i \neq 1$,

$$R^* = ((v_i v_{n-2} v_{n-1} v_1 v_2 \ldots v_{i-1} v_i)) \oplus ((v v_{i+1} v_{i+2} \ldots v_{n-3} v)).$$

(B) In the case $n = 6$, the proof is similar. In Example 5 it was shown that W_5 is magic. The description of an appropriate skeleton R^* is easier than in part (A), for one case suffices.

(C) The proof that these magic bases are irreducible can be accomplished by a series of algebraic arguments. For example, consider W_5. Deleting an outer edge produces F_4, which was shown to be not magic in Example 3. Deleting a spoke produces B_5 shown to be not magic in Figure 10.

Exercise 3. If n is odd, $n \geqslant 5$, show that the fan F_{n-2} is an irreducible magic base for K_n. (Comparison with Theorem 8 justifies the remark that K_n may have more than one irreducible magic base.)

Exercise 4. The graph G with two components K_2 and K_n, $n \geqslant 5$, is magic; but G^* is not magic. (Hence the edge-completion process does not always maintain the magic property.)

9. Super-magic graphs. We say that G is *super-magic* if and only if there exists an α in $S(G)$ which satisfies the condition

(M.5) *the set* $\{\alpha(e_i)\}$ *consists of consecutive positive integers.*

We can appeal to the literature (cf. **2**) concerning the magic squares of number theory and use the correspondence explained in Example 1 to assert that $K(n, n)$ is super-magic for all $n \geqslant 3$. Before discussing other examples we note some necessary conditions.

If the integers in (M.5) are indicated as $a, a + 1, \ldots, a + (E - 1)$, then the conditions (M.1) and (M.5) require

(9) $$n\sigma(\alpha) = 2\sum\alpha(e_i) = 2(Ea + E(E - 1)/2), \qquad \text{for at least one } a \geqslant 1.$$

The relation (9) imposes congruence conditions on the parameter a which must be satisfied if $\sigma(\alpha)$ is to be an integer.

Suppose ρ' is the maximum degree in G. The condition (M.1) at a vertex of degree ρ' requires

(10′) $$\rho' a + \rho'(\rho' - 1)/2 \leqslant \sigma(\alpha),$$

since the minimum sum of ρ' integers in the set defined by (M.5) must not exceed $\sigma(\alpha)$.

Suppose ρ'' is the minimum degree in G. The condition (M.1) at a vertex of degree ρ'' requires

(10″) $$\rho''(a + E - 1) - \rho''(\rho'' - 1)/2 \geqslant \sigma(\alpha),$$

since the maximum sum of ρ'' integers in the set defined by (M.5) must not be less than $\sigma(\alpha)$.

For example, consider the basket B_4. The relation (9) requires $5\sigma(\alpha) = 2(7a + 21)$; hence $a \equiv 2 \pmod 5$. The condition (10″) with $\rho'' = 2$ requires $13 \geqslant 4a$. Since $a \geqslant 1$, the only possibility is $a = 2$. By trial we find that B_4 is super-magic (see Figure 12) and in the sense of isomorphism there is only one solution.

For the wheel W_4 the relation (9) requires $5\sigma(\alpha) = 2(8a + 28)$; hence $a \equiv 4 \pmod 5$. The condition (10′) with $\rho' = 4$ requires $4a \leqslant 26$. The only possibility is $a = 4$. We find that W_4 is super-magic in essentially one way (see Figure 12). In similar fashion we find that W_5 is super-magic in essentially one way (see Figure 12).

In contrast, we can show that W_n is not super-magic for $n \geqslant 6$, for (9) and (10′) can be combined to require

$$(n + 1)(na + n(n - 1)/2) \leqslant 2(2na + 2n(2n - 1)/2),$$

which is equivalent to $n^3 - 8n^2 + 3n + 2n(n - 3)a \leqslant 0$. But using the condition $a \geqslant 1$, we find that

$$n^3 - 8n^2 + 3n + 2n(n - 3)a \geqslant n(n(n - 6) - 3) > 0$$

when $n \geqslant 7$. In the case $n = 6$, the congruence condition (9) requires $a \equiv 5 \pmod 7$, and with $a \geqslant 5$, the condition (10′) again provides a contradiction.

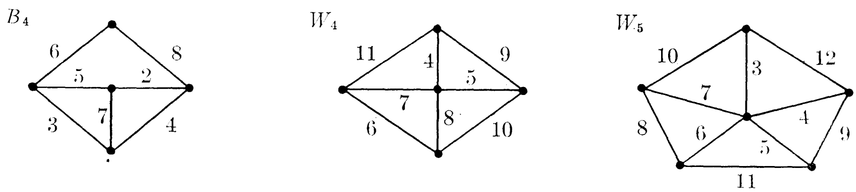


FIGURE 12.

If G is regular of degree ρ, the function β, with $\beta(e) = 1$ for every e in G, belongs to $S(G)$ with $\sigma(\beta) = \rho$. If α is in $S(G)$ and has property (M.5), then $\alpha + x\beta$ is in $S(G)$ and has property (M.5) with the corresponding set ranging from $a + x$ to $a + x + E - 1$, providing x is an integer and $a + x \geqslant 1$. Consequently, if G is regular, then G is either not super-magic, or is super-magic for *all* $a \geqslant 1$. Thus in using (9) we may assume $a = 1$ and test

$$n\sigma(\alpha) = E(E + 1). \tag{9'}$$

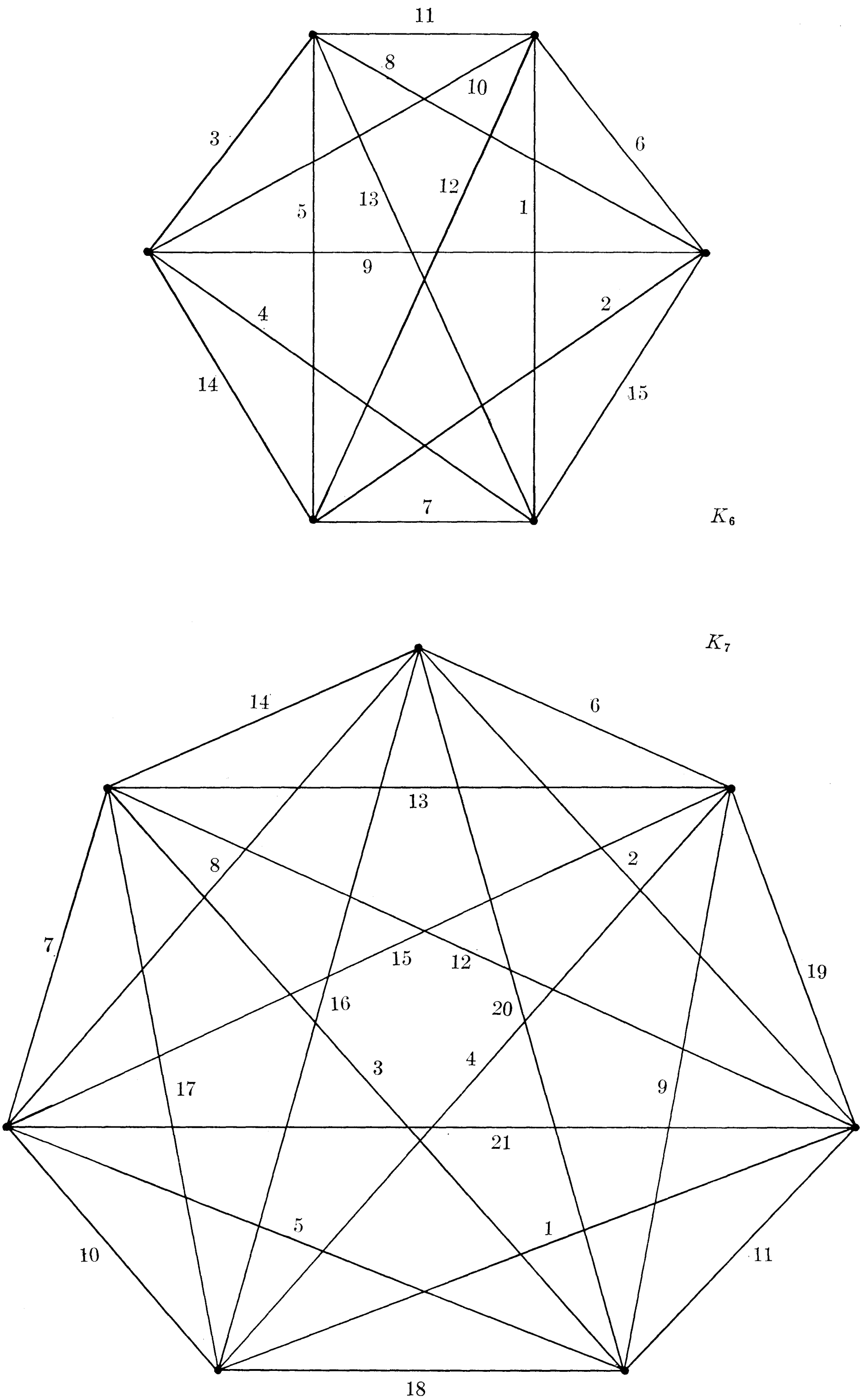
For example, for K_n we have $E = n(n - 1)/2$ and find that (9′) reduces to

$$4\sigma(\alpha) = (n - 1)(n^2 - n + 2); \tag{9''}$$

hence $\sigma(\alpha)$ is not an integer when $n \equiv 0 \pmod 4$.

We can show that K_5 is not super-magic, for there are only a few combinations to try, and none is successful.

In another paper **(3)** we show that K_n is super-magic when $n > 5$ and $n \not\equiv 0 \pmod 4$. Here we are content with showing solutions for K_6 and K_7 (see Figure 13).


11
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K_6
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2
7
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19
16
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17
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21
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1
10
11
18

FIGURE 13.

Exercise 5. For the graphs corresponding to the five Platonic solids, show that only the octahedron is super-magic.

10. Prime-magic graphs. We say that G is *prime-magic* if and only if there exists an α in $S(G)$ which satisfies the condition

(M.6) *the set* $\{\alpha(e_i)\}$ *consists of distinct positive rational integers which are primes.*

Except for the trivial K_2, the simplest prime-magic graphs are the fan F_3 and the complete graph K_5 (see Figure 14).

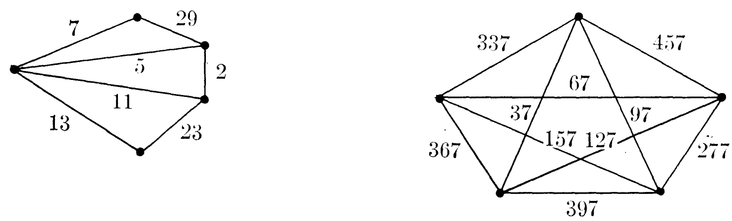

FIGURE 14.

There are infinitely many graphs that are magic, but not prime-magic. In Example 3 we showed that the fan F_{2K+1} is magic for $K \geqslant 1$; but we can show that F_{2K+1} is not prime-magic for $K \geqslant 2$. Since there is only one positive even prime, and F_{2K+1} has two vertices of degree 2, properties (M.1) and (M.6) require that $\sigma(\alpha)$ be even. But the sum of three odd primes is odd, and F_{2K+1}, for $K \geqslant 2$, has more than two vertices of degree 3; hence properties (M.1) and (M.6) require that $\sigma(\alpha)$ be odd, a contradiction. This answers, in the negative, a conjecture of Sedlacek that every magic graph is prime-magic.

Perhaps a qualified conjecture was intended—it may be that every *regular* magic graph is prime-magic. Sedlacek showed that the cube and octahedron are prime-magic. We can use the ideas in the following theorem to show that the icosahedron is prime-magic; but we have not yet been able to show that the dodecahedron is more than magic. The difficulty surrounding any general statement about prime-magic graphs is obvious—the irregular distribution of the primes.

For relatively prime positive integers a and b, define a set of integers:

$$\Pi(a, b) = \{k | k \geqslant 0, ak + b \text{ prime}\}.$$

According to Dirichlet, the set $\Pi(a, b)$ is infinite.

THEOREM 9. *If G is regular and if G is magic under an α whose values are in $\Pi(a, b)$, then G is prime-magic.*

Proof. Given G is magic under $\alpha(e)$. Let β be the function with $\beta(e) = 1$ for all e in G. Define α' in $A(G)$ by $\alpha' = a\alpha + b\beta$. Since G is regular, say of degree ρ,

$\sigma^{\alpha'}(v) = a\sigma^{\alpha}(v) + \rho b$ for every vertex of G. Since α satisfies (M.1), it follows that α' is in $S(G)$, with $\sigma(\alpha') = a\sigma(\alpha) + \rho b$. Since α satisfies (M.3) and since $a \neq 0$, if $e_i \neq e_j$, then

$$\alpha'(e_i) = a\alpha(e_i) + b \neq a\alpha(e_j) + b = \alpha'(e_j);$$

hence α' satisfies (M.3). Finally, by hypothesis each $\alpha(e)$ is in $\Pi(a, b)$; hence $\alpha'(e) = a\alpha(e) + b$ is a positive prime. Thus G is prime-magic.

For example, if we note that K_5 is magic under an α whose values 1, 2, 3, 4, 5, 9, 11, 12, 13, 15 are all in $\Pi(30, 7)$, then we can apply Theorem 9 to obtain the solution in Figure 14.

For a more elaborate example, consider the graph G of the truncated tetrahedron. By the method of Corollary 6.1 it is easy to find that G is magic under the function γ shown in the first part of Figure 15. Comparison with $\Pi(6, -1)$ shows that only values that need to be changed are 11 and 55. However, the simplest skeleton, with regular components involving these two edges, involves at least four other edges. We try the skeleton U, with associated $a\delta$, shown in the second part of the figure. To avoid integers of the type $x \equiv 1 \pmod 5$ which are not in $\Pi(6, -1)$ when $x > 1$, we find we must use $a \equiv 4 \pmod 5$. The choice $a = 29$ results in $\alpha = \gamma + 29\delta$ having all values in $\Pi(6, -1)$; and G is magic under α. Applying Theorem 9, we obtain the function $\alpha' = 6\alpha - \beta$ under which G is prime-magic.

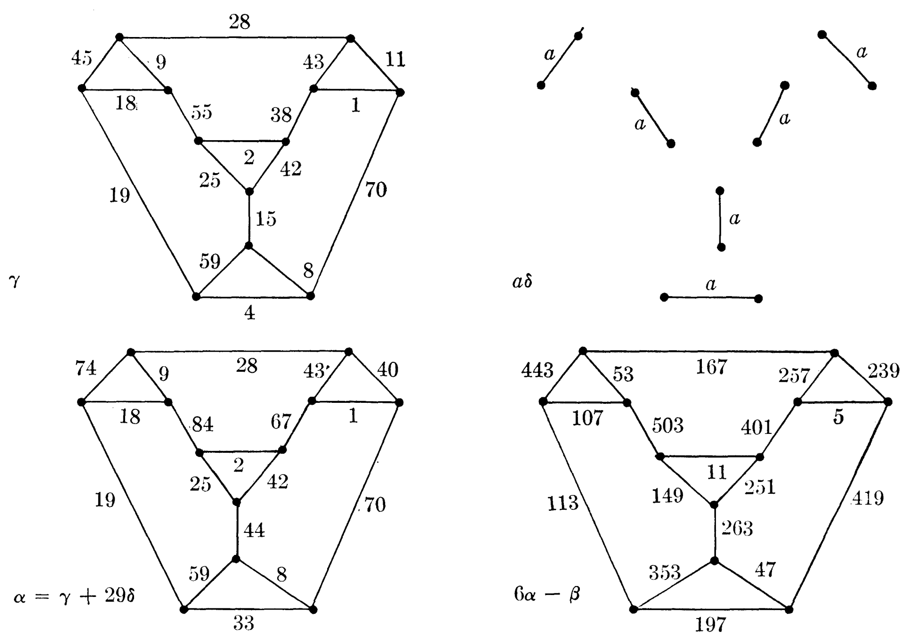

FIGURE 15.

It is possible for a graph to be both super-magic and prime-magic, witness $K(3, 3)$—shown in *dual* form below:

$$\begin{array}{ccc} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{array} \qquad\qquad \begin{array}{rrr} 13 & 73 & 53 \\ 107 & 29 & 3 \\ 19 & 37 & 83 \end{array}$$

It is claimed (**2**, p. 211) that $K(n, n)$ is prime-magic for $3 \leqslant n \leqslant 12$; however, the examples given used 1 as a prime, which does not seem quite fair to this writer.

For other combinations we note that B_4 and W_4 are super-magic, but not prime-magic; K_5 is prime-magic, but not super-magic; and W_{2K} is magic, but neither super-magic, nor prime-magic, for $K \geqslant 3$.

Summary. For each undirected, finite graph G, without multiple edges, without loops, and without isolated vertices, we have defined, over the real field, vector spaces $A(G)$, $S(G)$, $Z(G)$, ζ which in the sense of inclusion stand in the relation

$$A(G) \supseteqq S(G) \supseteqq Z(G) \supseteqq \zeta.$$

(The first equality is required only in the case K_2 where $E = 1$.) Concerning dimension we have shown that $\dim A(G) = E$,

$$\dim Z(G) \leqslant \dim S(G) \leqslant 1 + \dim Z(G),$$

and

$$\max\,(0, D(G) + d) \leqslant \dim S(G) \leqslant C(G) + d.$$

We have classified the graphs of admissible type as being trivially-magic, zero-magic, semi-magic, pseudo-magic, magic, super-magic, and prime-magic where the categories are related in the sense of inclusion as shown in Figure 16.

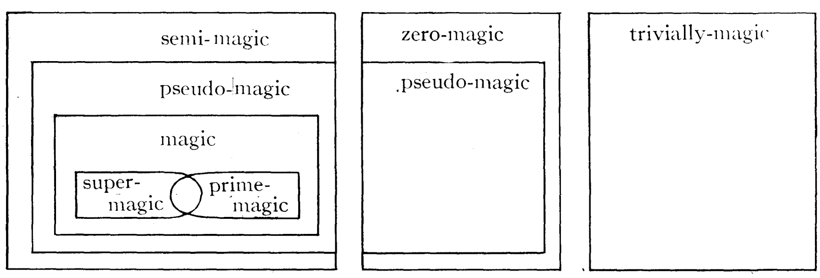

FIGURE 16.

Except possibly for the prime-magic case, the classes are infinite as the following examples show:
trivially-magic: stars $K(1, r)$, $r \geqslant 2$;

zero-magic, not pseudo-magic: kites with tailing arcs of odd length Q_{2t+1}, $t \geqslant 0$;
zero-magic and pseudo-magic: complete bipartite $K(2, t)$, $t \geqslant 3$;
semi-magic, not pseudo-magic: arcs of odd length C_{2t+1}, $t \geqslant 1$;
semi-magic and pseudo-magic, not magic: fans F_{2t}, $t \geqslant 2$;
magic, not super-magic, not prime-magic: wheels W_{2t}, $t \geqslant 3$;
super-magic: complete bipartite $K(n, n)$, $n \geqslant 3$; complete K_n, $n > 5$, $n \not\equiv 0 \pmod 4$.

For the prime-magic case, the classes are at least non-empty:

super-magic, not prime-magic: basket B_4, wheel W_4;
super-magic and prime-magic: $K(3, 3)$, $K(4, 4)$, octahedron;
prime-magic, not super-magic: fan F_3, K_5.

Added in proof.

COROLLARY 2.4. *If G is a connected graph containing a circuit of odd order, then* $\dim S(G) = C(G)$.

Proof. Suppose the connected graph G contains a circuit H of odd order. Either G itself contains only the one circuit H, or G can be obtained by edge completion from a proper subgraph U containing only the one circuit H. Because H is of odd order an algebraic argument using Lemma 2 shows that $\dim S(U) = 1$. Since $C(U) = 1$, Lemma 1 shows that $\dim S(G) = C(G)$.

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Michigan State University