J. Austral. Math. Soc. (Series A) **68** (2000), 252–260

ESSENTIAL IDEALS OF INCIDENCE ALGEBRAS

EUGENE SPIEGEL

(Received 19 August 1998; revised 17 August 1999)

Communicated by C. F. Miller

Abstract

It is determined when there exists a minimal essential ideal, or minimal essential left ideal, in the incidence algebra of a locally finite partially ordered set defined over a commutative ring. When such an ideal exists, it is described.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 16D15, 16D25.

In [2], Green and Van Wyk consider the existence of a minimal essential ideal of a structural matrix ring, and question when this ideal is the same as the Brown-McCoy radical of the ring. A structural matrix ring is the incidence algebra of a finite preordered set. In this note we describe the minimal essential ideal and minimal essential left ideal of the incidence algebra, $I(X, R)$, when X is a locally finite partially ordered set and R a commutative ring with identity. Recall that $I(X, R)$ is the set of all functions $f : X \times X \to R$ with $f(x, y) = 0$ unless $x \leq y$, together with the operations

$$(f + g)(x, y) = f(x, y) + g(x, y),$$
$$fg(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y),$$
$$(rf)(x, y) = rf(x, y)$$

for

$$f, g \in I(X, R), \quad r \in R, \quad x, y \in X.$$

If $x, y \in X$, with $x \leq y$, let e_{xy} denote the element of $I(X, Y)$ given by

$$e_{xy}(u, v) = \begin{cases} 1 & \text{if } x = u \text{ and } y = v; \\ 0 & \text{otherwise.} \end{cases}$$

An ideal, A, of a ring T, is called *essential*, or *large*, if $A \cap B \neq \{0\}$ for any ideal $B \neq \{0\}$. Similarly, the left ideal A is an *essential left ideal* if $A \cap B \neq \{0\}$ for any non-zero left ideal B. Of course, T is essential in T, the intersection of two essential ideals is essential, and any ideal containing an essential ideal is essential. Similar statements hold for essential left ideals.

Suppose, now, that X is a locally finite partially ordered set and R a commutative ring with identity. Associate to X the partially ordered set, $I(X)$, ordered by inclusion, of all non-empty intervals, $[x, y]$, with $x, y \in X$. Further, let $\mathrm{Ess}(R)$ be the partially ordered set, ordered by inclusion, of all essential ideals of R. If $\mathrm{Max}(I(X))$ is the collection of all maximal elements of $I(X)$, call a function $\phi : \mathrm{Max}(I(X)) \to \mathrm{Ess}(R)$, an essential function. Suppose ϕ is an *essential function* and let

$$A_\phi = \{f \in I(X, R) \mid f(x, y) \in \phi([x, y]) \text{ if } [x, y] \in \mathrm{Max}(I(X)), f(x, y) = 0 \text{ otherwise}\}.$$

It is straightforward to verify that A_ϕ is an ideal. Notice that when $[x, y] \in \mathrm{Max}(I(X))$, and K is an ideal of $I(X, R)$, then $e_{xx} K e_{yy}$ is an ideal of $I(X, R)$, namely,

$$e_{xx} K e_{yy} = \{f(x, y) e_{xy} \mid f \in K\}.$$

This follows by the maximality of $[x, y]$. Indeed, $g e_{xy} = g(x, x) e_{xy}$ and $e_{xy} g = g(y, y) e_{xy}$, for any $g \in I(X, R)$.

We now note some additional ideals of $I(X, R)$. Let n be a positive integer and

$$Z_n(X, R) = \{f \in I(X, R) \mid f(x, y) = 0 \text{ if } |[x, y]| \leq n\}.$$

Again it is easy to verify that $Z_n(X, R)$ is an ideal of $I(X, R)$. The following lemma shows that the ideals that we have defined give rise to essential ideals.

LEMMA 1. *Suppose X is a locally finite partially ordered set and R a commutative ring with identity. Let n be a positive integer and ϕ an essential function. Then $A_\phi + Z_n(X, R)$ is an essential ideal of $I(X, R)$.*

PROOF. Let $J = A_\phi + Z_n(X, R)$. Certainly J is an ideal. We check that it is essential. Let K be a non-zero ideal of $I(X, R)$. Suppose that $0 \neq f \in K$ and $f(x, y) \neq 0$. If $[x, y]$ is contained in a maximal interval $[u, v]$, then $u \leq x \leq y \leq v$ and $w = e_{ux} f e_{yv} = f(x, y) e_{uv} \in K$. In particular, $e_{uu} K e_{vv} = \{g(u, v) e_{uv} \mid g \in K\}$,

https://doi.org/10.1017/S1446788700001981 Published online by Cambridge University Press

which is contained in K, is a non-zero ideal of $I(X, R)$. Let $C = \{g(u, v) \mid g \in K\}$ and $D = \{h(u, v) \mid h \in A_\phi\}$. As C is a non-zero ideal of R, and $D = \phi([u, v])$ is an essential ideal of R, we have $C \cap D$ is a non-zero ideal of R. Then there is a $g \in K$, and an $h \in A_\phi$, with $g(u, v) = h(u, v) \neq 0$. Hence $e_{uu} g e_{vv} = e_{uu} h e_{vv} \in K \cap J$. We have thus shown that $J \cap K \neq \{0\}$ when $[x, y]$ is contained in a maximal interval. Suppose, now, that $[x, y]$ is not contained in a maximal interval. Then there is a sequence of intervals

$$[x, y] = [x_0, y_0] \subset [x_1, y_1] \subset [x_2, y_2] \subset \cdots,$$

with $[x_i, y_i]$ a proper subset of $[x_{i+1}, y_{i+1}]$, for $i = 1, 2, \ldots$. Further,

$$|[x_{n+1}, y_{n+1}]| \geq n + 1$$

and $f(x, y) e_{x_{n+1} y_{n+1}} \in Z_n(X, R) \subseteq J$. But $f(x, y) e_{x_{n+1} y_{n+1}} = e_{x_{n+1} x} f e_{y y_{n+1}} \in K$, so that, in this case too, $J \cap K$ is non-zero. The lemma now follows. □

If there is a minimal essential ideal of $I(X, R)$, then the intersection of a collection of essential ideals is still essential. The following computes the intersection of the ideals of the previous lemma. We will denote the minimal essential ideal of a ring T, when it exists, by $E(T)$. Similarly, $E_L(T)$ denotes the minimal essential left ideal of T.

LEMMA 2. *Suppose X is a locally finite partially ordered set and R a commutative ring with identity. Let ϕ be an essential function. Then*

$$\bigcap_{n=1}^{\infty} (A_\phi + Z_n(X, R)) = A_\phi.$$

PROOF. Let $f \in \bigcap_{n=1} (A_\phi + Z_n(X, R))$ and suppose $f(x, y) \neq 0$. Let $m = |[x, y]|$. Since $f \in (A_\phi + Z_m(X, R))$, we can find $g \in A_\phi$ and $h \in Z_m(X, R)$ with $f = g + h$. As $h(x, y) = 0$ we have that $g(x, y) = f(x, y)$. Hence $[x, y]$ is a maximal interval and $f(x, y) \in \phi([x, y])$. It follows that $f \in A_\phi$. □

The next lemma tells us when A_ϕ is essential.

LEMMA 3. *Suppose X is a locally finite partially ordered set and R a commutative ring with identity. Let ϕ be an essential function. Then A_ϕ is an essential ideal if and only if each interval of X is contained in a maximal interval. In particular, if one A_ϕ is essential, they all are.*

PROOF. Assume each interval of X is contained in a maximal interval. Let K be a non-zero ideal of $I(X, R)$. To show that A_ϕ is essential we check that $K \cap A_\phi \neq \{0\}$. Let f be a non-zero element of K and let $x, y \in X$ be such that $f(x, y) \neq 0$. Further, let $[u, v]$ be a maximal interval of X which contains $[x, y]$. Then $e_{ux} f e_{yv} = f(x, y)e_{uv} \in e_{uu} K e_{vv} = \{g(u, v)e_{uv} \mid g \in K\}$. Further, $e_{uu}Ke_{vv}$ is an ideal of $I(X, R)$ contained in K. Let $B = \{g(u, v) \mid g \in K\}$. Then B is a non-zero ideal of R having a non-zero intersection with the essential ideal $\phi([u, v])$. Since $\{0\} \neq \{re_{uv} \mid r \in (\phi([u, v]) \cap B)\} \subset (A_\phi \cap K)$, we have that A_ϕ is essential.

Conversely, suppose A_ϕ is essential and, looking for a contradiction, there exists an interval, $I_0 = [x_0, y_0]$, in X, which is not contained in a maximal interval. Let K be the ideal of $I(X, R)$ generated by $e_{x_0y_0}$. As A_ϕ is essential, we can find $0 \neq h \in K \cap A_\phi$, and thus a maximal interval, $[u, v]$, with $h(u, v) \neq 0$. Since K is generated by $\{f e_{x_0y_0} g \mid f, g \in I(X, R)\}$, we must have an $f_1, g_1 \in I(X, R)$ with $(f_1 e_{x_0y_0} g_1)(u, v) \neq 0$. But $(f_1 e_{x_0y_0} g_1)(u, v) = f_1(u, x_0)g_1(y_0, v)$ and, if this is to be non-zero, $[x_0, y_0] \subseteq [u, v]$. This is a contradiction as it says that $[x_0, y_0]$ is contained in a maximal interval. The lemma is then established. □

We now give a criterion for an incidence algebra to have a minimal essential ideal. For notational convenience, when $f \in I(X, R)$ and A is a subset of R, write $Af = \{af \mid a \in A\}$.

THEOREM 1. *Let X be a locally finite partially ordered set and R a commutative ring with identity. Then $I(X, R)$ has a minimal essential ideal, $E(I(X, R))$, if and only if R has a minimal essential ideal, $E(R)$, and each interval of X is contained in a maximal interval. If $E(I(X, R))$ exists, then*

$$E(I(X, R)) = \langle E(R)e_{uv} \mid [u, v] \text{ a maximal interval}\rangle.$$

PROOF. Suppose $E(I(X, R))$ exists. Let ϕ be an essential function for X. From Lemma 1 and Lemma 2, A_ϕ is essential and thus, by Lemma 3, each interval of X is contained in a maximal interval. We now check that R has a minimal essential ideal. To do this it is sufficient to show that the intersection of any class of essential ideals of R is again essential. Let $\{K_i \mid i \in I\}$ be a class of essential ideals of R. Here I is an index set. Further, let $[u, v]$ be a maximal interval in X, and ϕ_i the essential function given by $\phi_i([u, v]) = K_i$ and $\phi_i([x, y]) = R$ for any other maximal interval, $[x, y]$ of X. From Lemma 3, A_{ϕ_i} is essential in $I(X, R)$, and since $E(I(X, R))$ exists, $\bigcap_{i\in I} A_{\phi_i}$ is essential. But it is easy to see that

$$\bigcap_{i\in I} A_{\phi_i} = \left\{f \in I(X, R) \mid f(u, v) \in \bigcap_{i \in I} K_i \text{ and } f(x, y) = 0 \text{ if } [x, y] \text{ not maximal}\right\}.$$

Suppose $B = \bigcap_{i\in I} K_i$ is not essential. Then there is a non-zero ideal, C, of R, such that $C \cap B = \{0\}$. Let $L = \{ce_{uv} \mid c \in C\}$. Then L is a non-zero ideal of $I(X, R)$ and,

https://doi.org/10.1017/S1446788700001981 Published online by Cambridge University Press

as $\bigcap_{i\in I} A_{\phi_i}$ is essential, there is a non-zero $f \in (L \cap (\bigcap_{i\in I} A_{\phi_i}))$. Simultaneously we must have $f = ce_{uv}$, with $c \in C$, and $f = be_{uv}$, with $b \in B$. This is not possible as $B \cap C = \{0\}$. We conclude that B is essential, and R contains a minimal essential ideal.

Conversely, suppose that $E(R)$ exists and that each interval of X is contained in a maximal interval. Let $D = \langle E(R)e_{uv} \mid [u, v] \text{ maximal}\rangle$. Note that $D = \bigoplus E(R)e_{uv}$, the sum ranging over all maximal intervals $[u, v]$ in X. We first check that D is essential. Let K be a non-zero ideal of $I(X, R)$ and f a non-zero element of K. Let $x, y \in X$ be such that $f(x, y) \neq 0$ and $[u, v]$ a maximal interval containing $[x, y]$. Then $e_{ux} f e_{yv} = f(x, y)e_{uv}$ and the non-zero ideal $e_{uu}Ke_{vv} = \{g(u, v)e_{uv} \mid g \in K\} \subseteq K$. Since $E(R)$ is essential, $E(R) \cap \{g(u, v) \mid g \in K\} \neq \{0\}$, and so $E(R)e_{uv} \cap e_{uu}Ke_{vv} \neq \{0\}$. Hence D is essential.

To complete the proof we need only check that D is the minimal essential ideal of $I(X, R)$. Let M be an essential ideal of $I(X, R)$, C a non-zero ideal of R, and $[u, v]$ a maximal interval in X. Then $K_C = \{ce_{uv} \mid c \in C\}$ is a non-zero ideal of $I(X, R)$, and so $K_C \cap e_{uu}Me_{vv} \neq \{0\}$. As $e_{uu}Me_{vv} = \{m(u, v)e_{uv} \mid m \in M\}$, then $L = \{m(u, v) \mid m \in M\}$ is an ideal of R which has a non-zero intersection with C. Since C is an arbitrary ideal of R, L is essential. Hence $E(R) \subseteq L$. Therefore, $E(R)e_{uv} \subseteq M$ and $D \subseteq M$. □

A point, $x \in X$, is isolated if the connected component of x, in its Hasse diagram, is $\{x\}$. The following corollary shows that $E(I(X, R))$ is often nilpotent.

COROLLARY 1. *Let X be a locally finite partially ordered set and R a commutative ring with identity. If E(I(X, R)) exists then*

$$(E(I(X, R)))^2 = \bigoplus_{x \text{ isolated}} (E(R))^2 e_{xx}.$$

In particular, if X has no isolated points, $(E(I(X, R)))^2 = \{0\}$.

Green and Van Wyk [2] considered when the minimal essential ideal of a structural matrix ring equals the maximal small ideal. The maximal small ideal is the Brown-McCoy radical [3], which, in the incidence ring case under discussion, coincides with the Jacobson radical [5]. If $J(T)$ denotes the Jacobson radical of the ring T, then $J(I(X, R) = \{f \in I(X, R) \mid f(x, x) \in J(R) \text{ for } x \in X\}$ (see [1]). As we have a description of both $E(I(X, R))$ (when it exists) and $J(I(X, R))$, the following result is easily verified. Recall first that a partially ordered set X is of bound n, if the longest chain of distinct elements of X is n.

THEOREM 2. *Let X be a locally finite partially ordered set and R a commutative ring with identity. Suppose I(X, R) has a minimal essential ideal. Then* $E(I(X, R)) = J(I(X, R))$ *if and only if one of the following holds*:

https://doi.org/10.1017/S1446788700001981 Published online by Cambridge University Press

(i) *X is a finite antichain, and $E(R) = J(R)$;*

(ii) *X has no isolated points, $J(R) = \{0\}$, $E(R) = R$, and X is a finite partially ordered set of bound 2.*

PROOF. Suppose $E(I(X, R)) = J(I(X, R))$. If $f \in E(I(X, R))$, from Theorem 1 it follows that $f(u, v) = 0$ for all but a finite number of $[u, v]$. Thus X is a finite partially ordered set. Assume first that $J(R) \neq \{0\}$. Let $x \in X$. Then $J(R)e_{xx} \subseteq J(I(X, R))$ and so, by Theorem 1, $[x, x]$ is a maximal interval and $J(R) \subseteq E(R)$. It then follows that $E(R)e_{xx} \subseteq E(I(X, R))$ and $E(R) = J(R)$. Assume, now, that $J(R) = \{0\}$. Then $E(R)e_{xx} \cap J(I(X, R)) = \{0\}$ and thus x is not an isolated point. If $y \in X$ is such that $x < y$ then $e_{xy} \in J(I(X, R))$ guarantees that $[x, y]$ is a maximal interval, and that $1 \in E(R)$. It follows that X is of bound 2 and $E(R) = R$. The converse of the theorem is straightforward. □

In the following we briefly describe when $I(X, R)$ has a minimal essential left ideal within the lattice of left ideals of $I(X, R)$. The left ideal A of the ring T is an essential left ideal if $A \cap B \neq \{0\}$ for any non-zero left ideal B of T. If M is a left T-module, then the submodule N is an essential left submodule of M if $N \cap V \neq \{0\}$, for each non-zero submodule V of M.

As before, X denotes a locally finite partially ordered set and R a commutative ring with identity. Let $\mathrm{Min}(X)$ be the collection of all minimal elements of X and $\mathrm{Max}(X)$ the collection of all maximal elements of X. Of course, $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$ are antichains of X, and each interval of X is contained in a maximal interval if and only if $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$ are each maximal antichains.

Let $L = \{f \in I(X, R) \mid f(x, y) = 0 \text{ if } x \notin \mathrm{Min}(X)\}$ and $Z_n(X, R)$ the ideal defined before Lemma 1. It is easy to check that L is a left ideal and, for n a positive integer, that $L + Z_n(X, R)$ is again a left ideal. Suppose K is a non-zero left ideal of $I(X, R)$, and $0 \neq f \in K$. Further, suppose $f(u, v) \neq 0$, for some $u, v \in X$, with u related to an element, $x \in \mathrm{Min}(X)$. Then $0 \neq e_{xu}f \in (K \cap L)$. If no such u exists, by an argument similar to that in Lemma 1, we obtain that $f \in (Z_n(X, R) \cap K)$. This shows that $L + Z_n(X, R)$ is an essential left ideal of $I(X, R)$. Further, by an argument parallel to that in Lemma 2, we obtain that the intersection of the left ideals $L + Z_n(X, R)$, as n ranges over the positive integers, is L. We summarize our observations in the following lemma.

LEMMA 4. *Suppose X is a locally finite partially ordered set, R a commutative ring with identity, and n a positive integer. Then $L + Z_n(X, R)$ is an essential left ideal of $I(X, R)$. Further,*

$$\bigcap_{n=1}^{\infty}(L + Z_n(X, R)) = L.$$

https://doi.org/10.1017/S1446788700001981 Published online by Cambridge University Press

We now observe when L is an essential left ideal.

LEMMA 5. *Suppose X is a locally finite partially ordered set and R a commutative ring with identity. Then L is an essential left ideal of $I(X, R)$ if and only if* $\mathrm{Min}(X)$ *is a maximal antichain.*

PROOF. Assume $\mathrm{Min}(X)$ is a maximal antichain. By the remarks preceding Lemma 4, if f is a non-zero element of a non-zero left ideal, K, of $I(X, R)$ and $f(u, v) \neq 0$, then $e_{xu}f \in (K \cap L)$, for any minimal element $x \leq u$ of X. This shows that L is an essential left ideal.

Conversely, suppose there exists an $x_0 \in X$ incompatible with all elements of $\mathrm{Min}(X)$. Then it is easy to check that the left ideal generated by $e_{x_0x_0}$ does not have any non-zero elements in common with L. □

We need some additional terminology before presenting a criterion for the existence of a minimal essential left ideal of $I(X, R)$. If $M = {}_RM$ is a left R-module, the submodule T of M is essential if $T \cap N \neq \{0\}$, for any non-zero submodule N of M. We say that ${}_RM$ has a minimal essential submodule, $E_L(M)$, if the intersection of all its essential submodules is essential. In order that $I(X, R)$ have a minimal essential ideal, we observed in Theorem 1 that R must have a minimal essential ideal, $E(R)$. This, of course, is equivalent to saying that R, as a left R-module, has a minimal essential submodule, and it is this latter formulation which leads to a necessary condition for $I(X, R)$ to have a minimal essential left ideal.

Let κ be a cardinal number and let $\Pi_\kappa R$ denote the product of κ copies of the commutative ring R. We consider ${}_R\Pi_\kappa R$, that is, $\Pi_\kappa R$ regarded as a left R-module. If ${}_RA$ and ${}_RB$ are left R-modules then $A \oplus B$ has a minimal essential submodule, if and only if A and B each do, and $E_L(A \oplus B) = E_L(A) \oplus E_L(B)$. In particular, if $\kappa_1 \leq \kappa_2$ are cardinal numbers then the existence of $E_L({}_R\Pi_{\kappa_2}R)$ guarantees the existence of $E_L({}_R\Pi_{\kappa_1}R)$. Further, it is easy to check that if $E_L({}_R\Pi_\kappa R)$ exists then R has a minimal essential ideal, $E(R)$, and

$$_R\oplus_\kappa E(R) \subseteq E_L({}_R\Pi_\kappa R) \subseteq {}_R\Pi_\kappa E(R).$$

Let $x \in \mathrm{Min}(X)$ and let $\kappa_x = |\{y \in X \mid x \leq y\}|$. We call R $\mathrm{Min}(X)$ essential if $E_L({}_R\Pi_{\kappa_x}R)$ exists, for each $x \in X$. We can now describe when $I(X, R)$ has a minimal essential left ideal.

THEOREM 3. *Let X be a locally finite partially ordered set and R a commutative ring with identity. Then $I(X, R)$ has a minimal essential left ideal, $E_L(I(X, R))$, if and only if* $\mathrm{Min}(X)$ *is a maximal antichain and R is* $\mathrm{Min}(X)$ *essential.*

https://doi.org/10.1017/S1446788700001981 Published online by Cambridge University Press

PROOF. Suppose that $I(X, R)$ has a minimal essential left ideal $E_L(I(X, R))$. Since the ideal L, of Lemma 4, is the intersection of essential left ideals, it must be essential. Lemma 5 then tells us that $\mathrm{Min}(X)$ is a maximal antichain. We now check that R is $\mathrm{Min}(X)$ essential. Let $x \in \mathrm{Min}(X)$, $S(x) = \{y \in X \mid x \le y\}$ and $\kappa_x = |S(x)|$. It is sufficient to see that ${}_R\Pi_{\kappa_x} R$ has a minimal essential submodule. Let J_x be an index set of cardinality κ_x and $\phi_x : J_x \to S(x)$ a bijective mapping. Call $f \in I(X, R)$ an $S(x)$ function if $f(u, v) = 0$ for $u \ne x$. If f is an $S(x)$ function and $g \in I(X, R)$, then, for $y \in X$, $gf(x, y) = g(x, x)f(x, y)$. It follows that the left ideal generated by f agrees with the left R-module generated by f. For each $j \in J_x$, let $R_j = R$ and $T(x) = {}_R\prod_{j\in J_x} R_j$. We regard $T(x)$ as the collection of functions, g, from J_x to R with $g(j) \in R_j$. If f is an $S(x)$ function, let $\alpha_x(f) \in T(x)$ be the element defined by $\alpha_x(f)(j) = f(x, \phi_x(j))$. It is easy to see that α_x is a bijective R-module mapping from the R-submodule of $S(x)$ functions to $T(x)$. Further, if $\alpha_x(f) = t$, then the cyclic R-submodule of $T(x)$ generated by t corresponds, under α_x^{-1}, with the cyclic R-submodule generated by f, which, in turn, agrees with the left ideal of $I(X, R)$ generated by f. Let $V = e_{xx}E_L(I(X, R))$. Then V is the collection of all $S(x)$ functions in $E_L(I(X, R)$. Let $\alpha_x(V) = \{\alpha_x(f) \mid f \in V\}$. A straightforward verification shows that $\alpha_x(V)$ is an R-submodule of $T(x)$. If t_1 is a non-zero element of $T(x)$, then $\alpha_x^{-1}(t)$ is an $S(x)$ function and thus the left ideal of $I(X, R)$ that it generates has a non-zero intersection with $E_L(I(X, R)$. It follows that there is an $r \in R$ with $r\alpha_x^{-1}(t)$ a non-zero element in V. Hence $0 \ne rt \in \alpha_x(V)$ and $\alpha_x(V)$ is an essential submodule of $T(x)$. We check that it is the minimal essential R-submodule of $T(x)$. Suppose U is an essential submodule of T. For $y \in \mathrm{Min}(X)$, with $y \ne x$, let $W(y)$ be the collection of all $S(y)$ functions contained in $E_L(I(X, R)$. Further, let K be the left ideal generated by

$$\left(\bigcup_{\substack{y\in \mathrm{Min}(X)\\ y\ne x}} W(y)\right)\bigcup \alpha_x^{-1}(U).$$

Notice that the collection of all $S(x)$ functions in K coincides with $\alpha_x^{-1}(U)$. We check that K is an essential left ideal of $I(X, R)$. Then it follows that $E_L(I(X, R) \subseteq K$ and so $V \subseteq \alpha_x^{-1}(U)$. Hence $\alpha_x(V) \subseteq U$ and $\alpha_x(V)$ is the minimal essential R-submodule of $T(x)$.

Let f be a non-zero function of $I(X, R)$ and $f(u, v) \ne 0$, for $u, v \in X$. There is a $y \in \mathrm{Min}(X)$ with $y \le u$. Suppose, first, that $y \ne x$. The left ideal generated by $e_{yu}f(u, v)$ has a non-zero intersection with $E_L(I(X, R))$, and so there is a non-zero $S(y)$ function, g, common to these two ideals. Since $g \in W(y)$, then $g \in K$. Suppose now that $y = x$. Then $e_{xu}f$ is a non-zero $S(x)$ function and thus $t_1 = \alpha_x(e_{xu}f)$ is a non-zero element of $T(x)$. Since U is an essential module, there is an $r_1 \in R$ with $r_1\alpha_x(e_{xu}f)$ a non-zero element of U. But $0 \ne r_1e_{xu}f \in \alpha_x^{-1}(U) \subseteq K$ and, thus, in either case, the left ideal generated by f has a non-zero intersection with K. It follows that K is essential and that R is $\mathrm{Min}(X)$ essential.

Conversely, suppose that R is $\mathrm{Min}(X)$ essential and that $\mathrm{Min}(X)$ is a maximal antichain. Let $x \in \mathrm{Min}(X)$. Let P_x be the essential submodule of the product of κ_x copies of R regarded as a left R-module. Then $\alpha_x^{-1}(P_x)$ is a left ideal of $I(X, R)$ consisting of $S(x)$ functions. Let

$$E_L(I(X, R)) = \bigoplus_{x \in \mathrm{Min}(X)} \alpha_x^{-1}(P_x).$$

Using similar methods to the previous part, it is straightforward to check that $E_L(I(X, R))$ is the minimal essential left ideal of $I(X, R)$. □

The following consequence of the previous theorem and its proof gives a description of the minimal essential left ideal in a special situation.

COROLLARY 2. *Suppose R is a commutative ring with identity and X a locally finite partially ordered set having the property that, for $x \in X$, $|\{y \in X \mid x \leq y\}| < \infty$. Then $I(X, R)$ has a minimal essential left ideal, $E_L(I(X, R))$, if and only if* $\mathrm{Min}(X)$ *is a maximal antichain and R has a minimal essential ideal, $E(R)$.*

Suppose $E_L(I(X, R))$ exists. For $x \in$ $\mathrm{Min}(X)$, *let*

$$A(x) = \{f \in I(X, R) \mid f(x, v) \in E(R), f(u, v) = 0 \text{ otherwise}\}.$$

Then $A(x)$ is a left ideal of $I(X, R)$ and

$$E_L(I(X, R)) = \bigoplus_{x \in \mathrm{Min}(X)} A(x).$$

References

[1] P. Doubilet, G.-C. Rota and R. P. Stanley, 'On the foundation of combinatorial theory VI: The idea of generating function', in: *Finite operator calculus* (Academic Press, New York, 1975) pp. 267–318.

[2] B. W. Green and L. Van Wyk, 'On the small and essential ideals in certain classes of rings', *J. Austral. Math. Soc.* (*Series A*) **46** (1989), 262–271.

[3] N. V. Loi and R. Wiegandt, 'Small ideals and the Brown-McCoy radical', *Radical theory* (*Eger 1982*), Colloq. Math. Soc. János Bolyai 38 (North Holland, Amsterdam, 1985).

[4] L. Rowen, *Ring theory*, vol. I (Academic Press, Boston, 1988).

[5] E. Spiegel and C. J. O'Donnell, *Incidence algebras* (Marcel Dekker, New York, 1977).

Department of Mathematics
University of Connecticut
Storrs, CT 06269
USA
e-mail: spiegel@math.uconn.edu

https://doi.org/10.1017/S1446788700001981 Published online by Cambridge University Press