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# NUMERICAL CRITERIA FOR VERY AMPLENESS OF DIVISORS ON PROJECTIVE BUNDLES OVER AN ELLIPTIC CURVE

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ABSTRACT. Let $D$ be a divisor on a projectivized bundle over an elliptic curve. Numerical conditions for the very ampleness of $D$ are proved. In some cases a complete numerical characterization is found.

1. **Introduction.** Ampleness of divisors on algebraic varieties is a numerical property. On the other hand it is in general very difficult to give numerical necessary and sufficient conditions for the very ampleness of divisors. In [4] the author gives a sufficient condition for a line bundle associated with a divisor $D$ to be normally generated on $X = \mathbb{P}(E)$ where $E$ is a vector bundle over a smooth curve $C$. A line bundle which is ample and normally generated is automatically very ample. Therefore the condition found in [4], together with Miyaoka's well known ampleness criterion, gives a sufficient condition for the very ampleness of $D$ on $X$. This work is devoted to the study of numerical criteria for very ampleness of divisors $D$ which do not satisfy the above criterion, in the case of $C$ elliptic. With this assumption Biancofiore and Livorni [3] (see also [2, Proposition 8.5.8] for a generalization) gave a necessary and sufficient condition when $E$ is indecomposable, $\operatorname{rk} E = 2$ and $\deg E = 1$. Gushel [6] also gave a complete characterization of the very ampleness of $D$ assuming that $E$ is indecomposable and $|D|$ embeds $X$ as a scroll. This work deals with the general situation and addresses the cases still open.

The main technique used here is a very classical one. A suitable divisor $A$ on $X$ is chosen such that there exists a smooth $S \in |A|$ containing every pair of points, possibly infinitely near. Appropriate vanishing conditions are established to assure that the natural restriction map $H^0\big(X, \mathcal{O}_X(D)\big) \longrightarrow H^0\big(S, \mathcal{O}_X(D)|_S\big)$ is surjective. In this way we get that a divisor $D$ of $X$ is very ample if and only if $D|_S$ is very ample. In this context $S$ is chosen as $S = \mathbb{P}(E')$ where $E'$ is a quotient of $E$, thus with rank smaller than rank $E$. Therefore an inductive process on the rank can be set up. This process is not always easy to carry on. For example if $E$ is assumed to be indecomposable there is no guarantee that $E'$ will still be indecomposable. Since ampleness is inherited by quotients, we will require at some stage that $E$ be ample. The paper is organized as follows. Section 2 contains notation, known and preliminary results used in the sequel. In Section 3 the case of rank $E = 2$ is

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fully treated. We recover Biancofiore and Livorni's results and deal with the case of $E$ decomposable. Section 4 deals with the case of rank $E = 3$ while Section 5 contains the study of case rank $E \geq 4$. In particular in the case of rank $E = 3$ we get the following result (see Section 2.1 for notation):

THEOREM. *Let $E$ be a rank 3 vector bundle on an elliptic curve $C$ and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$.*

*(a) If $E$ is indecomposable then*

*(a1) if $d = 0 \pmod 3$, $D$ is very ample if and only if $b + a\mu^-(E) \geq 3$,*

*(a2) if $d = 1 \pmod 3$, $D$ is very ample if $b + a\mu^-(E) > 1$,*

*(a3) if $d = 2 \pmod 3$, $D$ is very ample if $b + a\mu^-(E) > \frac{4}{3}$,*

*(b) if $E$ is decomposable, then $D$ is very ample if and only if $b + a\mu^-(E) \geq 3$ except when $E = E_1 \oplus E_2$, with $\operatorname{rk} E_1 = 1$, $\operatorname{rk} E_2 = 2$, $\deg E_2$ odd and $\deg E_1 > \frac{\deg E_2}{2}$. In the latter case the condition is only sufficient.*

Notice that the above theorem shows the existence, among others, of a smooth threefold of degree 20 embedded in $\mathbb{P}^9$ as a fibration of Veronese surfaces over an elliptic curve, choosing $a = 2$, $b = -1$ and $d = 4$.

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## 2. General results and preliminaries.

2.1. *Notation.* The notation used in this work is mostly standard from Algebraic Geometry. Good references are [9] and [5]. The ground field is always the field $\mathbb{C}$ of complex numbers. Unless otherwise stated all varieties are supposed to be projective. $\mathbb{P}^n$ denotes the $n$-dimensional complex projective space and $\mathbb{C}^*$ the multiplicative group of non zero complex numbers. Given a projective $n$-dimensional variety $X$, $\mathcal{O}_X$ denotes its structure sheaf and $\mathrm{Pic}(X)$ denotes the group of line bundles over $X$. Line bundles, vector bundles and Cartier divisors are denoted by capital letters as $L, M, \ldots$. Locally free sheaves of rank one, line bundles and Cartier divisors are used interchangeably as customary. Let $L, M \in \mathrm{Pic}(X)$, let $E$ be a vector bundle of rank $r$ on $X$, let $\mathcal{F}$ be a coherent sheaf on $X$ and let $Y \subset X$ be a subvariety of $X$. Then the following notation is used:

$LC$ the intersection number of $L$ with a curve $C$,

$L^n$ the degree of $L$,

$|L|$ the complete linear system of effective divisors associated with $L$,

$L|_Y$ the restriction of $L$ to $Y$,

$L \sim M$ the linear equivalence of divisors,

$L \equiv M$ the numerical equivalence of divisors,

$\mathrm{Num}(X)$ the group of line bundles on $X$ modulo the numerical equivalence,

$E^*$ the dual of $E$,

$\mathbb{P}(E)$ the projectivized bundle of $E$,

$H^i(X, \mathcal{F})$ the $i$-th cohomology vector space with coefficient in $\mathcal{F}$,

$h^i(X, \mathcal{F})$ the dimension of $H^i(X, \mathcal{F})$.

If $C$ denotes a smooth projective curve of genus $g$, and $E$ a vector bundle over $C$ of $\deg E = c_1(E) = d$ and $\operatorname{rk} E = r$, we need the following standard definitions:

$E$ is *normalized* if $h^0(E) \neq 0$ and $h^0(E \otimes L) = 0$ for any invertible sheaf $L$ over $C$ with $\deg L < 0$.

$E$ has slope $\mu(E) = \frac{d}{r}$.

$E$ is *semistable* if and only if for every proper subbundle $S$, $\mu(S) \leq \mu(E)$. It is *stable* if and only if the equality is strict.

The Harder-Narasimhan filtration of $E$ is the unique filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$$

such that $\frac{E_i}{E_{i-1}}$ is semistable for all $i$, and $\mu_i(E) = \mu(\frac{E_i}{E_{i-1}})$ is a strictly decreasing function of $i$.

We recall now some definitions from [4] which we will use in the following: let $0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$ be the Harder-Narasimhan filtration of a vector bundle $E$ over $C$. Then

$\mu^-(E) = \mu_s(E) = \mu(\frac{E_s}{E_{s-1}})$,

$\mu^+(E) = \mu_1(E) = \mu(E_1)$,

or alternatively

$\mu^+(E) = \max\{\mu(S) \mid 0 \longrightarrow S \longrightarrow E\}$,

$\mu^-(E) = \min\{\mu(Q) \mid E \longrightarrow Q \longrightarrow 0\}$.

We have also $\mu^+(E) \geq \mu(E) \geq \mu^-(E)$ with equality if and only if $E$ is semistable. In particular if $C$ is an elliptic curve, an indecomposable vector bundle $E$ on $C$ is semistable and hence $\mu(E) = \mu^-(E)$. Moreover if $F$, $G$ are indecomposable and hence semistable vector bundles on an elliptic curve $C$ and $F \longrightarrow G$ is a non zero map, it follows that $\mu(F) \leq \mu(G)$.

2.2. *General results.* Let $C$ be a smooth projective curve of genus $g$, $E$ a vector bundle of rank $r$, with $r \geq 2$, over $C$ and $\pi: X = \mathbb{P}(E) \longrightarrow C$ the projective bundle associated to $E$ with the natural projection $\pi$. With standard notations denote with $\mathcal{T} = \mathcal{O}_{\mathbb{P}(E)}(1)$ the tautological sheaf and with $\mathcal{F}_P = \pi^* \mathcal{O}_C(P)$ the line bundle associated with the fiber over $P \in C$. Let $T$ and $f$ denote the numerical classes respectively of $\mathcal{T}$ and $\mathcal{F}_P$.

Let $D \sim a\mathcal{T} + \pi^* B$, with $a \in \mathbb{Z}$, $B \in \operatorname{Pic}(C)$ and $\deg B = b$, then $D \equiv aT + bf$. Moreover $\pi_* D = S^a(E) \otimes \mathcal{O}_C(B)$ and hence $\mu^-(\pi_* D) = a\mu^-(E) + b$ (see [4]).

Regarding the ampleness, the global generation, and the normal generation of $D$, the following criteria are known:

THEOREM 2.1 (MIYAOKA [10]). *Let $E$ be a vector bundle over a smooth projective curve $C$ of genus $g$, and $X = \mathbb{P}(E)$. If $D \equiv aT + bf$ is a line bundle over $X$, then $D$ is ample if and only if $a > 0$ and $b + a\mu^-(E) > 0$.*

PROPOSITION 2.2 (GUSHEL [7], PROPOSITION 3.3). *Let $D \sim a\mathcal{T} + \pi^* B$ where $a > 0$ and $B \in \operatorname{Pic}(C)$, be a divisor on a projective bundle $\pi: X = \mathbb{P}(E) \longrightarrow C$. Then:*

*i) if $a = 1$, the bundle $\pi_*(D)$ is generated by global sections if and only if the divisor $D$ is,*

*ii) if $a \geq 2$, and the vector bundle $\pi_*(D)$ is generated by global sections, then also the divisor $D$ is.*

LEMMA 2.3 (GUSHEL [6], PROPOSITION 3.2). *Let $E$ be an indecomposable vector bundle over an elliptic curve $C$. $E$ is globally generated if and only if* $\deg E > \operatorname{rank} E$.

LEMMA 2.4 (SEE *e.g.* [4], LEMMA 1.12). *Let $E$ be a vector bundle over $C$ of genus $g$.*

*i) if $\mu^-(E) > 2g - 2$ then $h^1(C, E) = 0$,*

*ii) if $\mu^-(E) > 2g - 1$ then $E$ is generated by global sections.*

For the following theorem we need a definition:

DEFINITION 1 (BUTLER [4]). Let $E$ be a vector bundle over a variety $Y$, and let $\pi: X = \mathbb{P}(E) \longrightarrow Y$ be the natural projection. A coherent sheaf $\mathcal{F}$ over $X$ is said to be *$t\pi$-regular* if, for all $i > 0$,

$$\mathcal{R}^i\pi_*\big(\mathcal{F}(t-i)\big) = 0.$$

THEOREM 2.5 (BUTLER [4]). *Let $E$ be a vector bundle over a smooth projective curve $C$ of genus $g$, and $X = \mathbb{P}(E)$. If $D$ is a $(-1)\pi$-regular line bundle over $X$, with $\mu^-(\pi_*D) > 2g$, then $D$ is normally generated.*

REMARK 2.6. Let $D$ be a divisor of $X = \mathbb{P}(E)$, with $E$ vector bundle on a smooth projective curve of genus $g$. As $h^i\big(\mathcal{F}_P, D_{|\mathcal{F}_P}(-1-i)\big) = 0$ for $i \geq 1$, the $(-1)\pi$-regularity of $D$ is satisfied, hence the condition $a\mu^-(E) + b > 2g$ implies that $D$ is normally generated. If a line bundle $D$ on a projective variety $X$ is ample and normally generated it is very ample. Hence from Theorem 2.1 and 2.5 we get that $D$ is very ample on $X = \mathbb{P}(E)$ if

$$b + a\mu^-(E) > 2g. \tag{1}$$

Hence, if $g = 1$, the very ampleness of $D \equiv aT + bf$ is an open problem only in the range

$$0 < b + a\mu^-(E) \leq 2. \tag{2}$$

2.3. *Preliminaries.* The following result is standard from the theory of vector bundles (see [9]):

LEMMA 2.7. *Let $E$ be an indecomposable vector bundle of* rank *$r$ on an elliptic curve. If $E$ is normalized then* $0 \leq \deg E \leq r - 1$.

LEMMA 2.8. *Let $E = \bigoplus_{i=1}^n E_i$ be a decomposable vector bundle over an elliptic curve $C$, with $E_i$ indecomposable vector bundles. Then $\mu^-(E) = \min \mu(E_i)$.*

PROOF. For the proof we need the following three claims.

CLAIM 1. *Let $E = \bigoplus_i E_i$ be as above, then $\mu(E) \geq \min \mu(E_i)$.*

PROOF. Let us denote by $r = \mathrm{rk}(E)$, $r_i = \mathrm{rk}(E_i)$, $d = \deg(E)$, $d_i = \deg(E_i)$. Let us consider the vectors $\underline{v}_i$ in $\mathbb{R}^2$ whose coordinates are $(r_i, d_i)$ and the vector $\underline{v} = \sum_i \underline{v}_i$. Let $\alpha_i$ be the angle between the $r$-axis and $\underline{v}_i$. Let $\alpha$ be the angle between the $r$-axis and $\underline{v}$. It is $\mu(E) = \frac{d}{r} = \mathrm{tg}(\alpha) \geq \min_i \mathrm{tg}(\alpha_i) = \min_i(\frac{d_i}{r_i}) = \min_i \mu(E_i)$. ∎

CLAIM 2. *Let $E = \bigoplus_i E_i$ be as above, and $\mu(E_i) = \frac{d_i}{r_i} = h \in \mathbb{Q}$, for all $i$. Then $\mu^-(E) = h$.*

PROOF. Notice that under this hypothesis $\mu(E) = h$. Moreover, by definition, it is $\mu^-(E) = \min\{\mu(Q) \mid E \longrightarrow Q \longrightarrow 0\}$. If $Q$ is decomposable in the direct sum of indecomposable vector bundles $Q_k$, the existence of a surjective map $E \longrightarrow Q \longrightarrow 0$ implies the existence of surjective maps $E \longrightarrow Q_k \longrightarrow 0$ for all $k$ and consequently from Claim 1, $\mu^-(E) = \min\{\mu(Q) \mid E \longrightarrow Q \longrightarrow 0$, and $Q$ indecomposable$\}$.

Now let $Q_o$ be an indecomposable vector bundle which realizes the minimum, *i.e.* $\mu(Q_o) = \mu^-(E)$. From $\bigoplus_i E_i \longrightarrow Q_o \longrightarrow 0$ it follows that there exists at least an index $i_0$ such that the map $E_{i_0} \longrightarrow Q$ is not zero and $\mu(E_{i_0}) \leq \mu(Q_o)$. Therefore it is $h \leq \mu^-(E)$. As $h \geq \mu^-(E)$, the claim is proved. ∎

CLAIM 3. *Let $E = \bigoplus_i E_i$ be as in Claim (2). Then $E$ is semistable.*

PROOF. It is enough to prove that for any $S$ vector bundle on $C$ such that there exists a map $0 \longrightarrow S \longrightarrow E$ then $\mu(S) \leq \mu(E) = h$. If we consider the dual map $E^* \longrightarrow S^* \longrightarrow 0$ we have $\mu(S^*) \geq \mu^-(E^*) = \mu^-(\bigoplus_i E_i{}^*)$ and, as $\mu(E^*{}_i) = -\frac{d_i}{r_i} = -h$, from Claim 2 applied to $E^*$ we have $\mu^-(E^*) = -h$. Hence $\mu(S^*) = -\mu(S) \geq -h$ and $\mu(S) \leq h$. ∎

The lemma can now be proved.

Let $E = \bigoplus_i E_i$ be as in the hypothesis of the lemma, and denote by $\mu_i = \mu(E_i)$. We can choose an ordering such that $E = E_1 \oplus E_2 \oplus E_3 \cdots$ and $\mu_1 \geq \mu_2 \geq \mu_3 \cdots$. Let $E = \bigoplus_{k=1}^s A_k$ be a new decomposition of $E$ such that each $A_k$ is an indecomposable vector bundle or a sum of indecomposable vector bundles $E_i$ with the same $\mu_i$. In this way we get a strictly decreasing sequence $\mu(A_1) > \mu(A_2) > \cdots > \mu(A_s)$, and by claim (3) each $A_k$ is semistable. Moreover the sequence $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ with $F_i = A_1 \oplus A_2 \oplus \cdots \oplus A_i$ with $1 \leq i \leq s$, is the Harder-Narasimhan filtration of $E$ because the sequence of the slopes $\mu(\frac{F_i}{F_{i-1}}) = \mu(A_i)$ is strictly decreasing and each $\frac{F_i}{F_{i-1}} = A_i$ is semistable for all $i = 1 \cdots s$. Hence we get $\mu^-(E) = \mu(\frac{E_s}{E_{s-1}}) = \mu(A_s) = \min \mu(E_i)$. ∎

LEMMA 2.9. *Let $D \sim a\mathcal{T} + \pi^* B$ be a line bundle in $X = \mathbb{P}(E)$ over a curve $C$ of genus $g = 1$, with $B \in \mathrm{Pic}(C)$, $a \geq 1$ and $\deg B = b$.*

*i) If $a = 1$ $D$ is globally generated if and only if $b + \mu^-(E) > 1$.*

*ii) If $a \geq 2$ $D$ is globally generated if $b + a\mu^-(E) > 1$.*

PROOF. To prove ii) it is sufficient to apply Proposition 2.2 and Lemma 2.4. To prove i) notice that if $E$ is indecomposable it is enough to apply Proposition 2.2 and Lemma 2.3, observing that an indecomposable vector bundle $E$ over an elliptic curve is semistable and hence $\mu^-(E) = \mu(E)$. Let now $E$ be decomposable and hence $E \otimes B$ decomposable over $C$. In particular let $E \otimes B = \bigoplus_{q=1}^s A_q$ be a decomposition of $E \otimes B$ in indecomposable vector bundles $A_q$ over $C$. By Lemma 2.3 every $A_q$, for $q = 1, \cdots, s$, is globally generated if and only if $\deg A_q > \operatorname{rk} A_q$, *i.e.* if and only if $\mu(A_q) > 1$, for all $q$. From Lemma 2.3, Lemma2.8 and Proposition 2.2 we get the following chain of equivalences which conclude the proof: $\mu^-(E) + b > 1 \Leftrightarrow \mu^-(E \otimes B) = \min_q \mu(A_q) > 1 \Leftrightarrow \mu(A_q) > 1$ for all $q \Leftrightarrow A_q$ is globally generated for all $q \Leftrightarrow \pi_* D$ is globally generated on $C \Leftrightarrow D$ is globally generated on $X$. ∎

The above lemma is partially contained in [6, Proposition 3.3]. Unfortunately the proof presented there is based on [6, Proposition 1.1(iv)], which is not correct, as the following counterexample shows. Let $E$ be an indecomposable vector bundle over an elliptic curve with $\deg E = 1$ and $\operatorname{rank} E = 2$. Then $2\mathcal{T} = \mathcal{O}_{\mathbb{P}(E)}(2)$ is generated by global sections, according to [2, Proposition 8.5.8]. On the other hand let $\pi_*(2\mathcal{T}) = S^2E = \oplus_q A_q$, where $A_q$ is indecomposable for all $q$. Then $S^2E$ is generated by global sections if and only if $A_q$ is such, for all $q$. From Lemma 2.3 it follows that $S^2E$ is globally generated if and only if $\mu(A_q) > 1$ for all $q$, *i.e.* if and only if $\mu^-(S^2E) > 1$, *i.e.* if and only if $2\mu^-(E) = 2\mu(E) > 1$ which is false.

If we consider an indecomposable vector bundle of degree $d = 0$, we have the following proposition. It is contained in [7, Theorem 3.9], but we prefer to give here a simpler proof.

PROPOSITION 2.10. *Let $E$ be an indecomposable* rank *$r$ vector bundle over an elliptic curve $C$ with* $\deg E = 0 \pmod r$*), and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then $D$ is very ample on $X$ if and only if $b + a\mu^-(E) = b + a\mu(E) \geq 3$.*

PROOF. It is enough to consider the case in which $E$ is normalized, as if $E$ is not normalized we can consider its normalization $\bar{E} = E \otimes L$ with $\deg L = l$. If $D \equiv aT + bf$ in $\operatorname{Num} \mathbb{P}(E)$ then in $\operatorname{Num} \mathbb{P}(\bar{E})$ we get

$$(3) \qquad D \equiv a\bar{T} + (b - al)\bar{f}, \quad \bar{d} = \deg \bar{E} = d + rl, \quad \mu(\bar{E}) = \mu(E) + al.$$

Let $E$ be normalized, hence $d = 0$ and $E = F_r$ in the notation of [1] (recall that $F_1 = O_C$). According to (2) the only cases to be considered are $b = 1$ and $b = 2$ and hence $D \equiv aT + f$ or $D \equiv aT + 2f$. We want to show that in both these cases $D$ is not very ample. Assume the contrary and proceed by induction on $r$. Let $r = 2$. As $DT = 1$ or $2$, the smooth elliptic curve $\Gamma$, which is the only element of $|\mathcal{T}|$, is embedded by $\phi_{|D|}$ as a line or a conic which is a contradiction. Assume now the proposition true for $F_{r-1}$ and recall that there is a short exact sequence (see [1] p. 432)

$$(4) \qquad 0 \longrightarrow O_C \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow 0.$$

Let $T' = T|_Y$ and $f' = f|_Y$ the generators of $\operatorname{Num}(Y)$ where $Y = \mathbb{P}(F_{r-1}) \subset X = \mathbb{P}(E)$. If $D$ is very ample, $D|_Y$ is very ample too; but $D|_Y \equiv aT' + bf'$ and it is not very ample by induction hypothesis. Hence $D$ is very ample if and only if $b \geq 3$. ∎

The following lemma, which gives a sufficient condition for the very ampleness of a divisor $D$ on $X = \mathbb{P}(E)$, will be needed later on.

LEMMA 2.11. *Let $E$ be a* rank $r$ *vector bundle over a curve $C$ and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$, with $a \geq 1$. If $\pi_* D$ is a very ample vector bundle on the curve $C$, then $D$ is very ample on $\mathbb{P}(E)$. Moreover if $a = 1$, $D$ is very ample on $X$ if and only if $\pi_* D$ is very ample on $C$.*

PROOF. We give only a sketch of the proof. A divisor $D \equiv aT + bf$ on $X$ defines a map $\varphi_{|D|}$ in a suitable projective space such that $X' = \varphi_{|D|}(X)$ is a bundle on $C$ whose fibers are the Veronese embedding of the fibers of $X = \mathbb{P}(E)$. Moreover each fiber of $X'$ is embedded in a fiber of the projective bundle $\mathbb{P}\big(S^a(E) \otimes \mathcal{O}_C(B)\big)$. It follows that the very ampleness of $S^a(E) \otimes \mathcal{O}_C(B)$ and hence of its tautological bundle implies that the map $\varphi_{|D|}$ gives an embedding and hence that $D$ is very ample. The case $a = 1$ follows immediately from the above considerations. ∎

2.4. *The case $a = 1$.* We want to investigate the very ampleness of $D \equiv aT + bf$ in dependence of $a$ and $b$. As we have remarked at the end of Section 2.2, the problem is open only when $0 < b + a\mu^-(E) \leq 2$. Let us begin with the case $a = 1$. In this case we have the following theorem:

THEOREM 2.12 (GUSHEL [6], THEOREM 4.3). *Let $D \sim \mathcal{T} + \pi^* B$ be a divisor on $\mathbb{P}(E)$, where $E$ is an indecomposable and normalized vector bundle of* rank $r$ *over an elliptic curve $C$. If $b = \deg B$, the divisor $D$ is very ample if and only if:*

*i)* $b \geq 3$ *if* $\deg E = 0$,

*ii)* $b \geq 2$ *if* $0 < \deg E < r$.

Now it is easy to prove the following (see (3)):

PROPOSITION 2.13. *In the above assumptions and notations, if $E$ is indecomposable but not normalized, it follows that $D$ is very ample if and only if the following conditions hold:*

*i)* $b + \mu(E) \geq 3$ *if* $d = 0 \pmod r$.

*ii)* $b + \mu(E) \geq 2$ *otherwise.*

REMARK 2.14. The previous results consider the case in which $E$ is indecomposable. If $E$ is decomposable, by Lemma 2.12, we can argue as follows: firstly in this case, as $a = 1$, $D$ is very ample if and only if $\pi_*(D)$ is very ample. Secondly we have $D \sim \mathcal{T} + \pi^* B$, $\pi_*(D) \simeq E \otimes \mathcal{O}_C(B) = \bigoplus E_j \otimes \mathcal{O}_C(B)$, with $E_j$ indecomposable vector bundles. Moreover $E \otimes \mathcal{O}_C(B)$ is very ample if and only if every $E_j \otimes \mathcal{O}_C(B)$ is very ample. Let $\deg E_j = d_j$ and $\operatorname{rk} E_j = r_j$, and assume that $d_j = 0 \pmod{r_j}$, possibly only for $j = \cdots t$. Then $D$ is very ample if and only if $b + \frac{d_j}{r_j} \geq 3$ for $j = 1 \cdots t$, and $b + \frac{d_j}{r_j} \geq 2$ for the remaining $j$'s, by Proposition 2.13.

Having dealt above with the case $a = 1$, from now on the blanket assumption $a \geq 2$ will be in effect.

3. **Rank** 2. Let $E$ be a rank 2 vector bundle on an elliptic curve $C$ and let $D \equiv aT+bf$ be a line bundle on $X = \mathbb{P}(E)$. Assume that $E$ is indecomposable. If $E$ is normalized then $\deg E = 0, 1$ by Lemma 2.7. If $\deg E = 0$, from Proposition 2.10 it follows that $D$ is very ample if and only if $b \geq 3$. If $\deg E = 1$, necessary and sufficient conditions for the very ampleness of $D$ are given by the following theorem, reformulated under our assumption that $a \geq 2$.

THEOREM 3.1 (BIANCOFIORE-LIVORNI [3], THEOREM 6.3). *Let $D \sim a\mathcal{T} + \pi^* B$ be a divisor on $\mathbb{P}(E)$, where $E$ is an indecomposable normalized vector bundle of rank* 2 *and degree* 1 *over an elliptic curve $C$. If $b = \deg B$, the divisor $D$ is very ample if and only if $b + \frac{a}{2} > 1$.*

The following proposition can now be easily proved (see (3)).

PROPOSITION 3.2. *In the above hypothesis, if $E$ is indecomposable but not normalized, $D$ is very ample if and only if the following conditions hold:*

*ii)* $b + a\mu^-(E) \geq 3$ *if* $d = 0 \pmod 2$.

*i)* $b + a\mu^-(E) > 1$ *if* $d = 1 \pmod 2$.

The case $E$ decomposable is treated by the following theorem.

THEOREM 3.3. *Let $D \sim a\mathcal{T} + \pi^* B$ be a divisor on $\mathbb{P}(E)$, where $E$ is a decomposable vector bundle of* rank 2 *over an elliptic curve $C$, $b = \deg B$. The divisor $D$ is very ample if and only if $b + a\mu^-(E) \geq 3$.*

PROOF. To prove the sufficient condition let $E$ be decomposable as $H \oplus G$ where $H$ and $G$ are line bundles on $C$ with $\deg H = h \geq \deg G = g$. By Lemma 2.8 it is $\mu^-(E) = g$. By Lemma 2.11, a sufficient condition for the very ampleness of $D$ on $X$ is that $\pi_*(D)$ is very ample as a vector bundle on $C$. In our hypothesis

$$\pi_*(D) = S^a(E) \otimes \mathcal{O}_C(B) = \bigoplus_{q=0}^{a} H^{\otimes q} \otimes G^{\otimes a-q} \otimes B.$$

Now $\pi_*(D)$ is very ample if each element of its decomposition has degree $\geq 3$, *i.e.* if $qh + (a-q)g + b \geq 3$, for all $q = 0, \ldots, a$. As the minimum of $qh + (a-q)g + b$ is realized for $q = 0$, $\pi_*(D)$ is very ample if and only if $ag + b = b + a\mu^-(E) \geq 3$. This condition is also necessary for the very ampleness of $D$. Indeed the projective bundle $\mathbb{P}(G)$, by the exact sequence

$$0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$$

[6], Proposition 1.1, gives an elliptic curve $\Gamma$ on $X$, $\Gamma \in |\mathcal{T} + \pi^*(H^*)|$. Notice that $h^0(X, \mathcal{T} + \pi^*(H^*)) > 0$. If $D$ is very ample it must be $D\Gamma = (aT + bf)(T - hf) = ag + b = b + a\mu^-(E) \geq 3$. ∎

4. **Rank** 3. In this section and in the next one, we will prove the very ampleness of a divisor on a smooth variety following a classical method, based on the following lemmata.

LEMMA 4.1. *Let $X$ be a smooth variety, $D$ a divisor in* Pic($X$) *and let $A$ be an other element of* Pic($X$)*, such that $h^1\big(X, O_X(D-A)\big) = 0$. If, for each pair of points $R, Q \in X$ (possibly infinitely near) it is possible to find a smooth element $S \in |A|$ containing $R$ and $Q$, then $D$ is very ample on $X$ if and only if $D|_S$ is very ample on $S$.*

PROOF. If $D$ is very ample then obviously $D|_S$ is very ample. On the other hand, pick any two points $R, Q \in X$ (distinct or infinitely near) and choose $S \in |A|$ such that $R, Q \in S$. As $D|_S$ is very ample there exist sections of $D|_S$ separating $R$ and $Q$. Now look at the following exact sequence

$$0 \longrightarrow O_X(D-A) \longrightarrow O_X(D) \longrightarrow O_S(D|_S) \longrightarrow 0.$$

From the assumptions above we get that the map $H^0\big(X, O_X(D)\big) \longrightarrow H^0\big(S, O_S(D|_S)\big)$ is surjective and hence $D$ is very ample on $X$ if and only if $D|_S$ is very ample on $S$. ∎

LEMMA 4.2. *Let $E$ be an ample vector bundle over an elliptic curve $C$ such that $\deg E < \operatorname{rk} E$. Let $X = \mathbb{P}(E)$, let $P$ be a fixed point of $C$, $D \sim a\mathcal{T} + \pi^*B$ and $A = \mathcal{B} + \mathcal{F}_P$ be line bundles on $X$, with $b = \deg B$, $b-1+(a-1)\mu^-(E) > 0$ and $h^0\big(X, O_X(A)\big) \geq \deg E+3$. Then the hypothesis of Lemma 4.1 are satisfied for $A$.*

PROOF. If $(a-1)\mu^-(E) + b - 1 > 0$ then by Lemma 2.4 it is $h^1\big(X, O_X(D-A)\big) = 0$. Moreover being $h^0\big(X, O_X(A)\big) \geq \deg E + 3$, for each pair of points $R, Q \in X$ there exists a linear subsystem $\mathcal{L} \subset |\mathcal{A}|$, with $\dim \mathcal{L} \geq \deg \mathcal{E}$, all the elements of which contain $R$ and $Q$. Moreover in $\mathcal{L}$ there is at least one smooth element $S$. In fact, assume that all the elements of $\mathcal{L}$ are singular. Note that any singular element of $\mathcal{L}$ must be reducible as $\Gamma \cup \mathcal{F}_P$, with $\Gamma \in |\mathcal{T}|$, $\Gamma$ smooth, because we have that any divisor numerically equivalent to $T-f$ is not effective as $\deg E < \operatorname{rk} E$. As $h^0\big(X, O_X(\mathcal{F}_P)\big) = 1$, for all $P \in C$, by Bertini's theorem all the elements of $\mathcal{L}$ are singular only if $\mathcal{F}_P$ is fixed and $\Gamma$ varies in a subsystem of $|\mathcal{T}|$ of dimension $\deg E$. This is impossible as $h^0\big(X, O_X(\mathcal{T})\big) = \deg E$. ∎

LEMMA 4.3. *Let $E$ be an ample vector bundle over an elliptic curve $C$ such that $\deg E < \operatorname{rk} E$. Let $X = \mathbb{P}(E)$, and let $D \sim a\mathcal{T} + \pi^*B$ be a line bundle on $X$, with $b = \deg B$, $b + (a-1)\mu^-(E) > 0$ and $h^0\big(X, O_X(\mathcal{T})\big) \geq 3$. Then the hypothesis of Lemma 4.1 are satisfied, with $A = \mathcal{T}$.*

PROOF. If $(a-1)\mu^-(E) + b > 0$ then Lemma 2.4 gives $h^1\big(X, O_X(D-A)\big) = 0$. Moreover as each element of $|\mathcal{T}|$ is smooth, because we have that any divisor numerically equivalent to $T-f$ is not effective as $\deg E < \operatorname{rk} E$, the condition $h^0\big(X, O_X(\mathcal{T})\big) \geq 3$ shows that it is possible to find a smooth element $S \in |\mathcal{T}|$ containing each fixed pair of points $R, Q \in X$. ∎

The following lemma will be very useful to obtain the vanishing condition required by Lemma 4.1 in many borderline cases. The notation used here is the classical notation used by Atiyah in [1].

LEMMA 4.4. *Let $E$ be an indecomposable vector bundle over an elliptic curve $C$ with* $\operatorname{rank} E = r$ *and* $\deg E = d$. *Let $X = \mathbb{P}(E)$ and let $\pi: X \longrightarrow C$ be the natural projection. Let $D = a\mathcal{T} + \pi^*(B)$ for a line bundle $B$ with* $\deg B = b$ *and let $A = \mathcal{T} + \pi^*\big(\mathcal{O}_C(P)\big)$ where $P$ is a point in $C$.*

*If $\frac{(a-1)d}{r} + b - 1 = 0$, it is possible to choose $P \in C$ such that $h^1(X, D - A) = 0$.*

PROOF. It is enough to show that $h^1\big(C, S^{a-1}E \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$.

Since $\deg S^{a-1}E \otimes B \otimes \mathcal{O}_C(-P) = 0$ by Riemann Roch it is enough to show that $h^0\big(S^{a-1}E \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$. Because $S^{a-1}(E)$ is a direct summand of $E^{\otimes(a-1)}$ it is enough to show that $h^0\big(E^{\otimes(a-1)} \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$.

Let $h = \gcd(d, r)$. Then by [1] Lemma 2.4 and 2.6 it is

$$E = E' \otimes F_h \tag{5}$$

where $d' = \deg E' = \frac{d}{h}$ and $r' = \operatorname{rank} E' = \frac{r}{h}$ so that $\gcd(d', r') = 1$, $F_h$ is as in [1] Theorem 5 and $E'$ is indecomposable. Being $r'$ and $d'$ relatively prime, the condition $\frac{(a-1)d'}{r'} + b - 1 = 0$ shows that $r'$ divides $(a-1)$. Therefore following [8] Proposition 1.4 it follows that

$$E'^{\otimes(a-1)} = \bigoplus_i (F_{r_i} \otimes L_i) \tag{6}$$

Therefore putting (5) and (6) together we get

$$E^{\otimes(a-1)} = (E' \otimes F_h)^{\otimes(a-1)} = \bigoplus_i (F_{r_i} \otimes L_i) \otimes F_h^{\otimes(a-1)}.$$

Theorem 8 in [1] shows that tensor powers of $F_l$ 's are direct sums of $F_k$'s so we conclude that

$$E^{(a-1)} = \bigoplus_j (F_{r_j} \otimes L_j).$$

It is then enough to show that for all $j$ it is $h^0\big(F_{r_j} \otimes L_j \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$.

Let $\mathcal{L}_{j,P} = L_j \otimes B \otimes \mathcal{O}_C(-P)$. Recall that the $F_{r_j}$ are obtained as successive extensions of each other by $\mathcal{O}_C$, *i.e.* for every $r$ we have the sequence (4) (see proof of Proposition 2.11).

This shows that it is $h^0(F_{r_j} \otimes \mathcal{L}_{j,P}) = 0$ unless $\mathcal{L}_{j,P} = \mathcal{O}_C$. It is then enough to choose a point $P$ such that $L_j \otimes B \otimes \mathcal{O}_C(-P) \neq \mathcal{O}_C$ for all $j$. Since $B$ is a fixed line bundle and $j$ runs over a finite set, a $P$ that works for all $j$ can certainly be found. ∎

The following theorem collects our results for the case $\operatorname{rk} E = 3$.

THEOREM 4.5. *Let $E$ be a* rank 3 *vector bundle on an elliptic curve $C$ and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$.*

*(a) If $E$ is indecomposable then*
   - *(a1) if $d = 0 \pmod 3$, $D$ is very ample if and only if $b + a\mu^-(E) \geq 3$,*
   - *(a2) if $d = 1 \pmod 3$, $D$ is very ample if $b + a\mu^-(E) > 1$,*
   - *(a3) if $d = 2 \pmod 3$, $D$ is very ample if $b + a\mu^-(E) > \frac{4}{3}$,*

*(b) if $E$ is decomposable, then $D$ is very ample if and only if $b + a\mu^-(E) \geq 3$ except when $E = E_1 \oplus E_2$, with $\operatorname{rk} E_1 = 1$, $\operatorname{rk} E_2 = 2$, $\deg E_2$ odd and $\deg E_1 > \frac{\deg E_2}{2}$. In the latter case the condition is only sufficient.*

PROOF. Firstly we consider the case $E$ indecomposable and normalized. By Lemma 2.7 and Proposition 2.10 only the cases $d = 1$ and $d = 2$ need to be considered.

CASE 1. $d = 1$.

Let $A$ be as in Lemma 4.2 and notice that $h^0(A) = \deg E + 3$. By Remark 2.6. We can assume $b+\frac{a}{3} = 1+\frac{\varepsilon}{3}$, where $\varepsilon = 1,2,3$. If $\varepsilon = 2,3$ then $(a-1)\mu^-(E)+b-1 = b+\frac{a}{3}-\frac{4}{3} > 0$. If $\varepsilon = 1$ Lemma 4.4 still allows the use of Lemma 4.1. Therefore Lemma 4.1 and 4.2 can be applied. Let $S$ be a smooth element in $|\mathcal{T} + \mathcal{F}_P|$. It is enough to check when $D|_S$ is very ample. By [6], Proposition 1.1, $S = \mathbb{P}(E')$ where $E'$ is a rank 2 vector bundle on the curve $C$, with $\deg E' = 2$, defined by the sequence

$$0 \longrightarrow O_C(-P) \longrightarrow E \longrightarrow E' \longrightarrow 0. \tag{7}$$

As Num($S$) is generated by $T'$ and $f'$, with $T' = T|_S$ and $f' = f|_S$, $D|_S \equiv aT' + bf'$ and moreover by Section 3, $D|_S$ is very ample in our range if and only if $b + a\mu^-(E') \geq 3$. If $E'$ is indecomposable then $\mu^-(E') = \mu(E') = 1$. If $E'$ is decomposable, *i.e.* $E' = H \oplus G$, both $\mu(H) \geq \mu(E) = \frac{1}{3}$ and $\mu(G) \geq \mu(E) = \frac{1}{3}$ because from (7) there exist non zero surjective morphisms $E \longrightarrow H$ and $E \longrightarrow G$.

Hence $\deg H = \deg G = 1$, $\mu(H) = \mu(G) = 1$ and $\mu^-(E') = 1$. In every case the condition $b + a\mu^-(E') \geq 3$ is satisfied in the range under consideration.

CASE 2. $d = 2$.

Let $A$ and $S$ be as in Lemma 4.1 and 4.2. In this case it is $h^0(X, A) = 5 \geq \deg E + 3$. By Remark 2.6 we can assume $b + a\frac{2}{3} = 1 + \frac{\varepsilon}{3}$ with $\varepsilon = 1,2,3$. If $\varepsilon = 3$ the hypothesis of Lemma 4.2 are satisfied. If $\varepsilon = 2$ Lemma 4.4 still allows the use of Lemma 4.1. Therefore it suffices to investigate the very ampleness of $D|_S$. Let $S = \mathbb{P}(E')$ with $E'$ a rank 2 vector bundle on $C$, with $\deg E' = 3$ defined again by (7). If $E'$ is indecomposable, $D|_S \equiv aT' + bf'$ is very ample if and only if $b + a\mu^-(E') > 1$, *i.e.* $b + \frac{3}{2}a > 1$ *i.e.* for all $D$ in the range under consideration.

If $E'$ is decomposable then $E' = H \oplus G$, with $\deg H$ and $\deg G \geq \mu(E) = \frac{2}{3}$. Hence we can assume $\deg H = 1$ and $\deg G = 2$. By Lemma 2.8, $\mu^-(E) = 1$ and the very ampleness condition is $b + a \geq 3$ which is satisfied by every $D$ in the range under consideration.

Notice that in the case $b + \frac{2}{3}a = \frac{4}{3}$, *i.e.* $\varepsilon = 1$, a very ampleness result for all $D$ in our range cannot be expected. For example, $D \sim 2\mathcal{T}$ is not very ample as $D|_Y$ is not very ample for each smooth surface $Y \in |\mathcal{T}|$ by Section 3.

If $E$ is indecomposable but not normalized, the result is obtained by similar computations (see (3)).

To prove (b), let now $E$ be decomposable. Firstly we prove the sufficient condition. By Proposition 2.2 it suffices to prove that $\pi_*(D)$ is very ample. Let us consider $\pi_*(D) = S^a(E) \otimes O_C(B) = \oplus_q A_q$ where $A_q$ is an indecomposable vector bundle on $C$ for all $q$. $S^a(E) \otimes O_C(B)$ is very ample if and only if $A_q$ is very ample for all $q$ *i.e.* if $\mu(A_q) \geq 3$ for all $q$. This condition is satisfied if $\min_q \mu(A_q) = \mu^-\big(S^a(E) \otimes O_C(B)\big) = b + a\mu^-(E) \geq 3$ which is what we wanted to show.

To prove the necessary condition, two cases will be considered:

i) $E$ is sum of three line bundles, $E = W \oplus G \oplus H$ respectively of degrees $w \leq g \leq h$. By Lemma 2.8 $\mu^-(E) = w$, and $d = \deg E = w + g + h$. From [6], Proposition 1.1 it follows that $\mathbb{P}(G \oplus H)$ is a subvariety of $X$ corresponding to a line bundle numerically equivalent to $T - wf$ while $\mathbb{P}(W)$ is an elliptic curve $\Gamma$ on $X$, isomorphic to $C$, which is numerically equivalent to $T^2 + xTf$, for some $x \in \mathbb{Z}$. As the cycles corresponding to $\mathbb{P}(W)$ and $\mathbb{P}(G \oplus H)$ do not intersect, from $(T^2 + xTf)(T - wf) = 0$ we get $x = -(g + h)$. If $D$ is a very ample line bundle on $X$, $D|_\Gamma$ is very ample, hence $D\Gamma = b + a\mu^-(E) \geq 3$.

ii) $E = H \oplus G$ where $\operatorname{rk} H = 1$, $\operatorname{rk} G = 2$, $h = \deg H$, $g = \deg G$. As in i), we get that $Z = \mathbb{P}(H)$ is numerically equivalent to $T^2 - gTf$ and the very ampleness of $D|_Z$ implies $b + ah \geq 3$. If $h \leq \frac{g}{2}$ this concludes the proof. Otherwise let us denote by $Y$ the smooth surface $\mathbb{P}(G)$. As usual $\operatorname{Num}(Y)$ is generated by $T' = T|_Y$ and $f' = f|_Y$. The very ampleness of $D$ implies the one of $D|_Y$ and by Section 3, $D|_Y$ is very ample if and only if

$b + a\frac{g}{2} \geq 3$ if $g$ is even and,

$b + a\frac{g}{2} > 1$ if $g$ is odd.

If $g$ is even, a necessary condition for the very ampleness of $D$ is $b + a\frac{g}{2} \geq 3$ *i.e.* $b + a\mu^-(E) \geq 3$ which is the desired condition.

If $g$ is odd, necessary conditions for the very ampleness of $D$ are both $b + ah \geq 3$ and $b + a\frac{g}{2} > 1$. Hence only the sufficient condition $b + a\mu^-(E) \geq 3$ is obtained in this case. ∎

5. **Rank $r$.** To deal with the case of $E$ vector bundle on an elliptic curve $C$, of rank $r > 3$ we need to recall first the following

THEOREM 5.1 ([8]). *Let $E$ be a vector bundle of* rank $r$ *on an elliptic curve $C$. $E$ is ample if and only if every indecomposable direct summand $E_i$ of $E$ has* $\deg E_i > 0$.

Our method based on Lemma 4.1, 4.2 and 4.3 forces us to investigate first the case $\deg E = 3$, then $\deg E = 1, 2$ and finally $\deg E \geq 4$.

5.1. $\deg E = 3$.

THEOREM 5.2. *Let $E$ be a vector bundle over an elliptic curve $C$, with* $\deg E = 3$ *and* $\operatorname{rank} E = 4$, *and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

*i) If $E$ is indecomposable, and $b + \frac{3}{4}a > \frac{3}{4}$, $D$ is very ample if and only if $b + a \geq 3$.*

*ii) If $E$ is decomposable and ample, and $b + \frac{a}{3} > \frac{1}{3}$, $D$ is very ample if $b + \frac{a}{2} > 2$.*

PROOF. i) If $\deg E = h^0(\mathcal{T}) = 3$ then we can apply Lemma 4.3 and 4.1 with $A = \mathcal{T}$, as $b + (a - 1)\mu^-(E) = b + \frac{3}{4}a - \frac{3}{4} > 0$ by hypothesis. Hence $D$ is very ample if and only if $D|_S$ is very ample, where $S$ is a suitable element of $|\mathcal{T}|$. There exists a vector bundle $E'$ with $\operatorname{rk} E' = 3$, $\deg E' = 3$, given by

$$0 \longrightarrow O_C \longrightarrow E \longrightarrow E' \longrightarrow 0, \tag{8}$$

such that $S = \mathbb{P}(E')$ and $\mathrm{Num}(S)$ is generated by $T'$ and $f'$, where $T' = T|_S$ and $f' = f|_S$ so that $D|_S \equiv aT' + bf'$.

By Section 4, $D|_S$ is very ample if and only if $b+a \geq 3$. Indeed if $E'$ is indecomposable the necessary and sufficient condition for the very ampleness is $b + a \geq 3$. If $E' = \bigoplus_i E'_i$ then $\mu(E'_i) \geq \frac{3}{4}$ for all $i$. Hence the only possibilities for a decomposition of $E'$ are:

1) $E' = F \oplus G \oplus H$, with $F, G, H$ line bundles all of degree 1.

2) $E' = H \oplus G$, with $\operatorname{rank} G = 2$, $\operatorname{rank} H = 1$, $\deg G = 2$, $\deg H = 1$.

Again by Section 4 in both the above cases, the necessary and sufficient condition for the very ampleness of $D|_S$ is $b + a \geq 3$.

ii) Let us suppose that $E$ is decomposable and ample. Then the possible decompositions for $E$ give $\mu^-(E) = \frac{2}{3}, \frac{1}{3}, \frac{1}{2}$. Note that the condition $b + \frac{a}{3} > \frac{1}{3}$ allows us to apply Lemma 4.3 with $A = \mathcal{T}$ in any case. If $S$ is the usual element of $|\mathcal{T}|$, we get that $D$ is very ample on $X$ if and only if $D|_S$ is. If $S = \mathbb{P}(E')$, $E'$ could be decomposable. In this case $\mu^-(E') \geq \frac{1}{2}$. Therefore the condition $b + \frac{a}{2} > 2$ guarantees the very ampleness of $D|_S$ by Section 4. ∎

Because $a \geq 2$, Theorem 5.2 immediately gives the following:

COROLLARY 5.3. *Let $E$ be an ample vector bundle over an elliptic curve $C$, with $\deg E = 3$ and $\operatorname{rank} E = 4$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$ with $b + \frac{a}{3} > \frac{1}{3}$. If $b + \frac{a}{2} > 2$ then $D$ is very ample.*

THEOREM 5.4. *Let $E$ be an ample vector bundle over an elliptic curve $C$, with $\deg E = 3$ and $r = \operatorname{rank} E \geq 4$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$ with $b + a\mu^-(E) > \frac{3}{5}$. Then $D$ is very ample if $b + \frac{a}{2} > 2$ and $b + \frac{a}{3} > \frac{1}{3}$.*

PROOF. Proceed by induction on $r = \operatorname{rank} E$. If $r = 4$ the inductive hypothesis is verified by Corollary 5.3. Let $r \geq 5$. It is $h^0(X, \mathcal{T}) = h^0(C, E) = 3$. Notice that $\mu^-(E) \leq \mu(E) \leq \frac{3}{r} \leq \frac{3}{5}$. Therefore $b + (a-1)\mu^-(E) > 0$ and Lemma 4.3 can be applied, with $A = \mathcal{T}$. Let $S = \mathbb{P}(E')$ be as in (8) with $\deg(E') = 3$ and $\mathrm{rk}(E') = r - 1 \geq 4$. $\mathrm{Num}(S)$ is generated by $T' = T|_S$ and $f' = f|_S$, so that $D|_S \equiv aT' + bf'$. Notice that $E'$ is ample being a quotient of $E$. Notice that $\mu^-(E) \leq \mu^-(E')$. Indeed there exists a map from at least one direct summand $E_k$ of $E$ and the summand $E'_j$ of $E'$ which realizes $\mu^-(E')$ and so we get $\mu^-(E) \leq \mu(E_k) \leq \mu(E_j) = \mu^-(E')$. As $b + a\mu^-(E') \geq b + a\mu^-(E) > \frac{3}{5}$, by induction $D|_S$ is very ample if $b + \frac{a}{2} > 2$, $b + \frac{a}{3} > \frac{1}{3}$. ∎

5.2. $\deg E = 2$.

THEOREM 5.5. *Let $E$ be an indecomposable vector bundle over an elliptic curve $C$, with $\deg E = 2$, $\mathrm{rk} E = r \geq 4$. Let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

*i) If $r = 4$, $D$ is very ample if $b + \frac{a}{2} > 2$.*

*ii) If $r \geq 5$, $D$ is very ample if $b + \frac{a}{2} > 2$, $b + \frac{a}{3} > \frac{1}{3}$ and $b + a\frac{2}{r} > 1 + \frac{1}{r}$.*

PROOF. Let $A$ be as in Lemma 4.2 and notice that $h^0(A) = 2 + r > \deg E + 3$. By Remark 2.6 and the assumptions in i) and ii) we can assume

$$b + a\frac{2}{r} = 1 + \frac{\varepsilon}{r}, \tag{9}$$

where $\varepsilon = 1, 2 \ldots, r$. If $b + a\frac{2}{r} > 1 + \frac{2}{r}$ the assumptions of Lemma 4.1 and 4.2 are satisfied. If $b + a\frac{2}{r} = 1 + \frac{2}{r}$ the line bundle $D - A$ has degree 0 and Lemma 4.4 shows that Lemma 4.1 can be used if

$$b + a\frac{2}{r} \geq 1 + \frac{2}{r}. \tag{10}$$

Therefore condition (10) can be rewritten using (9) as

$$b + a\frac{2}{r} > 1 + \frac{1}{r}. \tag{11}$$

Let $S = \mathbb{P}(E')$, where $E'$ is as in (7), where $\deg E' = 3$, $\operatorname{rk} E' = r - 1 \geq 3$, $\operatorname{Num}(S)$ is generated by $T' = T|_S$ and $f' = f|_S$, and $D|_S \equiv aT' + bf'$. Notice that $E'$ is indecomposable or decomposable and ample because $\mu^-(E') \geq \mu^-(E) = \frac{2}{r} > 0$.

To prove i) assume $r = 4$. In this case $\operatorname{rk} E' = 3$ and $\deg E' = 3$. If $E'$ is indecomposable, as $b + a\mu^-(E') > b + a\mu^-(E) = b + \frac{a}{2} > 1$, by Theorem 4.5 $D|_S$ is very ample if $b + a \geq 3$. By the same theorem if $E'$ is decomposable and ample then $D|_S$ is very ample if $b + a\mu^-(E') > 2$. The possible values for $\mu^-(E')$ are 1 or $\frac{1}{2}$. Therefore by Theorem 4.5 it follows that $D|_S$ is very ample if

$$b + \frac{a}{2} > 2. \tag{12}$$

Putting (11) and (12) together it follows that in the case $r = 4$ $D$ is very ample if $b + \frac{a}{2} > \frac{5}{4}$ and $b + \frac{a}{2} > 2$ *i.e.* $b + \frac{a}{2} > 2$.

To prove ii) assume $r - 1 \geq 4$. Since $b + a\mu^-(E') \geq b + a\mu^-(E) = b + a\frac{2}{r} > 1 + \frac{1}{r} > \frac{3}{5}$ from Theorem 5.4 it follows that $D|_S$ is very ample if $b + \frac{a}{2} > 2$ and $b + \frac{a}{3} > \frac{1}{3}$. Therefore if $r \geq 5$, $D$ is very ample if $b + \frac{a}{2} > 2$, $b + \frac{a}{3} > \frac{1}{3}$ and $b + a\frac{2}{r} > 1 + \frac{1}{r}$. ∎

THEOREM 5.6. *Let $E$ be a decomposable and ample vector bundle of* rank $r \geq 4$ *over an elliptic curve $C$, with* $\deg E = 2$. *Let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

*i) If $r = 4$, $D$ is very ample if $b + \frac{a}{2} > 2$ and $b + a\mu^-(E) > \frac{3}{2}$.*
*ii) If $r \geq 5$, $D$ is very ample if $b + a\mu^-(E) > 1 + \frac{2}{r}$, $b + \frac{a}{3} > \frac{1}{3}$ and $b + \frac{a}{2} > 2$.*

PROOF. Let $A$ be as in Lemma 4.2 and notice that $h^0(A) = r + 2 \geq \deg E + 3$. Also notice that $\mu^-(E) \leq \mu(E) = \frac{2}{r}$. In our hypothesis we have $b + (a - 1)\mu^-(E) - 1 \geq b + a\mu^-(E) - \frac{2}{r} - 1 > 0$.

Therefore Lemma 4.1 and 4.2 can be applied. Let $S = \mathbb{P}(E')$, where $E'$ is as in (7) where $\deg E' = 3$ and $\operatorname{rk}(E') = r - 1 \geq 3$. If $r \geq 5$ we can apply Theorem 5.4 to $D|_S$. Indeed $b + a\mu^-(E') \geq b + a\mu^-(E) > 1 + \frac{2}{r} > \frac{3}{5}$. Hence $D|_S$ is very ample if $b + \frac{a}{3} > \frac{1}{3}$ and $b + \frac{a}{2} > 2$.

If $r = 4$ we can apply Theorem 4.5. If $E'$ is indecomposable then $D|_S$ is very ample if $b+a \geq 3$. If $E'$ is decomposable, noticing that $E'$ is ample, the condition is $b+a\mu^-(E') > 2$. In this case $\mu^-(E')$ can be 1 or $\frac{1}{2}$ and hence the worst sufficient condition becomes $b + \frac{a}{2} > 2$. ∎

Theorem 5.5 and 5.6 give the following corollary.

COROLLARY 5.7. *Let $E$ be an ample vector bundle over an elliptic curve $C$, with $\deg E = 2$ and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

*i) If $\operatorname{rank} E \geq 5$, then $D$ is very ample if $b+a\mu^-(E) > 1+\frac{2}{r}$, $b+\frac{a}{2} > 2$ and $b+\frac{a}{3} > \frac{1}{3}$.*
*ii) If $\operatorname{rank} E = 4$, $D$ is very ample if $b + a\mu^-(E) > \frac{3}{2}$ and $b + \frac{a}{2} > 2$.*

5.3. $\deg E = 1$. Note that if $\deg E = 1$, $E$ is ample if and only if it is indecomposable.

THEOREM 5.8. *Let $E$ be an indecomposable vector bundle over an elliptic curve $C$, with $\deg E = 1$, $\operatorname{rk} E = r \geq 4$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$ with $b + \frac{a}{r} > 1$. Then the following conditions hold:*

*i) If $r = 4$, $D$ is very ample if $b + \frac{a}{2} > 2$.*
*ii) If $r = 5$, $D$ is very ample if $b + \frac{a}{3} > \frac{3}{2}$ and $b + \frac{a}{2} > 2$.*
*iii) If $r \geq 6$, $D$ is very ample if $b + \frac{a}{r-2} > 1 + \frac{2}{r-1}$ and $b + \frac{a}{2} > 2$.*

PROOF. In our hypothesis it is $(a-1)\frac{1}{r} + b - 1 > 0$. Lemma 4.4 allows us to apply Lemma 4.1 and 4.2 when $(a-1)\frac{1}{r} + b - 1 \geq 0$, noticing that if $A = \mathcal{T} + \mathcal{F}_P$ it is $h^0(A) = 1 + r \geq 5$. If $S = \mathbb{P}(E')$, where $E'$ is as in (7), notice that $E'$ is ample because $E$ is. It is $\operatorname{rk} E' = r - 1 \geq 3$ and $\deg E' = 2$. Let us now distinguish the three cases according to the values of $r$. Theorem 5.4 and Corollary 5.7 will be used.

i) If rank $E = 4$, rank $E' = 3$, and $D|_S$ is very ample if $b + \frac{2}{3}a > \frac{4}{3}$ when $E'$ is indecomposable while if $E'$ is decomposable and ample the condition is $b+a\mu^-(E') > 2$. In the worst case $\mu^-(E') = \frac{1}{2}$ so we have $b + \frac{a}{2} > 2$.

ii) If rank $E = 5$, rank $E' = 4$, and $D|_S$ is very ample if $b+a\mu^-(E') > \frac{3}{2}$, and $b+\frac{a}{2} > 2$. As $E'$ is indecomposable or decomposable and ample, in the worst case $\mu^-(E') = \frac{1}{3}$ and so it is enough to ask $b + \frac{a}{3} > \frac{3}{2}$ and $b + \frac{a}{2} > 2$.

iii) If rank $E \geq 6$ then rank $E' \geq 5$, hence by Corollary 5.7 $D|_S$ is very ample if $b+a\mu^-(E') > 1+\frac{2}{r-1}$, $b+\frac{a}{3} > \frac{1}{3}$ and $b+\frac{a}{2} > 2$. If $E'$ is indecomposable then $\mu^-(E') = \frac{2}{r-1}$ while if $E'$ is decomposable and ample, then $\mu^-(E') = \min(\frac{1}{s}, \frac{1}{r-1-s})$ with $s = 1 \cdots r-2$. So the condition $b + a\mu^-(E') > 1 + \frac{2}{r-1}$ can be substituted by $b + \frac{a}{r-2} > 1 + \frac{2}{r-1}$ which implies the condition $b + \frac{a}{3} > \frac{1}{3}$. ∎

5.4. $\deg E \geq 4$.

THEOREM 5.9. *Let $E$ be an ample vector bundle over an elliptic curve $C$, with $\deg E = d \geq 4$, $\operatorname{rk} E = r \geq 4$ and $d < r$. Let $D \equiv aT + bf$ be a line bundle on*

*$X = \mathbb{P}(E)$. If $b + \frac{a}{d-1} > 2$ and $b + (a-1)\mu^-(E) > 0$ then $D$ is very ample.*

PROOF. The proof proceeds by induction on $r$. If $r = 4$ the smallest possible value of $\mu^-(E)$ is $\frac{1}{3}$. Since $d \geq 4$ it is $\frac{1}{d-1} \leq \frac{1}{3}$. Therefore $b + a\mu^-(E) \geq b + \frac{a}{3} \geq b + \frac{a}{d-1} > 2$. Lemma 2.11 and Theorem 2.5 conclude the proof of the initial inductive step. Let us suppose the statement true for $r-1$ and prove it for $r$. Let $A$ be as in Lemma 4.3 and notice that $h^0(A) = d \geq 4$. Therefore Lemma 4.3 and 4.1 can be applied. By considering $S = \mathbb{P}(E')$ where $E'$ is as in (8), we get that $E'$ is ample with $\deg E' = d \geq 4$, $\operatorname{rk} E' = r' = r - 1 \geq 4$. Because $\mu^-(E') \geq \mu^-(E)$ it is $b + (a-1)\mu^-(E') \geq b + (a-1)\mu^-(E) > 0$ and again $b + \frac{a}{d-1} > 2$. Hence we can conclude by induction hypothesis. ∎

REMARK 5.10. Note that in the above theorem, if $E$ is indecomposable, by normalizing $E$ we can always assume $d < r$.

In a very particular case we can say a little more:

PROPOSITION 5.11. *Let $E$ be an indecomposable vector bundle over an elliptic curve $C$, with $\deg E = d \geq 4$, $\operatorname{rk} E = d + 1$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. If $b + (a-1)\frac{d}{d+1} > 0$ and $b + a > 2$ then $D$ is very ample.*

PROOF. It is $h^0(X, \mathcal{T}) = d \geq 4$. Our hypothesis $b + (a-1)\frac{d}{d+1} > 0$ shows that Lemma 4.1 and 4.3 can be applied.

By considering $S = \mathbb{P}(E')$, where $E'$ is as in (8) it is $\deg E' = \operatorname{rk} E' = d$. A sufficient condition for the very ampleness of $D|_S$ is $b + a\mu^-(E') > 2$ by Lemma 2.11 and Theorem 2.5. We claim that in this case $\mu^-(E') = 1$. If $E'$ is indecomposable it is $\mu^-(E') = \mu(E') = 1$. If $E'$ is decomposable we can suppose that $E' = \oplus G_j$, with $r_j = \operatorname{rk} G_j \geq 1$, $d_j = \deg G_j \geq 1$ (as $\mu(G_j) \geq \mu(E) = \frac{d}{d+1}$) and $\sum d_j = \sum r_j = d$. Hence we have $\frac{d_j}{r_j} \geq \frac{d}{d+1}$, for all $j$, which implies $r_j - d_j \leq \frac{d_j}{d} < 1$, for all $j$. Hence $r_j - d_j \leq 0$, for all $j$ *i.e.* $d_j = r_j + s_j$ with $s_j \geq 0$, for all $j$. But $d = \sum d_j = \sum(d_j + s_j) = d + \sum s_j$ and $\sum s_j = 0$ *i.e.* $s_j = 0$, for all $j$. Hence $\mu^-(E') = 1$. ∎

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