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EMBEDDING THE COMPLEMENT OF TWO LINES IN A FINITE PROJECTIVE PLANE

Dedicated to George Szekeres on the occasion of his 65th birthday

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Abstract

In this paper we use a result from graph theory on the characterization of the line graphs of the complete bigraphs to show that if n is any integer $\geqq 2$ then any finite linear space having $p = n^2 - n$ or $p = n^2 - n + 1$ points, of which at least $n^2 - n$ have degree $n + 1$, and $q \leqq n^2 + n - 1$ lines is embeddable in an FPP of order n unless $n = 4$. If $n = 4$ there is only one possible exception for each of the two values of p, and for $p = n^2 - n$, this exception can be embedded in the FPP of order 5.

1. Introduction

Recently de Witte (1975c) has proved that any finite linear space (= FLS) with $p = n^2 - 1$ points, all of degree at most $n + 1$ can be embedded in a finite projective plane (= FPP) of order $n \geqq 4$). This result extended a theorem of Bose and Shrikhande (1973) (see also Bose (1973)). Bose and Shrikhande and also de Witte, in getting rid of the exceptional case in the Bose-Shrikhande theorem, made extensive use of graph theory. Bose and Shrikhande appealed to a well-known theorem [Theorem 8.6 of Harary (1969)] on the characterization of the line graphs of the complete graphs K_m for $m \neq 8$. De Witte considered the three graphs having the same parameters as, but differing from, the line graph of the complete graph K_8.

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It is the intention of this paper to apply another well-known theorem from graph theory to linear geometry. Using the theorem on the characterization of the line graphs of the complete bigraphs $K_{m,m}$ for $m \neq 4$ [Shrikhande (1959) and Theorem 8.7 of Harary (1969)], we will establish the following:

THEOREM. *If $\mathscr{L}$ is an FLS with $p = n^2 - n$ or $p = n^2 - n + 1$ points, of which at least $n^2 - n$ have degree $n + 1$ and $q \leqq n^2 + n - 1$ lines, where n is some integer $\geqq 2$, then $\mathscr{L}$ is embeddable in an FPP of order n unless $n = 4$. If $n = 4$ there is only one possible exception for each of the two values of p, and for $p = n^2 - n$, this exception can be embedded in the FPP of order 5.*

The above FLS's are precisely what would arise by deleting two lines (except possibly one point) from an FPP of order n. Thus the theorem comments on the reconstructibility of the FPP.

We will use the notation and terminology of de Witte (1975a). Let us recall those symbols and terms needed for this paper. By a *finite linear space* (= *FLS*) is meant a finite set of p so-called *points* together with a (finite) set of q subsets of points, called *lines*, such that every pair of distinct points is included in precisely one line and every line contains at least two points. The number of points on a line x (resp. lines through a point u) will be denoted by $a(x)$ (resp. $b(u)$) and called the *degree* of x (resp. u). A *k-line* (resp. *k-point*) is a line (resp. point) of degree k. Two lines *miss* each other if they are disjoint and are *parallel* if they are either identical or disjoint. One should note that disjointness and parallelism are not necessarily transitive. An FLS is called *trivial* if there is at most one line, and a *near-pencil* if it is non-trivial and $p - 1$ points are collinear.

The points and lines of an FLS will often be denoted by $u_\alpha (1 \leqq \alpha \leqq p)$ and x_σ $(1 \leqq \sigma \leqq q)$ respectively, introduced so that $\alpha \leqq \beta$ implies $b_\alpha = b(u_\alpha) \geqq b(u_\beta) = b_\beta$ and $\rho \leqq \sigma$ implies $a_\rho = a(x_\rho) \geqq a(x_\sigma) = a_\sigma$. The *incidence number* of a point u_α and a line x_σ will be denoted by $r_{\sigma\alpha}$, and is equal to 1 if u_α lies on x_σ and 0 otherwise. The *incidence matrix* associated with the FLS is the $q \times p$ matrix $(r_{\sigma\alpha})$.

2. Prerequisites

The results P1 to P4 below are well-known; for proofs, cf. de Witte (1975a). Lemmas 1 and 2 are deep results proved by Shrikhande (1959) and de Witte (1975b) respectively. For this reason they are stated without proof. Lemmas 3 and 4 use the same methods of proof as in Bose and Shrikhande (1973) and de Witte (1975c).

P1. $\Sigma_\alpha b_\alpha = \Sigma_\sigma a_\sigma$.

P2. For all α, $p = 1 + \Sigma_\sigma r_{\sigma\alpha}(a_\sigma - 1)$ if $p \geqq 1$.
P3. Any line x_σ meets $1 + \Sigma_\alpha r_{\sigma\alpha}(b_\alpha - 1)$ lines if $q \geqq 1$.
P4. If u_α does not lie on x_σ, then $b_\alpha - a_\sigma$ counts the number of lines passing through u_α and missing x_σ. Hence, if $b_1 = n + 1 \geqq 2$, then $a_1 \leqq n + 1$.

Lemma 1: *Let m be an integer $\geqq 1$, and G a graph, as defined by Harary (1969), satisfying the following four conditions:*
(a) there are m^2 vertices;
(b) every vertex has $2(m-1)$ neighbours;
(c) every two adjacent vertices have $m-2$ common neighbours;
(d) every two non-adjacent vertices have 2 common neighbours.
Then G is the line graph of the complete bipartite graph $K_{m,m}$ or the following exceptional graph with $m = 4$:

vertex	*neighbours*	*vertex*	*neighbours*
1	2 3 4 5 6 7	9	2 3 8 11 14 15
2	1 3 4 8 9 10	10	2 4 8 12 13 16
3	1 2 6 9 11 12	11	3 7 9 12 14 16
4	1 2 7 10 13 14	12	3 6 10 11 13 16
5	1 6 7 8 15 16	13	4 6 10 12 14 15
6	1 3 5 12 13 15	14	4 7 9 11 13 15
7	1 4 5 11 14 16	15	5 6 8 9 13 14
8	2 5 9 10 15 16	16	5 7 8 10 11 12

Lemma 2. *Let n be a positive integer $\geqq 2$ and $\mathscr{L}$ an FLS such that $n^2 \leqq p \leqq q \leqq n^2 + n + 1$. Then $\mathscr{L}$ is either a near-pencil or $\mathscr{L}$ is embeddable in an FPP of order n.*

Lemma 3. *In an FLS with $q = n^2 + n - 1$ and all points of degree $n + 1$, the following hold:*

(i) *any $(n-1)$-line misses $2(n-1)$ lines;*

(ii) *if two $(n-1)$-lines miss each other, then there are $n-2$ lines missing them both;*

(iii) *if two distinct $(n-1)$-lines meet each other, then there are 2 lines missing them both.*

Proof. By P3 any $(n-1)$-line misses

$$q - 1 - \sum_\alpha r_{\sigma\alpha}(b_\alpha - 1) = n^2 + n - 1 - 1 - (n-1)n = 2(n-1)$$

lines. Now suppose the $(n-1)$-lines x and y miss each other. By P4 there is one line different from y passing through each point of y and missing x. This accounts for n of the lines that miss x. Thus there must be $n-2$ lines missing both x and y. If x and y were distinct $(n-1)$-lines meeting one another at the point w, then by P4 there are 2 lines passing through each point of y different form w and missing x, and hence there are 2 lines that miss both x and y.

LEMMA 4. *Let $\mathscr{L}$ be an FLS with all points of degree $n+1$, all lines of degree n or $n-1$, every $(n-1)$-line meeting every n-line and with the graph defined on the set of $(n-1)$-lines by the disjointness relation equal to the line graph of a complete bipartite graph $K_{m,m}$ $(m \geqq 2)$. Then $\mathscr{L}$ is embeddable in a finite affine plane $(=FAP)$ of order $m = n(\geqq 3)$. Moreover $\mathscr{L}$ has $p = n^2 - n$ points and $q = n^2 + n - 1$ lines, of which $n-1$ are of degree n and n^2 are of degree $n-1$, and every point lies on one line of degree n and n lines of degree $n-1$.*

PROOF. Since $m \geqq 1$, the line graph of $K_{m,m}$ is non-empty. Hence, there are $(n-1)$-lines, and so $n \geqq 3$. It is clear that each of the $(n-1)$-lines may be represented by the ordered pair (i,j), where i,j are integers $1 \leqq i,j \leqq m$, such that two $(n-1)$-lines are disjoint iff their representations agree in one coordinate. Then $\mathscr{L}$ can be extended as follows. For every integer i, $1 \leqq i \leqq m$, a "new" point u_{p+i} is introduced and to each $(n-1)$-line (i,j) of $\mathscr{L}$ the point u_{p+i} is added. A "new" line $x_{q+1} = \{u_{p+1}, \cdots, u_{p+m}\}$ is also added. We claim that the resulting structure $\mathscr{L}'$ is an FAP of order $m = n(\geqq 3$ from above). To show that $\mathscr{L}'$ is first of all an FLS, we need only show that every two points are contained in precisely one line, since $m \geqq 2$. But to vertify this, we need only consider one new point and one old one, say u_{p+i} and u_α. If u_α does not lie on the $(n-1)$-line (i,j), then by P4 we know that there are two $(n-1)$-lines passing through u_α and missing (i,j). They must be of the form (i,k) and (l,j). Clearly u_α and u_{p+i} both belong to the extended $(n-1)$-line (i,k). Since any two distinct lines of $\mathscr{L}'$ containing u_{p+i} are disjoint in $\mathscr{L}$, we get the uniqueness of the extended $(n-1)$-line (i,k). Thus $\mathscr{L}'$ is an FLS. Now since u_{p+i} is added to all $(n-1)$-lines (i,j), $j = 1, \cdots, m$, we see that $b_{p+i} = m+1$. Thus if $m = n$, all lines of $\mathscr{L}'$ have degree n and all points have degree $n+1$, and $\mathscr{L}'$ is an FAP of order $m = n$. So we need only show that $m = n$. By considering P2 at an old point, we see that $p' = n^2$, and then by considering a new point we have

$$n^2 - 1 = p' - 1 = (m-1) + m(n-1) = mn - 1,$$

and thus $m = n$. Therefore, $\mathscr{L}'$ is an FAP of order $m = n$ $(\geqq 3)$. Hence $p = p' - n = n^2 - n$, $q = q' - 1 = n^2 + n - 1$, and by considering P1 and P2 we get the remainder of the conclusions.

3. Proof of the theorem

3.1. We will first prove the theorem for $p = n^2 - n$ and then use that to prove the remainder. By P3 we have $a_1 \leqq n$ and by P2 we see that there is at least one n-line passing through every point of $\mathscr{L}$. Now if two n-lines x and y meet at a point w, then through every point of y different from w there is a line missing x by P4 and hence by P3 we get a contradiction. Therefore, through every point of $\mathscr{L}$ there is precisely one line of degree n and n lines of degree $n-1$. This yields $q = n^2 + n - 1$ lines, of which $n-1$ have degree n and n^2 have degree $n-1$. In view of lemma 3 all four conditions of lemma 1 are satisfied, if G is the graph defined on the set of $(n-1)$-lines by the disjointness relation. Thus if G is not the exceptional graph listed in lemma 1, then G is the line graph of a complete bipartite graph $K_{m,m}$ and by lemma 4 we must have that $\mathscr{L}$ is embeddable in an FAP of order n (and hence in an FPP of order n). If G is the exceptional graph of lemma 1, then $n = 4$ and, as we will see in section 4, $\mathscr{L}$ must be given by the incidence matrix in the upper left-hand corner of figure 1. We call this FLS the Shrikhande-FLS. The whole incidence matrix in figure 1 is the FPP of order 5 and thus the theorem is proved for $p = n^2 - n$. (As can be seen from figure 1 the Shrikhande-FLS is obtained from the FPP of order 5 by selecting three non-concurrent lines and deleting all points that do not lie on precisely one of them).

3.2. We may now consider $p = n^2 - n + 1$ and assume that all but possibly one of the points has degree $n+1$. Let w denote this one point whose degree may differ from $n+1$.

3.2.1. Let us first suppose that $a_1 \geqq n+1$. Then x_1 must pass through w if $b(w) \neq n+1$. If $b(w) = n+1$ we could have chosen w as any point in $\mathscr{L}$, so even in this case we may suppose that w lies on x_1. By P4 we see that $a_1 = n+1$, and then from P2 and P3 we get $b(w) = n-1$. Hence $q = n^2 + n - 1$. Since $b(w) = n-1$ any line not passing through w has degree at most $n-1$ by P4. Thus by P2 through every $(n+1)$-point there is one line of degree $n+1$ and n lines of degree $n-1$. By deleting w from $\mathscr{L}$ we obtain an FLS, $\mathscr{L}'$, with $n^2 - n$ points all of degree $n+1$ and $n^2 + n - 1$ lines. Thus by 3.1 if $\mathscr{L}'$ is not the Shrikhande-FLS, it can be embedded in an FPP of order n. The set of $(n+1)$-lines of $\mathscr{L}$ clearly correspond to a set of disjoint n-lines of $\mathscr{L}'$. Since this set of n-lines must meet pairwise in the FPP, they must correspond to concurrent lines in the FPP, say concurrent at w'. By mapping w into w', we can embed $\mathscr{L}$ into this FPP in an obvious manner. However, if $\mathscr{L}'$ is the Shrikhande-FLS, then $\mathscr{L}$ is, in fact, non-desarguesian and the embeddability question remains open. To see that it is non-desarguesian, consider the following: lines 1, 2, 3 meet at w (the numbers refer to row and column numbers in figure 1); the two triangles perspective at w are determined by the points 4, 5, 9 and 3, 8, 11; the respective sides of the triangles meet at points 12, 1, 7, which are non-collinear.

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Figure 1: The exceptional Shrikhande-FLS embedded in the FPP of order 5.

3.2.2. Now suppose that $a_1 \leqq n$. Then by P2 we have that $b(w) \geqq n$ and also that through any $(n+1)$-point there are at least 2 lines of degree n. Whenever two n-lines meet at an $(n+1)$-point, we see by P3 and P4 that one of them must pass through w and $b(w) \leqq n$. Thus $b(w) = n$ and there are exactly 2 lines of degree n passing through each $(n+1)$-point. Then by P2

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we may conclude that $\mathscr{L}$ contains only n-lines and $(n-1)$-lines. Now let us extend $\mathscr{L}$ in the following manner:

For each n-line x passing through w, we will add a "new" point $[x]$ to $\mathscr{L}$ such that $[x]$ should lie on precisely those lines of $\mathscr{L}$ that are parallel to x. In addition we will add a "new" line, x_{q+1}, to $\mathscr{L}$, where x_{q+1} contains precisely the n "new" points. We must now verify that the resulting structure, $\mathscr{L}'$, is an FLS. Since $\mathscr{L}$ has $(n-1)$-lines, we have $n \geqq 3$ and hence every line of $\mathscr{L}'$ contains at least two points. To verify that every two distinct points determine a unique line, we need only check it when at least one of the points is "new", say $[x]$. But by P4 there was one line through every $(n+1)$-point parallel to any given n-line in $\mathscr{L}$ and thus $[x]$ is joined to any "old" point by a unique line. If we have two distinct "new" points, say $[x]$ and $[y]$, then they are joined by x_{q+1}. If they are joined by an "old" line, then there is a line z that misses both x and y in $\mathscr{L}$. But from above we must have $a(z) \geqq n-1$ and hence by P4 again $b(w) \geqq n+1$, since x and y meet at w. This is impossible and therefore $\mathscr{L}'$ is an FLS with $p' = p + n = n^2 + 1$ points and $q' = q + 1 \leqq n^2 + n$ lines. By lemma 2, $\mathscr{L}'$ and so also $\mathscr{L}$ can be embedded in an FPP of order n, unless $\mathscr{L}'$ is a near-pencil. Since $\mathscr{L}'$ is obviously not a near-pencil, we are through.

4. Uniqueness of the Shrikhande-FLS

In this section we will show that any FLS with $p = 12$, $q = 19$, all points having degree 5, and every point lying on precisely one line of degree 4 and four lines of degree 3 must in fact be the Shrikhande-FLS. Let the three 4-lines be denoted by a, b, c and the points be distributed as $a = \{1,2,3,4\}$, $b = \{5,6,7,8\}$, $c = \{9,10,11,12\}$. The 3-lines will have the interrelations described in the exceptional graph of lemma 1, e.g. line 1 is disjoint from lines 2,3,4,5,6,7. Clearly every 3-line meets each of a, b, c. Without loss of generality we may assume that line 1 (denoted L1) is given by L1 $= \{1,5,9\}$. Similarly we may assume that L2 $= \{2,6,10\}$ and L3 $= \{3,7,11\}$ since L1, L2, L3 are pairwise disjoint. Now L8 meets L1, say at 1. L8 also meets L3 at either 7 or 11, but since choosing 7 over 11 only results in the changing of the roles of b and c we may further suppose that L3 and L8 meet at 11. Since L8 meets b, L8 $= \{1,8,11\}$. L9 cannot contain any of 1,2,3,6,7,8,10,11 and is thus $= \{4,5,12\}$. Now L10 cannot pass through any of 1,2,6,8,10,11 and must contain either 9 or 12. Therefore it cannot contain 5, since then it would have two points in common with either L1 or L9. Thus it must contain 7 and, by considering L3, point 4 and finally 9, by considering L9. This determines L4 $= \{3,8,12\}$. Now consider L5. It cannot pass through 1,5,8,9,11. By considering L2 it cannot pass through both 6 and 10 and hence must pass through at least one of 7 or 12.

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But by considering L3 and L4 we see that L5 cannot pass through 3. Since L5 meets both L3 and L4 we obtain L5 = $\{2, 7, 12\}$. The remaining point-sets can be readily determined now in increasing numerical order. The matrix in the upper left-hand corner of figure 1 is then obtained as follows: column i corresponds to point i, and row j corresponds to line $j-3$ for $j \geqq 4$, and rows 1,2,3 correspond to lines a, b, c respectively.

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