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ELEMENTARY OPERATORS AND SUBHOMOGENEOUS C^*-ALGEBRAS

ILJA GOGIĆ

Department of Mathematics, University of Zagreb, Bijenička cesta 30, Zagreb 10000, Croatia (ilja@math.hr)

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Abstract Let A be a C^*-algebra and let Θ_A be the canonical contraction form the Haagerup tensor product of $M(A)$ with itself to the space of completely bounded maps on A. In this paper we consider the following conditions on A: (a) A is a finitely generated module over the centre of $M(A)$; (b) the image of Θ_A is equal to the set $\mathrm{E}(A)$ of all elementary operators on A; and (c) the lengths of elementary operators on A are uniformly bounded. We show that A satisfies (a) if and only if it is a finite direct sum of unital homogeneous C^*-algebras. We also show that if a separable A satisfies (b) or (c), then A is necessarily subhomogeneous and the C^*-bundles corresponding to the homogeneous subquotients of A must be of finite type.

Keywords: C^*-algebra; subhomogeneous; elementary operators

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1. Introduction and preliminaries

Throughout this paper A will denote a C^*-algebra and A_h will denote the self-adjoint part of A. The centre of A is denoted by $Z(A)$, and the set of all ideals of A is denoted by $\mathrm{Id}(A)$ (in this paper, by an ideal we shall always mean a closed two-sided ideal). For $J \in \mathrm{Id}(A)$, we denote the annihilator of J by $J^\perp$ (i.e. $J^\perp = \{x \in A \colon xJ = 0\}$) and we denote the quotient map $A \to A/J$ by q_J^A. By $\hat{A}$ and $\mathrm{Prim}(A)$, respectively, we denote the spectrum of A (i.e. the set of all classes of irreducible representations of A) and the primitive spectrum of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology.

Let $A \otimes_h A$ be the Haagerup tensor product of A with itself. If B is a C^*-subalgebra of A, we shall always assume that $B \otimes_h B \subseteq A \otimes_h A$, by the injectivity of the Haagerup tensor product [**11**, Proposition 1.4.3]. If $M(A)$ denotes the multiplier algebra of A and $\mathrm{ICB}(A)$ denotes the space of all completely bounded maps $T \colon A \to A$ that preserve every ideal of A (i.e. $T(J) \subseteq J$, for each $J \in \mathrm{Id}(A)$), then there is a canonical contraction $\Theta_A \colon M(A) \otimes_h M(A) \to \mathrm{ICB}(A)$ that is given on elementary tensors by

$$\Theta_A(a \otimes b)(x) := axb \quad (a, b \in M(A),\ x \in A).$$

The subspace $\Theta_A(M(A) \otimes M(A))$ of $\operatorname{Im}\Theta_A$ (i.e. the image of Θ_A) is denoted by $\mathrm{E}(A)$: the set of *elementary operators* on A. The *length* $\ell(T)$ of $T \in \mathrm{E}(A)$ is defined as the smallest number d such that

$$T = \Theta_A\biggl(\sum_{k=1}^{d} a_k \otimes b_k\biggr),$$

for some $a_k, b_k \in M(A)$. If $\sup\{\ell(T)\colon T \in \mathrm{E}(A)\} < \infty$, we say that $\mathrm{E}(A)$ is of *finite length*.

The operators that belong to $\operatorname{Im}\Theta_A$ have some nice properties. For example, each $T \in \operatorname{Im}\Theta_A$ is a strong central bimodule homomorphism of A (i.e. T^{**} preserves every ideal of A^{**}, where A^{**} denotes the von Neumann envelope of A, and T^{**} is the second adjoint of T on A^{**} (see [**1**, § 5.3])). But in general it is difficult to recognize when the operator $T \in \mathrm{ICB}(A)$ belongs to $\operatorname{Im}\Theta_A$. In [**8**] Magajna considered the problem when the image $\operatorname{Im}\Theta_A$ is as large as possible, and hence equal to the space $\mathrm{ICB}(A)$. He showed that (for a separable A) this happens if and only if A is a finite direct sum of homogeneous C^*-algebras of finite type, and then we have $\mathrm{ICB}(A) = \mathrm{E}(A)$. Recall that A is said to be *n-homogeneous* if all irreducible representations of A are of the same (finite) dimension n. In this case, by [**3**], $\Delta := \mathrm{Prim}(A) = \hat{A}$ is a (locally compact) Hausdorff space, and there exists a locally trivial C^*-bundle E over Δ with fibres isomorphic to $\mathrm{M}_n(\mathbb{C})$ such that A is isomorphic to the C^*-algebra $\Gamma_0(E)$ of all continuous sections of E that vanish at infinity. If Δ admits a finite open covering $\{U_i\}_{1\leqslant i\leqslant m}$ such that each restriction bundle $E|_{U_i}$ is trivial (i.e. isomorphic to $U_i \times \mathrm{M}_n(\mathbb{C})$, as a C^*-bundle), then we say that E, and hence A, are of *finite type*. Note that if A is a finite direct sum of homogeneous C^*-algebras of finite type, then so is $M(A)$ (by [**1**, Lemma 1.2.21] and [**8**, Remark 3.3]), and hence, by the Serre–Swan Theorem [**7**, Theorem I.6.18], $M(A)$ is a finitely generated (projective) module over its centre. It is then easy to check that for such A, $\mathrm{E}(A)$ must be of finite length (see the proof of Theorem 2.7).

In this paper we consider the following conditions on A:

(a) A is a finitely generated module over $Z(M(A))$;

(b) $\mathrm{E}(A)$ is of finite length; and

(c) $\operatorname{Im}\Theta_A$ is as small as possible; that is, it is equal to $\mathrm{E}(A)$.

We show that A satisfies (a) if and only if A is a finite direct sum of unital homogeneous C^*-algebras. If A satisfies (b) or (c), we show that A is necessarily subhomogeneous. Recall that A is said to be *n-subhomogeneous* ($n \in \mathbb{N}$) if $\sup\{\dim\pi\colon [\pi] \in \hat{A}\} = n$. In this case, by [**9**, 6.2.5], A has a finite *composition series*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_p = A$$

of ideals of A such that each quotient J_i/J_{i-1} is a homogeneous C^*-algebra. The ideal J_1 is called the *n-homogeneous ideal* of A (since it is the largest ideal of A that is n-homogeneous, as a C^*-algebra). We also show that if A is separable and satisfies (b) or (c), then each quotient J_i/J_{i-1} is of finite type. In this case we say that A is *subhomogeneous of finite type*. Note that in a separable and unital case, this notion coincides with Phillips's notion [**10**] of *recursive subhomogeneous algebra*. We also conjecture that conditions (b) and (c) are equivalent, but we were not able to prove this.

Finally, we state the next two results, which will be very useful to us. The first is a combination of the results in [**10**, Proposition 2.9], [**8**, Lemma 3.2] and [**8**, Remark 3.3].

Theorem 1.1. *Let E be a locally trivial C^*-bundle over a locally compact Hausdorff space Δ with fibres $\mathrm{M}_n(\mathbb{C})$. The following conditions are equivalent:*

(i) *E is of finite type (as a C^*-bundle);*

(ii) *E is of finite type when regarded as a complex vector bundle (of rank n^2) over Δ, by forgetting the additional structure.*

In this case E can be extended to the locally trivial C^*-bundle F over the Stone–Čech compactification $\beta\Delta$ of Δ, and $\Gamma(F) \cong \Gamma_{\mathrm{b}}(E) = M(\Gamma_0(E))$, where $\Gamma_{\mathrm{b}}(E)$ denotes the C^*-algebra of all continuous bounded sections of E. Specifically, the multiplier algebra of an n-homogeneous C^*-algebra of finite type is also n-homogeneous.

Hence, to show that an $\mathrm{M}_n(\mathbb{C})$-bundle E is of finite type, it is sufficient to check that the underlying n^2-dimensional vector bundle is of finite type. The next lemma gives a useful way of doing this, and its proof can be found in [**8**, § 1].

Lemma 1.2. *Let E be a locally trivial vector bundle of constant rank over a paracompact Hausdorff space Δ. The following conditions are equivalent:*

(i) *E is of finite type;*

(ii) *there exists a finite subset $\{a_1, \dots, a_m\}$ of $\Gamma_{\mathrm{b}}(E)$ such that*

$$\operatorname{span}\{a_1(s), \dots, a_m(s)\} = E(s) \quad \text{for all } s \in \Delta.$$

2. Results

Definition 2.1. Let A be a C^*-algebra. We say that A is *finitely centrally generated* if A, as a module over $Z(M(A))$, is finitely generated.

Lemma 2.2. *Suppose that A is a finitely centrally generated C^*-algebra. Then A is unital.*

Proof. Let $e_1, \dots, e_m \in A$ be the generators of A as a module over $Z(M(A))$. Since A is the linear span of A_h, we may assume that each e_i is self-adjoint. Let

$$a := \left(\sum_{k=1}^{m} e_k^2\right)^{1/4}.$$

Note that

$$e_i e_i^* = e_i^2 \leqslant \sum_{k=1}^{m} e_k^2 = a^4 \quad \text{for all } 1 \leqslant i \leqslant m.$$

By [**6**, Exercise 4.6.39] there exist elements $b_1, \dots, b_m \in A$ such that

$$e_i = ab_i \quad \text{for all } 1 \leqslant i \leqslant m. \tag{2.1}$$

Let $x \in A$. By assumption, there exist elements $z_1(x), \dots, z_m(x) \in Z(M(A))$ such that $x = \sum_{k=1}^m z_k(x)e_k$. Using (2.1) it follows that

$$x = \sum_{k=1}^m z_k(x)e_k = \sum_{k=1}^m z_k(x)ab_k = a\sum_{k=1}^m z_k(x)b_k. \tag{2.2}$$

Specifically, for $x = a$ and $e := \sum_{k=1}^m z_k(a)b_k$ we have $a = ae = e^*a$. By (2.2), $e^*x = x$ for all $x \in A$, and hence e^* is a left unit of A. After taking adjoints we also conclude that e is a right unit of A. Hence, A is unital with the unit $e = e^*$. □

Remark 2.3. Suppose that A is a finitely centrally generated C^*-algebra with $A = \operatorname{span}_{Z(A)}\{e_1, \dots, e_m\}$, for some $e_i \in A$ (by Lemma 2.2, A is unital). Then, for each $J \in \operatorname{Id}(A)$ we have $A/J = \operatorname{span}_{Z(A/J)}\{q_J^A(e_1), \dots, q_J^A(e_m)\}$ (since $q_J^A(Z(A)) \subseteq Z(A/J)$) and, specifically, A/J is finitely centrally generated.

Theorem 2.4. *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) *A is finitely centrally generated;*

(ii) *A is a finite direct sum of unital homogeneous C^*-algebras.*

Proof. (ii) ⇒ (i). It is sufficient to prove this in the case when A is unital and homogeneous. Let E be a locally trivial C^*-bundle over the (compact) space $\Delta := \operatorname{Prim}(A)$ with fibres isomorphic to $\mathrm{M}_n(\mathbb{C})$ such that $A \cong \Gamma(E)$. By the Serre–Swan Theorem [**7**, Theorem I.6.18], $\Gamma(E)$ is a finitely generated (projective) module over $C(\Delta) \cong Z(\Gamma(E))$.

(i) ⇒ (ii). By Lemma 2.2 A is unital. Suppose that

$$A = \operatorname{span}_{Z(A)}\{e_1, \dots, e_m\} \quad \text{for some } e_1, \dots, e_m \in A. \tag{2.3}$$

Claim 1. A is n-subhomogeneous, where $n \leqslant \sqrt{m}$.

Indeed, if $\pi\colon A \to \mathfrak{B}(\mathcal{H}_\pi)$ is an irreducible representation of A on a Hilbert space $\mathcal{H}_\pi$, then $\pi(Z(A)) = \mathbb{C}1_{\mathcal{H}}$, and hence $\pi(A) = \operatorname{span}\{\pi(e_1), \dots, \pi(e_m)\}$. Since π is irreducible, it follows that $\dim \pi = \dim \mathcal{H}_\pi \leqslant \sqrt{m}$.

Claim 2. If J is the n-homogeneous ideal of A, then J is unital, and hence $A \cong J \oplus (A/J)$.

Let E be the locally trivial C^*-bundle over $\Delta := \operatorname{Prim}(J)$, with fibres $\mathrm{M}_n(\mathbb{C})$, such that $J = \Gamma_0(E)$. To prove that J is unital, it is sufficient to show this in the case when J is essential in A. Indeed, if J is not essential, then we can substitute A with $B := A/J^\perp$, since, by Remark 2.3, B is also finitely centrally generated and, by [**8**, Lemma 3.1], the n-homogeneous ideal K of B is essential in B, and $K \cong J$. Next, we note that J is of finite type. Indeed, since J is essential in A, we have $J \subseteq A \subseteq M(J)$. By [**8**, Lemma 3.2], $M(J) = \Gamma_{\mathrm{b}}(E)$. Since $E(s) = \Gamma_0(E)(s)$, for each $s \in \Delta$, (2.3) implies that

$$E(s) = \operatorname{span}\{e_1(s), \dots, e_m(s)\} \quad \text{for all } s \in \Delta.$$

By Lemma 1.2, E is of finite type as a vector bundle, and hence, by Theorem 1.1, E is of finite type as a C^*-bundle. Theorem 1.1 also implies that E can be extended to a locally trivial C^*-bundle F over the Stone–Čech compactification $\beta\Delta$ of Δ, and that $M(J) = \Gamma_{\mathrm{b}}(E) = \Gamma(F)$. Suppose that J is not unital, from which it follows that $J \neq A$. Then Δ is not compact and hence $\beta\Delta \setminus \Delta \neq \emptyset$. Using [**4**, Lemma VII.8.7] and considering the characters of the centre $Z(J) \cong C_0(\Delta)$, we have

$$J = \{a \in \Gamma(F) \colon a|_{\beta\Delta\setminus\Delta} = 0\}. \tag{2.4}$$

For each point $s \in \beta\Delta$ let π_s denote the evaluation of sections of F at s, considered as an irreducible representation of $M(J) = \Gamma(F)$. The restriction $\pi_s|_A$ at all points of $\beta\Delta \setminus \Delta$ then gives a reducible representation of A. Let r be the maximal dimension of an irreducible subrepresentation of $\pi_s|_A$ as s runs through $\beta\Delta \setminus \Delta$, and let s_0 be any point at which this maximum is achieved. Note that $r > 0$. Indeed, since $J \neq A$ there exists an irreducible representation σ of A such that $\sigma(J) = \{0\}$. Then σ is unitarily equivalent to a subrepresentation of $\pi_s|_A$ for some $s \in \beta\Delta$. Since $\sigma(J) = \{0\}$, $s \notin \Delta$. By local triviality of F, there exists a compact neighbourhood K of s_0 in $\beta\Delta$ such that $F|_K \cong K \times \mathrm{M}_n(\mathbb{C})$. Using a fixed isomorphism we shall identify these two bundles over K. By the definition of s_0, and by the same arguments as in the first part of the proof of [**8**, Lemma 4.1], there exists a compact neighbourhood H of s_0 such that $H \subseteq K$ and (after conjugation with a unitary element of $C(H, \mathrm{M}_n(\mathbb{C}))$, if necessary) such that $\pi_s|_A$ has the form

$$\pi_s(a) = \begin{pmatrix} \sigma_s(a) & 0 \\ 0 & \rho_s(a) \end{pmatrix} \quad \text{for all } a \in A \text{ and } s \in H \setminus \Delta, \tag{2.5}$$

where $\sigma_s \colon A \to \mathrm{M}_r(\mathbb{C})$ is an irreducible representation and $\rho_s \colon A \to \mathrm{M}_{n-r}(\mathbb{C})$ is a representation of A (which is non-degenerate, but we will not need to use this fact). Let $U := H \cap \Delta$ and let

$$I_H := \{a \in M(J) \colon a|_H\} = 0, \quad A_H := A/(A \cap I_H) \quad \text{and} \quad J_H := J/(J \cap I_H).$$

Since $H \subseteq K$, A_H can be identified with a C^*-subalgebra of $C(H, \mathrm{M}_n(\mathbb{C})) \cong \Gamma(F|_H) \cong M(J)/I_H$ (the latter isomorphism follows from the Tietze Extension Theorem for sections of Banach bundles [**4**, Theorem II.14.8]). Note that by (2.4) J_H is the ideal of $C(H, \mathrm{M}_n(\mathbb{C}))$ (and of A_H) consisting of all $a \in C(H, \mathrm{M}_n(\mathbb{C}))$ such that $a|_{H\setminus U} = 0$. Since U is a dense and open subset of H, J_H is essential in $C(H, \mathrm{M}_n(\mathbb{C}))$. Under all these identifications, it follows from (2.5) that

$$a_{1,n}|_{H\setminus U} = 0 \quad \text{for all } a = (a_{i,j})_{1 \leqslant i,j \leqslant n} \in A_H. \tag{2.6}$$

If $e^{(k)} := q^A_{A\cap I_H}(e_k)$ $(1 \leqslant k \leqslant m)$, by Remark 2.3 A_H is also finitely centrally generated, with $A_H = \mathrm{span}_{Z(A_H)}\{e^{(1)}, \ldots, e^{(m)}\}$. Let $(E_{i,j})_{1\leqslant i,j\leqslant n}$ be the standard matrix units of $\mathrm{M}_n(\mathbb{C})$ considered as the constant elements of $C(H, \mathrm{M}_n(\mathbb{C}))$. Then for each function $f \in C_0(U) = \{g \in C(H) \colon g|_{H\setminus U} = 0\}$ the element $a_f := fE_{1,n}$ lies in $J_H \subseteq A_H$, so there exist the central elements $z_k \in Z(A_H)$ $(1 \leqslant k \leqslant m)$ such that $a_f = \sum_{k=1}^{m} z_k e^{(k)}$. Since $J_H \subseteq A_H$ is essential in $C(H, \mathrm{M}_n(\mathbb{C}))$, we have

$$Z(A_H) \subseteq Z(C(H, \mathrm{M}_n(\mathbb{C}))) = \{g1_n \colon g \in C(H)\}.$$

Hence, if $z_k = g_k 1_n$ $(g_k \in C(H),\ 1 \leqslant k \leqslant m)$, it follows that $f = \sum_{k=1}^{m} g_k e_{1,n}^{(k)}$. By (2.6), we have $e_{1,n}^{(k)} \in C_0(U)$ for all $1 \leqslant k \leqslant m$. Since $f \in C_0(U)$ was arbitrary, it follows that

$$C_0(U) = \mathrm{span}_{C(H)}\{e_{1,n}^{(1)}, \dots, e_{1,n}^{(m)}\} = \mathrm{span}_{C(\beta U)}\{e_{1,n}^{(1)}, \dots, e_{1,n}^{(m)}\}.$$

By Lemma 2.2, $C_0(U)$ is unital, so U is compact and hence equal to H, contradicting the fact that $s_0 \in H \setminus U$.

Claim 3. A is a finite direct sum of unital homogeneous C^*-algebras.

This follows directly by induction, from Claim 2 and Remark 2.3. □

We now concentrate on conditions (b) and (c) from § 1. We shall find the next auxiliary result very useful below.

Lemma 2.5. *Let A be a C^*-algebra. Let (a_k), (b_k) and (e_k) be sequences in $M(A)$ such that $e_k \in M(A)_h$ for all $k \in \mathbb{N}$, and such that the series $\sum_{k=1}^{\infty} a_k a_k^*$, $\sum_{k=1}^{\infty} b_k^* b_k$ and $\sum_{k=1}^{\infty} e_k^2$ are norm convergent. Let t and u be the tensors in $M(A) \otimes_h M(A)$ defined by $t := \sum_{k=1}^{\infty} e_k \otimes e_k$ and $u := \sum_{k=1}^{\infty} a_k \otimes b_k$. If $\Theta_A(t) = \Theta_A(u)$, then*

$$\overline{\mathrm{span}}\{q_P^{M(A)}(e_k) \colon k \in \mathbb{N}\} \subseteq \overline{\mathrm{span}}\{q_P^{M(A)}(b_k) \colon k \in \mathbb{N}\} \quad \text{for all } P \in \mathrm{Prim}(M(A)),$$

where $\overline{\mathrm{span}}$ denotes the closed linear span.

Proof. Suppose that $A \subseteq M(A) \subseteq A^{**}$, where A^{**} denotes the von Neumann envelope of A. Note that (by ultraweak continuity) the equality $\Theta_A(t) = \Theta_A(u)$ (of operators on A) implies the equality $\Theta_{A^{**}}(t) = \Theta_{A^{**}}(u)$ (of operators on A^{**}), and hence the equality

$$\Theta_{M(A)}(t) = \Theta_{A^{**}}(t)|_{M(A)} = \Theta_{A^{**}}(u)|_{M(A)} = \Theta_{M(A)}(u).$$

Therefore, we may assume that A is unital. Let $P \in \mathrm{Prim}(A)$. Since for each $J \in \mathrm{Id}(A)$ the diagram

$$\begin{array}{ccc} A \otimes_h A & \xrightarrow{\Theta_A} & \mathrm{ICB}(A) \\ {\scriptstyle q_J^A \otimes q_J^A}\downarrow & & \downarrow{\scriptstyle Q_J^A} \\ A/J \otimes_h A/J & \xrightarrow{\Theta_{A/J}} & \mathrm{ICB}(A/J) \end{array}$$

commutes, where Q_J^A denotes the induced map $\mathrm{ICB}(A) \to \mathrm{ICB}(A/J)$ defined by

$$Q_J^A(T)(q_J^A(x)) := q_J^A(T(x)) \quad \text{for all } T \in \mathrm{ICB}(A) \text{ and } x \in A, \tag{2.7}$$

the equality $\Theta_A(t) = \Theta_A(u)$ implies the equality $\Theta_{A/P}(q_P^A \otimes q_P^A(t)) = \Theta_{A/P}(q_P^A \otimes q_P^A(u))$. Since A/P is primitive, by [**1**, Corollary 5.4.10] $\Theta_{A/P}$ is an isometry (and hence injective), so the latter equality implies the equality $q_P^A \otimes q_P^A(t) = q_P^A \otimes q_P^A(u)$ of tensors in $A/P \otimes_h$

A/P. To simplify the notation, let $\dot{A} := A/P$, $\dot{x} := q_P^A(x)$ $(x \in A)$ and $\dot{v} := q_P^A \otimes q_P^A(v)$ $(v \in A \otimes_h A)$. Put

$$V := \overline{\overline{\operatorname{span}}}\{\dot{e}_k \colon k \in \mathbb{N}\} \quad \text{and} \quad W := \overline{\overline{\operatorname{span}}}\{\dot{b}_k \colon k \in \mathbb{N}\}.$$

To prove that $V \subseteq W$, it is sufficient to prove that each (bounded) linear functional on $\dot{A}$ that annihilates W also annihilates V. So, let $\varphi \in \dot{A}^*$ such that $\varphi|_W = 0$. If we first act on the equality

$$\sum_{k=1}^{\infty} \dot{e}_k \otimes \dot{e}_k = \dot{t} = \dot{u} = \sum_{k=1}^{\infty} \dot{a}_k \otimes \dot{b}_k$$

with the left slice map L_φ (see [**12**, § 4]) and then with the linear functional $\dot{x} \mapsto \overline{\varphi(\dot{x}^*)}$, we obtain

$$\sum_{k=1}^{\infty} |\varphi(\dot{e}_k)|^2 = \sum_{k=1}^{\infty} \overline{\varphi(\dot{a}_k^*)}\varphi(\dot{b}_k) = 0,$$

since $\varphi|_W = 0$. It follows that $\varphi(\dot{e}_k) = 0$ for all $k \in \mathbb{N}$, and hence $\varphi|_V = 0$. □

Theorem 2.6. *Let A be a C^*-algebra. If A satisfies one of the conditions*

(i) $\mathrm{E}(A)$ *is of finite length, or*

(ii) $\operatorname{Im} \Theta_A = \mathrm{E}(A)$,

then A is subhomogeneous.

Proof. Since A satisfies (i) or (ii) if and only if $M(A)$ satisfies (i) or (ii) (respectively), we may assume that A is unital. Suppose that $\sup\{\dim A/P \colon P \in \operatorname{Prim}(A)\} = \infty$, and let (P_n) be a sequence in $\operatorname{Prim}(A)$ such that $\dim A/P_n \geqslant n$, for all $n \in \mathbb{N}$ (if some primitive quotient A/P is infinite dimensional, we may put $P_n := P$ for all $n \in \mathbb{N}$). For each $n \in \mathbb{N}$ let $q_n := q_{P_n}^A$ and let $e_{k,n}$ $(1 \leqslant k \leqslant n)$ be self-adjoint elements of norm 1 in A_h such that the set $\{q_n(e_{1,n}), \dots, q_n(e_{n,n})\}$ is linearly independent in A/P_n. Let

$$t_n := \sum_{k=1}^{n} e_{k,n} \otimes e_{k,n} \quad (n \in \mathbb{N}) \qquad \text{and} \qquad t := \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=1}^{n} e_{k,n} \otimes e_{k,n}.$$

If $T_n := \Theta_A(t_n) \in \mathrm{E}(A)$ $(n \in \mathbb{N})$ and $T := \Theta_A(t)$, we claim that $\ell(T_n) = n$, for all $n \in \mathbb{N}$, and that $T \notin \mathrm{E}(A)$. Suppose that for some n we have $T_n = \Theta_A(u)$, where $u = \sum_{k=1}^{d} a_k \otimes b_k \in A \otimes A$ with $d < n$. By Lemma 2.5, we have

$$\operatorname{span}\{q_n(e_{1,n}), \dots, q_n(e_{n,n})\} \subseteq \operatorname{span}\{q_n(b_1), \dots, q_n(b_d)\},$$

which is impossible, since the set $\{q_n(e_{1,n}), \dots, q_n(e_{n,n})\}$ is linearly independent. Similarly, suppose that $T \in \mathrm{E}(A)$ and that $T = \Theta_A(u)$ for some $u = \sum_{k=1}^{d} a_k \otimes b_k \in A \otimes A$. If $r > d$, Lemma 2.5 implies that

$$V := \overline{\overline{\operatorname{span}}}\{q_r(e_{k,n}) \colon n, k \in \mathbb{N}\} \subseteq \operatorname{span}\{q_r(b_1), \dots, q_r(b_d)\},$$

which contradicts the fact that $\dim V \geqslant r > d$. □

We shall now look closely at what happens in the homogeneous case. To do this, we restrict ourselves to σ-unital algebras. We state the main result.

Theorem 2.7. *Let A be a σ-unital n-homogeneous C^*-algebra. The following conditions are equivalent:*

(i) *A is of finite type;*

(ii) *$\mathrm{E}(A)$ is of finite length;*

(iii) *$\mathrm{Im}\,\Theta_A = \mathrm{E}(A)$.*

The 'difficult part' of Theorem 2.7 is to prove that (ii) $\Rightarrow$ (i). The main idea is to show that $\sup\{\ell(T)\colon T \in \mathrm{E}(A)\}$ is closely related to the type of the underlying (vector) bundle E, where the type of E is a constant defined as follows.

Definition 2.8. Let E be a vector bundle over the base space Δ. If E is of finite type, then the smallest number $m \in \mathbb{N}$ for which there exists a finite open covering $\{U_i\}_{1\leqslant i\leqslant m}$ of Δ such that each restriction bundle $E|_{U_i}$ is trivial is called the *type* of E and is denoted by $\mathrm{type}(E)$. If E is not of finite type, we define $\mathrm{type}(E) := \infty$.

Lemma 2.9. *If $m, n \in \mathbb{N}$ and we suppose that $S_1, \dots, S_m$ are some sets such that $|S_i| = n$ and $S_i \neq S_j$, for all $1 \leqslant i, j \leqslant m$, $i \neq j$, then*

$$|S_1 \cup \cdots \cup S_m| \geqslant \frac{n}{e}\sqrt[n]{m}.$$

Proof. Let $S := S_1 \cup \cdots \cup S_m$ and $k := |S|$. By the hypothesis of the lemma, S has at least m different subsets of cardinality n, hence $m \leqslant \binom{k}{n}$. Since

$$\binom{k}{n} \leqslant \left(\frac{ke}{n}\right)^n$$

(see [**2**]), we have

$$k \geqslant \frac{n}{e}\sqrt[n]{m}.$$

□

Lemma 2.10. *Let E be a vector bundle of constant rank n over a locally compact Hausdorff space Δ. Suppose that $\mathcal{S}$ is a subset of $\Gamma_{\mathrm{b}}(E)$ such that*

$$\mathrm{span}\{a(s)\colon a \in \mathcal{S}\} = E(s) \quad \textit{for all } s \in \Delta. \tag{2.8}$$

Then

$$|\mathcal{S}| \geqslant \frac{n}{e}\sqrt[n]{\mathrm{type}(E)}.$$

Proof. If E is not of finite type, the claim follows directly from Lemma 1.2. Suppose that $m := \mathrm{type}(E) < \infty$. Obviously, $|\mathcal{S}| \geqslant n$, so if $m = 1$ the proof is trivial. Suppose that $m > 1$ and let $s_1 \in \Delta$ be an arbitrary point. By (2.8), there exists a subset $\mathcal{S}_1 \subseteq \mathcal{S}$

such that $|\mathcal{S}_1| = n$ and $\operatorname{span}\{a(s_1)\colon a \in \mathcal{S}_1\} = E(s_1)$. By the continuity of the sections in $\mathcal{S}$, the set

$$U_1 := \{s \in \Delta\colon \operatorname{span}\{a(s)\colon a \in \mathcal{S}_1\} = E(s)\}$$

is an open neighbourhood of s_1. Note that $\Delta \setminus U_1 \neq \emptyset$. Indeed, if $\mathcal{S}_1 = \{a_{j_1}, \dots, a_{j_n}\}$, then the map $\phi_1\colon U_1 \times \mathbb{C}^n \to E$ defined by

$$\phi_1(s, \lambda_1, \dots, \lambda_n) := \sum_{i=1}^{n} \lambda_i a_{j_i}(s)$$

is a local trivialization of E (i.e. ϕ_1 is an isomorphism of vector bundles $U_1 \times \mathbb{C}^n$ and $E|_{U_1}$). Hence, if $U_1 = \Delta$, it would follow that E is trivial, so $m = 1$. Choose an arbitrary point $s_2 \in \Delta \setminus U_1$. By (2.8), there exists a subset $\mathcal{S}_2 \subseteq \mathcal{S}$ such that $|\mathcal{S}_2| = n$ and $\operatorname{span}\{a(s_2)\colon a \in \mathcal{S}_2\} = E(s_2)$, and let

$$U_2 := \{s \in \Delta\colon \operatorname{span}\{a(s)\colon a \in \mathcal{S}_2\} = E(s)\}.$$

Again, U_2 is an open neighbourhood of s_2. Since $s_2 \notin U_1$, $\mathcal{S}_1 \neq \mathcal{S}_2$. If $m > 2$, by induction we would find a sequence $\mathcal{S}_1, \dots, \mathcal{S}_m$ of m distinct subsets of $\mathcal{S}$ such that $|\mathcal{S}_i| = n$ for all $1 \leqslant i \leqslant m$. Lemma 2.9 now implies that

$$|\mathcal{S}| \geqslant |\mathcal{S}_1 \cup \dots \cup \mathcal{S}_m| \geqslant \frac{n}{e} \sqrt[n]{m}.$$

□

Lemma 2.11. *Let E be a locally trivial vector bundle of constant rank over a locally compact σ-compact Hausdorff space Δ. The following conditions are equivalent:*

(i) $\operatorname{type}(E) < \infty$;

(ii) $\sup\{\operatorname{type}(E|_K)\colon K \subseteq \Delta,\ K \text{compact}\} < \infty$.

Proof. (i) ⇒ (ii). This is trivial.

(ii) ⇒ (i). Suppose that $\operatorname{type}(E) = \infty$. By [**10**, Lemma 2.4], there exists a countable open cover (U_i) of Δ such that each $\bar{U}_i$ is compact, and $\bar{U}_i \cap \bar{U}_j = \emptyset$ for all $|i-j| > 1$. By compactness of $\bar{U}_i$ and by local triviality of E, the restriction bundles $E|_{U_i}$ are of finite type. Suppose that there exists $N \in \mathbb{N}$ such that $\operatorname{type}(E|_{\bar{U}_i}) \leqslant N$ for all $i \in \mathbb{N}$. Put

$$V_1 := \biguplus_{i=1}^{\infty} U_{2i-1} \quad \text{and} \quad V_2 := \biguplus_{i=1}^{\infty} U_{2i},$$

where $\biguplus$ denotes the disjoint union. Since $\operatorname{type}(E|_{U_i}) \leqslant \operatorname{type}(E|_{\bar{U}_i}) \leqslant N$, for all $i \in \mathbb{N}$, and since all the sets in the corresponding union are disjoint, we have $\operatorname{type}(E|_{V_1}) \leqslant N$ and $\operatorname{type}(E|_{V_2}) \leqslant N$. Since $V_1 \cup V_2 = \Delta$, it follows that

$$\operatorname{type}(E) \leqslant \operatorname{type}(E|_{V_1}) + \operatorname{type}(E|_{V_2}) \leqslant 2N,$$

which contradicts the fact that $\operatorname{type}(E) = \infty$. □

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Proof of Theorem 2.7. Let E be a locally trivial C^*-bundle over $\Delta := \mathrm{Prim}(A)$ with fibres isomorphic to $\mathrm{M}_n(\mathbb{C})$ such that $A = \Gamma_0(E)$. Since A is σ-unital, note that Δ is σ-compact.

(i) $\Rightarrow$ (ii). By Theorem 1.1, $M(A)$ is also n-homogeneous, so Theorem 2.4 implies that it is finitely centrally generated. Suppose that $M(A) = \mathrm{span}_{Z(M(A))}\{e_1, \dots, e_m\}$, where $e_1, \dots, e_m \in M(A)$. Let $T \in \mathrm{E}(A)$, $T = \Theta_A(t)$ for some

$$t = \sum_{k=1}^{d} a_k \otimes b_k \in M(A) \otimes M(A).$$

If

$$a_k = \sum_{i=1}^{m} z_{k,i} e_i \quad \text{and} \quad b_k = \sum_{i=1}^{m} w_{k,i} e_i$$

for some $z_{k,i}, w_{k,i} \in Z(M(A))$, then

$$\begin{aligned} T &= \sum_{k=1}^{d} \Theta_A\biggl(\biggl(\sum_{i=1}^{m} z_{k,i} e_i\biggr) \otimes \biggl(\sum_{i=1}^{m} w_{k,i} e_i\biggr)\biggr) \\ &= \sum_{k=1}^{d} \sum_{i,j=1}^{m} z_{k,i} w_{k,j} \Theta_A(e_i \otimes e_j) \\ &= \Theta_A\biggl(\sum_{i,j=1}^{m} u_{i,j} e_i \otimes e_j\biggr) \\ &= \Theta_A\biggl(\sum_{j=1}^{m} \biggl(\sum_{i=1}^{m} u_{i,j} e_i\biggr) \otimes e_j\biggr), \end{aligned}$$

where $u_{i,j} := \sum_{k=1}^{d} z_{k,i} w_{k,j}$. Hence, $\ell(T) \leqslant m$.

(i) $\Rightarrow$ (iii). Assuming that (i) holds, it follows from [**8**, Theorem 1.1] that $\mathrm{E}(A) = \mathrm{ICB}(A)$ (note that separability is not needed for this part of [**8**, Theorem 1.1]). Condition (iii) is then immediate.

(ii) $\Rightarrow$ (i). Suppose that E is not of finite type. By Theorem 1.1, E is not of finite type as a vector bundle. For $m \in \mathbb{N}$ let $d_m := \lfloor (n^2/e) \sqrt[n^2]{m} \rfloor$. By Lemma 2.11, there exists a compact subset $K_m \subseteq \Delta$ such that $\mathrm{type}(E|_{K_m}) \geqslant m$. Let

$$J_m := \{a \in \Gamma_0(E) \colon a|_{K_m} = 0\} \quad \text{and} \quad A_m := A/J_m.$$

Using the Tietze Extension Theorem for sections of Banach bundles [**4**, Theorem II.14.8], we may identify $A_m = \Gamma(E|_{K_m})$. Since K_m is compact, by Lemma 1.2 there exist a finite number of self-adjoint sections $e_1, \dots, e_{r_m} \in \Gamma(E|_{K_m})_h$ such that

$$\mathrm{span}\{e_1(s), \dots, e_{r_m}(s) \colon s \in \Delta\} = E(s) \quad \text{for all } s \in K_m.$$

If $t := \sum_{k=1}^{r_m} e_k \otimes e_k$ and $\dot{T}_m := \Theta_{A_m}(t)$, then Lemma 2.5 implies that if $\dot{T}_m = \Theta_{A_m}(\sum_{k=1}^{d} a_k \otimes b_k)$ is another representation of $\dot{T}_m$, then we also have

$$\operatorname{span}\{b_1(s), \dots, b_d(s)\colon s \in \Delta\} = E(s) \quad \text{for all } s \in K_m.$$

Therefore, by Lemma 2.10 we have $\ell(\dot{T}_m) \geqslant d_m$. Finally, if $T_m \in \mathrm{E}(A)$ is any lift of $\dot{T}_m$ (that is, if $T_m \in \mathrm{E}(A)$ such that $\dot{T}_m = Q^A_{J_m}(T_m)$, where $Q^A_{J_m}$ is the map as in (2.7)), then obviously $\ell(T_m) \geqslant \ell(\dot{T}_m) \geqslant d_m$. Since $\lim_{m\to\infty} d_m = \infty$, it follows that $\mathrm{E}(A)$ is not of finite length.

(iii) $\Rightarrow$ (i). Suppose that E is not of finite type. Since Δ is σ-compact, there exists a sequence (e_k) of self-adjoint sections in $A = \Gamma_0(E)$ such that

$$\operatorname{span}\{e_k(s)\colon k \in \mathbb{N}\} = E(s) \quad \text{for all } s \in \Delta. \tag{2.9}$$

Let

$$t := \sum_{k=1}^{\infty} \frac{1}{k^2} e_k \otimes e_k \in A \otimes_h A.$$

Suppose that $T := \Theta_A(t) \in \mathrm{E}(A)$ and let $u = \sum_{k=1}^{d} a_k \otimes b_k \in M(A) \otimes M(A)$ such that $T = \Theta_A(u)$. By [**8**, Lemma 3.2], we have $M(A) = \Gamma_{\mathrm{b}}(E)$. Since, by Theorem 1.1, E is not of finite type as a vector bundle, Lemma 1.2 implies that there exists a point $s_0 \in \Delta$ such that

$$\operatorname{span}\{b_1(s_0), \dots, b_d(s_0)\} \subsetneqq E(s_0). \tag{2.10}$$

But Lemma 2.5 implies that

$$\operatorname{span}\{e_k(s_0)\colon k \in \mathbb{N}\} \subseteq \operatorname{span}\{b_1(s_0), \dots, b_d(s_0)\};$$

which contradicts (2.9) and (2.10). □

Remark 2.12. Let A be a σ-unital C^*-algebra and let $J \in \mathrm{Id}(A)$. By the noncommutative Tietze Extension Theorem [**13**, Theorem 2.3.9], the extension $(q^A_J)^\beta\colon M(A) \to M(A/J)$ of q^A_J is also surjective. It follows that the induced contraction $(q^A_J)^\beta \otimes (q^A_J)^\beta\colon M(A) \otimes_h M(A) \to M(A/J) \otimes_h M(A/J)$ is also surjective, and hence $Q^A_J(\mathrm{E}(A)) = \mathrm{E}(A/J)$ and $Q^A_J(\operatorname{Im}\Theta_A) = \operatorname{Im}\Theta_{A/J}$, where Q^A_J is the map as in (2.7). Hence, if $\operatorname{Im}\Theta_A = \mathrm{E}(A)$, then $\operatorname{Im}\Theta_{A/J} = \mathrm{E}(A/J)$, and if $\mathrm{E}(A)$ is of finite length, so is $\mathrm{E}(A/J)$.

Corollary 2.13. *Let A be a separable C^*-algebra. If A satisfies one of the conditions*

(i) $\mathrm{E}(A)$ *is of finite length,*

(ii) $\operatorname{Im}\Theta_A = \mathrm{E}(A)$,

then A is subhomogeneous of finite type.

Proof. By Theorem 2.6, A is subhomogeneous. Let $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_p = A$ be the standard composition series of A [**9**, 6.2.5]. We have to show that each homogeneous quotient J_i/J_{i-1} is of finite type. Using induction, Remark 2.12 and Theorem 2.7 it is sufficient to prove this in the case when $p = 2$. Let $J := J_1$. Again, Remark 2.12 and Theorem 2.7 imply that A/J is of finite type. To prove that J is also of finite type, it is sufficient to prove this in the case when J is essential in A. Indeed, if J is not essential, then we can substitute A with $B := A/J^{\perp}$, since by [**8**, Lemma 3.1] the n-homogeneous ideal K of B is essential in B and $K \cong J$. Suppose that J is not of finite type. By the proof of Theorem 2.7, there exists a sequence of tensors (t_m) in $J \otimes J$ and a tensor $t \in J \otimes_h J$ such that $\ell(\dot{T}_m) \to \infty$ as $m \to \infty$ and $\dot{T} \notin \mathrm{E}(J)$, where $\dot{T}_m := \Theta_J(t_m)$ and $\dot{T} := \Theta_J(t)$. Let $T_m := \Theta_A(t_m)$ and $T := \Theta_A(t)$. Since J is essential in A, we have $M(A) \subseteq M(J)$, and hence $\ell(T_m) \geqslant \ell(\dot{T}_m) \to \infty$ as $m \to \infty$ and $T \notin \mathrm{E}(A)$. □

Remark 2.14. We also note that the class of (separable) C^*-algebras that satisfies one of the conditions of the previous corollary is larger than the class of a finite direct sum of (separable) homogeneous C^*-algebras of finite type. For example, let A be the C^*-algebra from [**5**, Example 6.1], which consists of all elements $a \in C([0,\infty], \mathrm{M}_2(\mathbb{C}))$ such that

$$a(n) = \begin{pmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{pmatrix} \quad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. It is proved in [**5**, Lemma 6.6] that $\mathrm{E}(A)$ is closed in (completely bounded-) norm, and hence that $\operatorname{Im}\Theta_A = \mathrm{E}(A)$. Analysing the same proof, one can also see that $\mathrm{E}(A)$ is of finite length.

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