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MOD-C POSTNIKOV APPROXIMATION OF A 1-CONNECTED SPACE

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Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context [**4**]. They have also shown that if the set of morphisms is saturated then the Adams completion of an object is characterized by a certain couniversal property. We want to prove a stronger version of this result by dropping the saturation assumption on the set of morphisms; we also prove that the canonical map from an object to its Adams completion is an element of the set of morphisms under very moderate assumptions. These two results are fairly general in nature and are applicable to most cases of interest. Further using these two results and introducing "modulo a Serre class **C** of abelian groups" [**9**] we have obtained the mod-**C** Postnikov approximation of a 1-connected based CW-complex, with the help of a suitable set of morphisms.

1. Adams completion. Let $\mathbf{C}$ be a category and S a set of morphisms of $\mathbf{C}$. Let $\mathbf{C}[S^{-1}]$ denote the category of fractions of $\mathbf{C}$ with respect to S and $F:\mathbf{C} \to \mathbf{C}[S^{-1}]$ the canonical functor. Let $\mathbf{S}$ denote the category of sets and functions. Then for a given object Y of $\mathbf{C}$,

$$\mathbf{C}[S^{-1}](-, Y):\mathbf{C} \to \mathbf{S}$$

defines a contravariant functor. If this functor is representable by an object Y_S of $\mathbf{C}$, that is,

$$\mathbf{C}[S^{-1}](-, Y) \cong \mathbf{C}(-, Y_S)$$

then Y_S is called the (generalized) Adams completion of Y with respect to the set of morphisms S or simply the S-completion of Y. We shall often refer to Y_S as the completion of Y.

We now state Deleanu's theorem [**6**] that under certain conditions, the global Adams completion always exists.

1.1. THEOREM. *Let* $\mathbf{C}$ *be a cocomplete small* $\mathbf{U}$*-category* (*where* $\mathbf{U}$ *is a fixed Grothendieck universe*) *and* S *a set of morphisms of* $\mathbf{C}$ *that admits a calculus of left fractions. Suppose that the following compatibility condition with coproducts is satisfied*:

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(P) *If each* $s_i:X_i \to Y_i$, $i \in I$, *is an element of* S, *where the index set* I *is an element of* $\mathbf{U}$, *then*

$$\coprod_{i\in I} s_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$$

is an element of S.

Then every object of $\mathbf{C}$ *has an Adams completion with respect to the set of morphisms* S.

Given a set S of morphisms of $\mathbf{C}$, we define $\bar{S}$, the saturation of S, as the set of all morphisms u in $\mathbf{C}$ such that $F(u)$ is an isomorphism in $\mathbf{C}[S^{-1}]$. S is said to be saturated if $S = \bar{S}$.

Deleanu, Frei and Hilton have shown that if the set of morphisms S is saturated then the Adams completion of a space is characterized by a certain couniversal property ([**4**], Theorem 1.2). In most applications, however, the set of morphisms S is not saturated. We therefore present a stronger version of Deleanu, Frei and Hilton's characterization of Adams completion in terms of a couniversal property.

1.2. THEOREM. *Let* S *be a set of morphisms of* $\mathbf{C}$ *admitting a calculus of left fractions. Then an object* Y_S *of* $\mathbf{C}$ *is the* S*-completion of the object* Y *with respect to* S *if and only if there exists a morphism* $e:Y \to Y_S$ *in* $\bar{S}$ *which is couniversal with respect to morphisms in* S: *given a morphism* $s:Y \to Z$ *in* S *there exists a unique morphism* $t:Z \to Y_S$ *in* $\bar{S}$ *such that* $ts = e$. *In other words the following diagram is commutative*:

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dotted, "t"'] & \end{tikzcd}$$

Proof. Suppose that Y_S is the S-completion of Y with respect to S. Then we have a natural equivalence of functors

$$\mathbf{C}[S^{-1}](-, Y) \underset{\cong}{\overset{\tau}{\to}} \mathbf{C}(-, Y_S).$$

Set $\tau(1_Y) = e:Y \to Y_S$. First we will show that $e \in \bar{S}$, that is, $F(e)$ is an isomorphism in $\mathbf{C}[S^{-1}]$. Consider the following commutative diagram

$$\begin{tikzcd} \mathbf{C}[S^{-1}](Y, Y) \arrow[r, "\tau", "\cong"'] & \mathbf{C}(Y, Y_S) \\ \mathbf{C}[S^{-1}](Y_S, Y) \arrow[r, "\tau", "\cong"'] \arrow[u, "e^+"] & \mathbf{C}(Y_S, Y_S) \arrow[u, "e^*"'] \end{tikzcd}$$

where e^{+} is defined by $e^{+}(\alpha) = \alpha \circ F(e)$ for any $\alpha: Y_S \to Y$ in $\mathbf{C}[S^{-1}]$ and e^* is defined by $e^*(f) = f \circ e$ for any morphism $f: Y \to Y_S$ in $\mathbf{C}$. Let $\theta: Y_S \to Y$ in $\mathbf{C}[S^{-1}]$ be such that $\tau(\theta) = 1_{Y_S}$; thus

$$e^*\tau(\theta) = e = \tau e^{+}(\theta).$$

But $\tau(1_Y) = e$; therefore

$$e^{+}(\theta) = \theta \circ F(e) = 1_Y,$$

showing that $F(e)$ has a left inverse θ. To show that θ is also the right inverse for $F(e)$, we proceed as follows. Since S admits a calculus of left fractions, we write

$$\theta = [g, s] = F(s)^{-1}F(g)$$

with $s \in S$ and we express θ as follows

$$\begin{tikzcd}
Y_S \arrow[dr, "g"'] & U \arrow[d, "1_U"] \arrow[dr, "1_U"] & Y \arrow[d, "s"] \\
 & U \arrow[dr, "1_U"'] & U \arrow[d, "1_U"] \\
 & & U
\end{tikzcd}$$

We then have two commutative squares

$$\begin{tikzcd}
\mathbf{C}[S^{-1}](Y, Y) \arrow[r, "\tau", "\cong"'] & \mathbf{C}(Y, Y_S) \\
\mathbf{C}[S^{-1}](U, Y) \arrow[r, "\tau", "\cong"'] \arrow[u, "s^{+}"] \arrow[d, "g^{+}"'] & \mathbf{C}(U, Y_S) \arrow[u, "s^*"'] \arrow[d, "g^*"] \\
\mathbf{C}[S^{-1}](Y_S, Y) \arrow[r, "\tau", "\cong"'] & \mathbf{C}(Y_S, Y_S)
\end{tikzcd}$$

Note that $F(s)$ is an isomorphism; hence s^{+} is bijective. Let

$$\beta: U \to Y$$

be a morphism in $\mathbf{C}[S^{-1}]$ such that

$$s^{+}(\beta) = \beta \circ F(s) = 1_Y;$$

thus $\beta = F(s)^{-1}$. With this definition of β, we have

$$g^{+}(\beta) = \beta \circ F(g) = F(s)^{-1}F(g) = \theta.$$

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From the diagram

$$\begin{tikzcd} 1_Y \arrow[r, "\tau"] & e \\ \beta \arrow[u, "s^+"] \arrow[r, "\tau"] \arrow[d, "g^+"'] & h \arrow[u, "s^*"'] \arrow[d, "g^*"] \\ \theta \arrow[r, "\tau"] & 1_{Y_S} \end{tikzcd}$$

it follows that

(i) $s^*(h) = hs = e$ and

(ii) $g^*(h) = hg = 1_{Y_S}$.

Thus

$$F(e) \circ \theta = F(e)F(s)^{-1}F(g) = F(hs)F(s)^{-1}F(g)$$
$$= F(h)F(s)F(s)^{-1}F(g) = F(hg) = 1_{Y_S}.$$

We have thus shown that $e \in \bar{S}$. To show that e has couniversal property with respect to morphisms in S, let $s:Y \to Z$ be in S. Consider the following commutative diagram

$$\begin{tikzcd} \mathbf{C}[S^{-1}](Y, Y) \arrow[r, "\tau", "\cong"'] & \mathbf{C}(Y, Y_S) \\ \mathbf{C}[S^{-1}](Z, Y) \arrow[u, "s^+", "\cong"'] \arrow[r, "\tau", "\cong"'] & \mathbf{C}(Z, Y_S) \arrow[u, "s^*"'] \end{tikzcd}$$

Since s^+ is bijective, it follows that s^* is bijective. Thus there is a unique $t:Z \to Y_S$ such that

$s^*(t) = ts = e$

and we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dotted, "t"'] & \end{tikzcd}$$

Moreover, since $e \in \bar{S}$ and $s \in S \subset \bar{S}$, it follows that $t \in \bar{S}$. This completes the proof of the 'only if' part.

For the 'if' part, suppose that there is $e:Y \to Y_S$ in $\bar{S}$ having couniversal property with respect to morphisms in S. We show, first of all, that e has couniversal property with respect to morphisms in $\bar{S}$. Let $t:Y \to Z$ be a morphism in $\bar{S}$. Since $F(t) = [t, 1_Z]$ is an isomorphism, it has an inverse $[f, s]$ as shown in the diagram below, with $s \in S$.

$$\begin{tikzcd}
Y \arrow[dr, "t"'] & Z \arrow[d, "1_Z"] \arrow[dr, "f"] & Y \arrow[d, "s"] \\
& Z \arrow[dr, "f"'] & X \arrow[d, "1_X"] \\
& & X
\end{tikzcd}$$

Thus $[f \circ t, 1_X \circ s] = [1_Y, 1_Y]$, so we have a diagram

$$\begin{tikzcd}
& & Y \arrow[drr, "u"] & & \\
Y \arrow[urr, "1_Y"] \arrow[dr, "t"'] & Z \arrow[d, "1_Z"] \arrow[dr, "f"] & Y \arrow[u, "1_Y"'] \arrow[d, "s"] & & K \\
& Z \arrow[dr, "f"'] & X \arrow[d, "1_X"] & & \\
& & X \arrow[uurr, "v"'] & &
\end{tikzcd}$$

with $u = vs \in S$ and $vft = u$. Since $vft: Y \to K$ is in S and $e: Y \to Y_S$ has couniversal property, we have a unique morphism $w: K \to Y_S$ in $\bar{S}$ making the diagram

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "t"'] & Y_S \\
Z \arrow[d, "f"'] \arrow[ur, dotted, "wvf"'] & \\
X \arrow[d, "v"'] & \\
K \arrow[uuur, dotted, "w"'] &
\end{tikzcd}$$

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commutative; $wvf:Z \to Y_S$ is now the required morphism through which e factors. Now since $e \in \bar{S}$

$$e_+:\mathbf{C}[S^{-1}](-, Y) \to \mathbf{C}[S^{-1}](-, Y_S)$$

defined by $e_+(\theta) = F(e) \circ \theta$, defines an equivalence of functors. It will now be enough to show that the canonical functor F induces an equivalence

$$F_*:\mathbf{C}(-, Y_S) \to \mathbf{C}[S^{-1}](-, Y_S).$$

To show that F_* is surjective, let $\alpha:X \to Y_S$ be in $\mathbf{C}[S^{-1}]$. Then

$$\alpha = F(s)^{-1}F(g),$$

that is, α can be represented by

$$X \xrightarrow{g} Z \xleftarrow{s} Y_S,$$

with $s \in S$. The composite

$$Y \xrightarrow{e} Y_S \xrightarrow{s} Z$$

is easily seen to be in $\bar{S}$; by the couniversal property of e with respect to morphisms in $\bar{S}$ we have a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "se"'] & Y_S \\
Z \arrow[ur, dotted, "k"'] &
\end{tikzcd}$$

so that $kse = e$, with $k \in \bar{S}$, implying that $ks = 1_{Y_S}$. Hence

$$F(s)F(k) = 1_{Y_S}.$$

For the morphism $kg:X \to Y_S$ in $\mathbf{C}$, we have

$$F_*(kg) = F(k)F(g) = F(s)^{-1}F(g) = \alpha.$$

F_* is therefore surjective. To prove the injectivity of $F*$, let $f_1, f_2:X \to Y_S$ in $\mathbf{C}$ be such that $F_*(f_1) = F_*(f_2)$, so

$$[f_1, 1_{Y_S}] = [f_2, 1_{Y_S}].$$

This means we have a commutative diagram

$$\begin{tikzcd}
 & Y_S \arrow[dr, "u"] & \\
X \arrow[ur, "f_2"] \arrow[dr, "f_1"'] & Y_S \arrow[u, "1_{Y_S}"'] \arrow[d, "1_{Y_S}"] & L \\
 & Y_S \arrow[ur, "u"'] &
\end{tikzcd}$$

with $u \in S$ and $uf_1 = uf_2$. We also have a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "ue"'] & Y_S \\
L \arrow[ur, "t"'] &
\end{tikzcd}$$

Thus $tu = 1_{Y_S}$. We then have

$$f_1 = tuf_1 = tuf_2 = f_2.$$

This completes the proof of the theorem.

For most applications we have in mind we would like $e: Y \to Y_S$ to be in S. This is the case when S is saturated. However, in many cases of practical interest S is not saturated. Keeping in view the applications, we impose extra conditions on S which guarantees that $e \in S$.

1.3. THEOREM. *Let $S = S_1 \cap S_2$ be a set of morphisms of* $\mathbf{C}$ *admitting a calculus of left fractions. Let $e: Y \to Y_S$ be the canonical morphism as defined in Theorem* 1.2, *where Y_S is the S-completion of Y. Assume furthermore that S_1 and S_2 have the following properties*:

(i) *S_1, S_2 are closed under composition.*
(ii) *$fg \in S_1$ implies that $g \in S_1$.*
(iii) *$fg \in S_2$ implies that $f \in S_2$.*

Then $e \in S$.

Proof. Since $F(e)$ is an isomorphism in $\mathbf{C}[S^{-1}]$, assume that $[h, s]$, with $s \in S$, is the inverse of $F(e) = [e, 1_{Y_S}]$. We therefore have a diagram

$$\begin{tikzcd}
 & & Y \arrow[dr, "u"] & \\
Y \arrow[urr, "1_Y"] \arrow[dr, "e"'] & Y_S \arrow[d, "1_{Y_S}"] \arrow[dr, "h"] & Y \arrow[u, "1_Y"'] \arrow[d, "s"] & K \\
 & Y_S \arrow[dr, "h"'] & X \arrow[d, "1_X"] & \\
 & & X \arrow[uur, "v"'] &
\end{tikzcd}$$

with $u = vs \in S$ and $u = vhe$. It follows from condition (ii) that $e \in S_1$. Moreover, the couniversal property of e implies that we have a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "u"'] & Y_S \\
K \arrow[ur, "w"'] &
\end{tikzcd}$$

So $e = wu = wvhe$ implying that

$$wvh = 1_{Y_S} \in S \subset S_2.$$

Condition (iii) implies that $w \in S_2$. Therefore $e = wu \in S_2$ (because S_2 is closed under composition). Thus

$$e \in S_1 \cap S_2 = S.$$

This completes the proof of the theorem.

Both Theorems 1.2 and 1.3 can be dualised easily.

2. Modulo a Serre class C of abelian groups. Now we introduce "modulo a Serre class **C** of abelian groups" [**9**] to obtain the mod-**C** Postnikov approximation of a 1-connected based CW-complex, with the help of a suitable set of morphisms. From now onwards, we assume that **C** is a Serre class which is moreover an acyclic ideal of abelian groups [**9**].

Let **CW** denote the category of 1-connected based CW-complexes and based maps, and $\widetilde{\mathbf{CW}}$ the corresponding homotopy category. We assume that the underlying sets of the elements of $\widetilde{\mathbf{CW}}$ are elements of **U** where **U** is a fixed Grothendieck universe. We now fix suitable sets of morphisms in $\widetilde{\mathbf{CW}}$.

A map $\alpha: X \to Y$ in $\widetilde{\mathbf{CW}}$ is called a mod-$\mathbf{C}$ $(n + 1)$-equivalence if

$$\alpha_*: \pi_m(X) \to \pi_m(Y)$$

is a $\mathbf{C}$-isomorphism for $m \leqq n$ and a $\mathbf{C}$-epimorphism for $m = n + 1$.

Let S_n denote the set of all mod-$\mathbf{C}$ $(n + 1)$-equivalences in $\widetilde{\mathbf{CW}}$.

2.1. PROPOSITION. *S_n admits a calculus of left fractions.*

Proof. It is enough to prove that every diagram

$$Y \xleftarrow{\alpha} X \xrightarrow{\gamma} Z$$

in $\widetilde{\mathbf{CW}}$ ([**8**], Proposition I.1.8) with $\gamma \in S_n$, can be embedded in a weak push-out diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\delta"'] & W \end{tikzcd}$$

with $\delta \in S_n$. Suppose $\alpha = [f]$ and $\gamma = [s]$. Let $i_f: X \to M_f$ be the usual inclusion of X into M_f, the reduced mapping cylinder of f, and i_f is defined by $i_f(x) = [0, x]$. Let

$$j: Y \to M_f \quad \text{and} \quad r: M_f \to Y$$

be the maps defined by

$$j(y) = [y], \quad r([y]) = y, \quad r([s, x]) = f(x),$$

such that

$$r \circ j = 1_Y, \quad j \circ r \simeq 1_{M_f} \quad \text{and} \quad r \circ i_f = f.$$

Now we consider the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i_f"'] & Z \\ M_f & \end{tikzcd}$$

and form its push-out in $\mathbf{CW}$.

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "i_f"'] & Z \arrow[d, "u"] \\ M_f \arrow[r, dashed, "t"'] & W \end{tikzcd}$$

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Since i_f is a cofibration, so is u; we therefore have the following diagram

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[d, "s"'] & M_f \arrow[r, "p"] \arrow[d, "t"] & C \arrow[d, equal] \\ Z \arrow[r, "u"'] & W \arrow[r, "q"'] & C \end{tikzcd}$$

where C is the cokernel of i_f, as well as of u; p and q are the usual projections. We consider the exact homology sequences

$$\begin{tikzcd} \dots \arrow[r] & H_{m+1}(C) \arrow[r] \arrow[d, equal] & H_m(X) \arrow[r] \arrow[d, "s_*"] & H_m(M_f) \arrow[r] \arrow[d, "t_*"] & {} \\ \dots \arrow[r] & H_{m+1}(C) \arrow[r] & H_m(Z) \arrow[r] & H_m(W) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} H_m(C) \arrow[r] \arrow[d, equal] & H_{m-1}(X) \arrow[r] \arrow[d, "s_*"] & \dots \\ H_m(C) \arrow[r] & H_{m-1}(Z) \arrow[r] & \dots \end{tikzcd}$$

where H_* denotes the singular homology functor. Since

$$s_*:\pi_m(X) \to \pi_m(Z)$$

is a **C**-isomorphism for $m \leqq n$ and a **C**-epimorphism for $m = n + 1$, it follows from Theorem 9.6.22 [**9**] that

$$s_*:H_m(X) \to H_m(Z)$$

is a **C**-isomorphism for $m \leqq n$ and a **C**-epimorphism for $m = n + 1$. The mod-**C** Five lemma then implies that

$$t_*:H_m(M_f) \to H_m(W)$$

is a **C**-isomorphism for $m \leqq n$ and a **C**-epimorphism for $m = n + 1$. Hence

$$t_*:\pi_m(M_f) \to \pi_m(W)$$

is a **C**-isomorphism for $m \leqq n$ and a **C**-epimorphism for $m = n + 1$. Let $\beta = [u]$ and $\delta = [tj]$. Since j is a homotopy equivalence, j_* is an isomorphism of the corresponding homotopy groups; thus $\delta \in S_n$. We consider the following diagram in **CW**:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Z \arrow[d, "u"] \\ Y \arrow[r, "tj"'] & W \end{tikzcd}$$

We have

$$tjf = tjri_f \simeq t1_{M_f}i_f = ti_f = us.$$

Taking the corresponding homotopy classes, we have a diagram in $\widetilde{\mathbf{CW}}$

$$\begin{tikzcd} X \arrow[r, "\gamma = {[s]}"] \arrow[d, "\alpha = {[f]}"'] & Z \arrow[d, "\beta = {[u]}"] \\ Y \arrow[r, "\delta = {[tj]}"'] & W \end{tikzcd}$$

with $\delta \in S_n$. This indeed is a weak push-out diagram in $\widetilde{\mathbf{CW}}$. This completes the proof of the proposition.

2.2. PROPOSITION. *Let* $\{s_i:X_i \to Y_i, i \in I\}$ *be a subset of* S_n; *then*

$$\bigvee_{i \in I} s_i: \bigvee_{i \in I} X_i \to \bigvee_{i \in I} Y_i$$

is an element of S_n, *where the index set* I *is in* $\mathbf{U}$.

Proof. We consider the commutative diagram

$$\begin{tikzcd}[column sep=huge, row sep=large] \bigoplus_{i \in I} H_*(X_i) \arrow[r, "\{\alpha_{i_*}\}", "\cong"'] \arrow[d, "\bigoplus_{i \in I} s_{i_*}"', "\cong"] & H_*\left(\bigvee_{i \in I} X_i\right) \arrow[d, "\left(\bigvee_{i \in I} s_i\right)_*"] \\ \bigoplus_{i \in I} H_*(Y_i) \arrow[r, "\cong", "\{\beta_{i_*}\}"'] & H_*\left(\bigvee_{i \in I} Y_i\right) \end{tikzcd}$$

where

$$\alpha_i:X_i \to \bigvee_{i \in I} X_i \quad \text{and} \quad \beta_i: Y_i \to \bigvee_{i \in I} Y_i$$

are the canonical inclusions. Note that each horizontal row is an isomorphism, hence a $\mathbf{C}$-isomorphism. Moreover, since each s_{i_*} is a $\mathbf{C}$-isomorphism in dimension $\leqq n$ and a $\mathbf{C}$-epimorphism in dimension $n + 1$, so is $\bigoplus_{i \in I} s_{i_*}$, and from the commutativity of the diagram it follows that $(\vee_{i \in I} s_i)_*$ is also a **C**-isomorphism in dimension $\leqq n$ and a **C**-epimorphism in dimension $n + 1$. Thus

$$\bigvee_{i \in I} s_i \in S_n.$$

Let **U** be a fixed Grothendieck universe such that the category of CW-complexes and homotopy classes of maps between them is a **U**-category. Since S^1 can be given the structure of a CW-complex, $[S^1, S^1] \simeq \mathbf{Z}$ is an element of **U**, and it follows from the axioms of a Grothendieck universe that $\mathbf{Z}^+$, the set of positive integers, is also an element of **U**. We shall use this fact in proving the following proposition.

2.3. PROPOSITION. *For a given object X of the category $\widetilde{\mathbf{CW}}$ there exists a subset S_X of the set $\{s:X \to X' | s \in S_n\}$ such that S_X is an element of the universe* **U** *and for each $s:X \to X'$, $s \in S_n$, there exist an $s' \in S_X$ and a morphism u of $\widetilde{\mathbf{CW}}$ rendering the following diagram commutative*:

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ X' \arrow[r, "u"'] & X'' \end{tikzcd}$$

Proof. Given X in $\widetilde{\mathbf{CW}}$, we let

$$S_X = \{s:X \to Y | (Y, X) \text{ is a relative } CW\text{-complex with cells in dim} \geqq n + 2\}.$$

Clearly $S_X \subset S_n$. Moreover, if $s:X \to Y$ is in S_n, then we can find a CW-complex Z such that (i) (Z, X) has cells in dim $\geqq n + 2$, (ii) there is a map $u:Z \to Y$ which is a mod-**C** homotopy equivalence and which extends s. If v denotes the mod-**C** homotopy inverse of u and $s':X \to Z$ the usual inclusion, then the following diagram is easily seen to be homotopy commutative:

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

It will now be enough to prove that $S_X \in \mathbf{U}$. We write

$$A_k = \{s:X \to Y | (Y, X) \text{ has cells } e^m \text{ such that } n + 2 \leqq \dim(e^m) \leqq n + k + 1\}$$

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so that we have $S_X = \cup A_k$, k varying over the positive integers. We use induction to show that, for every $k \geqq 1$, $A_k \in \mathbf{U}$. For $k = 1$, we have

$$A_1 = \{s:X \to Y | (Y, X) \text{ is a relative } CW\text{-complex having cells in dim } n + 2 \text{ only}\}.$$

Therefore, Y must be of the form

$$Y = X \underset{a_i}{\cup} e_i^{n+2}$$

where $a_i:S^{n+1} \to X$ and $i \in I$ for some index set I. It is also evident that every family

$$\{a_i:S^{n+1} \to X\} \subset [S^{n+1}, X]$$

determines a space Y such that (Y, X) is a relative CW-complex with cells in dim $n + 2$ only. Thus,

$$A_1 \approx P[S^{n+1}, X],$$

where P denotes the power set. Since $[S^{n+1}, X] \in \mathbf{U}$, it follows from the axioms of a Grothendieck universe (see [**3**], p. 10) that

$$P[S^{n+1}, X] \in \mathbf{U};$$

thus $A_1 \in \mathbf{U}$.

We now assume inductively that $A_k \in \mathbf{U}$. To show that $A_{k+1} \in \mathbf{U}$, let $s:X \to Y$ be in A_k, i.e., (Y, X) is a relative CW-complex having cells e^m such that

$$n + 2 \leqq \dim(e^m) \leqq n + k + 1.$$

Let $\{a_i\}_{i \in I}$ be a family of maps with

$$a_i:S^{n+k+1} \to Y$$

for some index set I. It is then clear that the inclusion

$$X \hookrightarrow Y \underset{a_i}{\cup} e_i^{n+k+2}$$

is in A_{k+1}. Moreover, every map $s:X \to Z$ of A_{k+1} arises in this way. Therefore,

$$A_{k+1} = \underset{Y}{\cup} P[S^{n+k+1}, Y]$$

where the union is taken over all Y such that $s:X \to Y$ is in A_k. Since $A_k \in \mathbf{U}$ and $P[S^{n+k}, Y] \in \mathbf{U}$, we have $A_{k+1} \in \mathbf{U}$. Similarly, since the set of positive integers is an element of the universe $\mathbf{U}$, so is the union $\cup A_k = S_X$. This completes the proof of Proposition 2.3.

Since the category $\widetilde{\mathbf{CW}}$ as stated above is neither cocomplete nor small, Theorem 1.1 can not be used to show the existence of Adams completion of an object in the category $\widetilde{\mathbf{CW}}$ with respect to the set of morphisms S_n. However, we have the following result (Theorem 2.4) which is essentially Theorem 4.7 [**2**] and Theorem 3.8 [**1**] (it is also a generalization of the Theorem in [**5**]).

2.4. THEOREM. *Let* $\mathbf{U}$ *be a fixed Grothendieck universe. Let* $\widetilde{\mathbf{C}}$ *be the category defined as follows: the objects of* $\widetilde{\mathbf{C}}$ *are the based CW-complexes whose underlying sets are elements of* $\mathbf{U}$; *the morphisms of* $\widetilde{\mathbf{C}}$ *are based homotopy classes of base-point preserving maps between such CW-complexes. Let S be a family of morphisms of* $\widetilde{\mathbf{C}}$ *admitting a calculus of left fractions and satisfying the following axiom of compatibility with coproducts*:

(A) *If* $s_i:X_i \to Y_i$ *lies in S for each* $i \in I$, *where the index set I is an element of* $\mathbf{U}$, *then*

$$\bigvee_{i\in I} s_i:\bigvee_{i\in I} X_i \to \bigvee_{i\in I} Y_i$$

lies in S.

Assume that the family S and the object X of $\widetilde{\mathbf{C}}$ *satisfy the condition*:

(*) *There exists a subset* S_X *of the set* $\{s:X \to X' | s \in S\}$ *such that* S_X *is an element of the universe* $\mathbf{U}$ *and for each* $s:X \to X'$, $s \in S$, *there exist an* $s' \in S_X$ *and a morphism u of* $\widetilde{\mathbf{C}}$ *rendering the following diagram commutative.*

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ X' \arrow[r, "u"'] & X'' \end{tikzcd}$$

Then the Adams completion X_S *of X does exist.*

As remarked by Adams on page 34 of [**1**], this result remains valid if $\widetilde{\mathbf{C}}$ is the homotopy category of 1-connected CW-complexes (whose underlying sets belong to $\mathbf{U}$).

It is to be emphasized that condition (*) is essential in order to be able to apply E. H. Brown's representability theorem to prove this result.

Hence from Propositions 2.1, 2.2 and 2.3 it follows that the conditions of Theorem 2.4 are satisfied and so by Theorem 1.2 we obtain the following theorem.

2.5. THEOREM. *Every object X of the category* $\widetilde{\mathbf{CW}}$ *has an Adams completion* X_{S_n} *with respect to the set of morphisms* S_n *and there exists a morphism* $e_n:X \to X_{S_n}$ *in* $\bar{S}_n$ *which is couniversal with respect to morphisms in* S_n.

2.6. PROPOSITION. *The morphism* $e_n:X \to X_{S_n}$, *as constructed in Theorem* 2.5, *is in* S_n.

Proof. Let S_n^1 be the set of all morphisms $f:X \to Y$ in the category $\widetilde{\mathbf{CW}}$ such that

$$f_*:\pi_m(X) \to \pi_m(Y)$$

is a **C**-monomorphism for $m \leqq n$ and S_n^2 be the set of all morphisms $f:X \to Y$ in $\widetilde{\mathbf{CW}}$ such that

$$f_*:\pi_m(X) \to \pi_m(Y)$$

is a **C**-epimorphism for $m \leqq n + 1$. Clearly (i) $S_n = S_n^1 \cap S_n^2$, (ii) S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.3; hence $e_n \in S_n$.

3. A mod-C Postnikov approximation. Now we can obtain a tower for the mod-**C** Postnikov approximation of a 1-connected CW-complex, with the help of the sets of morphisms S_n. In the process, starting from a 1-connected based CW-complex X we get a tower of spaces, and the inverse limit of this tower gives us a space which in some sense is the mod-**C** Postnikov approximation of X. When $\mathbf{C} = \{0\}$, we of course get the usual Postnikov sections of X as different stages of the tower. In case $\mathbf{C} = \mathbf{C}_{P'}$, that is, the set of P'-torsion abelian groups, the inverse limit gives us the P-localization. When $P = \{p\}$, X is finitely generated and $\mathbf{C} = \mathbf{C}_{p'}$, the inverse limit gives us the p-profinite completion of X. It is hoped that this analysis of the completion will be helpful in other constructions (currently under investigations).

3.1. THEOREM. *Let X be a 1-connected based CW-complex. Then for $n \geqq 1$, there exist 1-connected based CW-complexes X_n, maps $e_n:X \to X_n$ and fibrations $p_{n+1}:X_{n+1} \to X_n$ such that*

(a) $e_{n_*}:\pi_m(X) \to \pi_m(X_n)$ *is a* **C**-*isomorphism for* $m \leqq n$ *and*

$$\pi_m(X_n) = 0 \quad \textit{for } m > n,$$

(b) $e_n = p_{n+1} \circ e_{n+1}$.

Proof. For each integer $n \geqq 1$, let X_n be the S_n-completion of X and $e_n:X \to X_n$ be the canonical map as stated in Proposition 2.6. Since $e_{n+1} \in S_{n+1}$, it follows that $e_{n+1} \in S_n$; hence by the couniversal property of e_n (Theorem 1.2), we have a map

$$p_{n+1}:X_{n+1} \to X_n$$

making the diagram

$$\begin{tikzcd}
X \arrow[r, "e_n"] \arrow[d, "e_{n+1}"'] & X_n \\
X_{n+1} \arrow[ur, dotted, "p_{n+1}"'] &
\end{tikzcd}$$

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commutative: $p_{n+1} \circ e_{n+1} = e_n$. The maps $\{p_n\}$ can of course be replaced by fibrations, in the usual manner. Since $e_n \in S_n$,

$$e_{n_*}:\pi_m(X) \to \pi_m(X_n)$$

is a **C**-isomorphism for $m \leqq n$. To show that $\pi_m(X_n) = 0$ for $m > n$, let $f:S^m \to X_n$ be a map with $m > n$. Let s denote the inclusion

$$X_n \hookrightarrow X_n \underset{f}{\cup} e^{m+1}.$$

Clearly s is an m-equivalence and hence a mod-**C** $(n + 1)$-equivalence (since $m > n$), so $s \in S_n$ and $s \circ e_n \in S_n$. By the couniversal property of e_n, we have a unique extension

$$t:X_n \underset{f}{\cup} e^{m+1} \to X_n$$

which makes the diagram

$$\begin{tikzcd}
X \arrow[r, "e_n"] \arrow[d, "e_n"'] & X_n \\
X_n \arrow[d, "s"'] & \\
X_n \underset{f}{\cup} e^{m+1} \arrow[uur, dotted, "t"'] &
\end{tikzcd}$$

commutative: $tse_n = e_n$; using the couniversal property of e_n again, we deduce that $ts = 1_{X_n}$;

$$\begin{tikzcd}
X \arrow[rr, "e_n"] \arrow[dd, "e_n"'] & & X_n \\
 & X_n \underset{f}{\cup} e^{m+1} \arrow[ur, "t"] & \\
X_n \arrow[ur, "s"] \arrow[uurr, "1_{X_n}"'] & &
\end{tikzcd}$$

hence $f \simeq 0$; so that $\pi_m(X_n) = 0$ for $m > n$. This completes the proof of the theorem.

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