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Turbulent Transport of Temperature Fields

In this chapter, we consider a turbulent transport of temperature field in an isotropic homogeneous and incompressible turbulence. We discuss the Kolmogorov theory of hydrodynamic turbulence and obtain spectrum of velocity fluctuations for fully developed turbulence using the dimensional analysis. We study isotropic and anisotropic spectra of temperature fluctuations in different subranges of turbulent scales and different Prandtl numbers applying the dimensional analysis. We derive mean-field equations for the temperature field and obtain expressions for turbulent heat flux, turbulent diffusion and level of temperature fluctuations for small and large Péclet numbers by means of various analytical methods, namely the dimensional analysis, the quasi-linear approach and the spectral tau approach (the relaxation approach).

1.1 Hydrodynamic Turbulence: Dimensional Analysis

In this section, we consider a theory of hydrodynamic isotropic homogeneous and incompressible turbulence using the dimensional analysis.

1.1.1 Governing Equations and Basic Parameters

The fluid velocity field in an incompressible flow is determined by the Navier[1]-Stokes[2] equation (Landau and Lifshits, 1987; Batchelor, 1967; Lighthill, 1986; Tritton, 1988; Faber, 1995; Falkovich, 2011):

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = -\frac{\nabla P}{\rho} + \nu\, \Delta \boldsymbol{U} + \boldsymbol{f}.} \tag{1.1}$$

[1] Claude-Louis Navier (1785–1836) was a French engineer and physicist well-known for his works in mechanics, fluid dynamics, theory of elasticity and structural analysis.

[2] Sir George Gabriel Stokes (1819–1903) was a mathematician and physicist (who was born in Ireland and worked at the University of Cambridge) well-known for his works in fluid dynamics, optics and mathematical physics.

Equation (1.1) is the second law of Newton[3] for a unit mass of a fluid:

$$\rho \, \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla P + \nabla \cdot \left(2\nu \, \rho \, \mathbf{S}^{(\mathrm{U})}\right) + \rho \boldsymbol{f}, \tag{1.2}$$

where according to the chain rule of differentiation of the function $\boldsymbol{U}[t, \boldsymbol{r}(t)]$, the substantial time derivative $\mathrm{d}\boldsymbol{U}/\mathrm{d}t$ for the moving fluid element is the sum of a local time derivative $\partial \boldsymbol{U}/\partial t$ and convective derivative $(\boldsymbol{U}\cdot\nabla)\boldsymbol{U}$. We take into account here that most fluids obey Newton's law of viscosity [see the second term on the right-hand side of Eq. (1.2)], where $\mathsf{S}_{ij}^{(\mathrm{U})} = \frac{1}{2}(\nabla_j U_i + \nabla_i U_j)$ are the components of the rate-of-strain-tensor $\mathbf{S}^{(\mathrm{U})}$ for incompressible fluid, ν is the kinematic viscosity, $\rho \boldsymbol{f}$ is the external force, that, e.g., creates a turbulent random velocity field, and P and ρ are the fluid pressure and density, respectively. The operators ∇ and $\Delta = \nabla^2$ in the Cartesian coordinates are defined as

$$\nabla = \boldsymbol{e}_x \, \frac{\partial}{\partial x} + \boldsymbol{e}_y \, \frac{\partial}{\partial y} + \boldsymbol{e}_z \, \frac{\partial}{\partial z}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.3}$$

and $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are unit vectors along the x-, y- and z-axes. When the viscosity ν tends to zero, Eq. (1.1) is reduced to the Euler[4] equation. The fluid pressure and density are the macroscopic variables that determine the internal state of the fluid, and they are related by the equation of state for the perfect gas, $P = (k_B/m_\mu) \, \rho \, T \equiv (R/\mu) \, \rho \, T$, where $k_B = R/N_A$ is the Boltzmann constant, R is the gas constant, N_A is the Avogadro number, $\mu = m_\mu N_A$ is the molar mass and m_μ is the mass of the molecules of the surrounding fluid. Generally for arbitrary fluid flows, the continuity equation which is the conservation law for the fluid mass reads

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \boldsymbol{U}) = 0 \, .} \tag{1.4}$$

This equation implies that for any volume, the change of the fluid mass inside the volume is compensated by the fluid flux through this volume.

For an incompressible fluid flow, the continuity equation (1.4) is reduced to

$$\boxed{\mathrm{div}\, \boldsymbol{U} \equiv \nabla \cdot \boldsymbol{U} = 0 \, ,} \tag{1.5}$$

where the fluid density ρ is constant in time and space. The second term $(\boldsymbol{U} \cdot \nabla)\boldsymbol{U}$ on the left-hand side of Eq. (1.1) is a nonlinear term that describes inertia. The

[3] Sir Isaac Newton (1642–1726) was an English mathematician, physicist and astronomer who made key contributions to the foundations of classical mechanics, optics and the infinitesimal calculus and built the first practical reflecting telescope.

[4] Leonhard Euler (1707–1783) is a Swiss mathematician, physicist, astronomer, geographer and engineer who made influential discoveries in mathematics (infinitesimal calculus and graph theory, topology, analytic number theory and mathematical analysis), mechanics, fluid dynamics, optics, astronomy and music theory.

dimensionless ratio of the nonlinear term to the viscous term in Eq. (1.1) is the Reynolds[5] number, which is a key parameter in the system:

$$\mathrm{Re} = \frac{|(\boldsymbol{U} \cdot \nabla)\boldsymbol{U}|}{|\nu \, \Delta \boldsymbol{U}|}. \tag{1.6}$$

For very large Reynolds numbers, the fluid flow is turbulent. There are many examples of turbulent flows in nature, laboratory experiments and industrial applications (Landau and Lifshits, 1987; Batchelor, 1953; Monin and Yaglom, 1971, 2013; Tennekes and Lumley, 1972; Frisch, 1995; Pope, 2000; Bernard and Wallace, 2002; Lesieur, 2008; Davidson, 2015). For instance, turbulence in laboratory experiments is produced, e.g., by oscillating grids, propellers, shear flows, etc. The atmospheric turbulence is produced by convective motions and large-scale shear flow (a non-uniform wind). Turbulence inside the Sun is produced by convection in the solar convective zone located under the solar surface. Turbulence in galaxies is produced by random supernova explosions. In astrophysics, turbulence can be also produced by shear motions and various plasma instabilities. Various pictures of turbulent flows can be found in the book by Van Dyke (1982).

In turbulent flows, the fluid velocity is a random field. Large-scale effects caused by small-scale turbulence can be studied using a mean-field approach. In the framework of this approach velocity field can be decomposed into the mean velocity and fluctuations, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where according to the Reynolds rule velocity fluctuations $\boldsymbol{u}$ have zero mean value, $\langle \boldsymbol{u} \rangle = 0$ and $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity. The angular brackets $\langle \ldots \rangle$ denote an averaging. Different kinds of averaging procedures will be discussed in the next section.

The Reynolds number defined by Eq. (1.6) can be estimated using the dimensional analysis. In particular, in Eq. (1.6) we replace operators $|\nabla|$ by ℓ_0^{-1} and Δ by ℓ_0^{-2}. This yields

$$\boxed{\mathrm{Re} = \frac{\ell_0 \, u_0}{\nu},} \tag{1.7}$$

where ℓ_0 is the integral (energy-containing or maximum) scale of turbulence and $u_0 = \left[\langle \boldsymbol{u}^2 \rangle_{\ell=\ell_0}\right]^{1/2}$ is the characteristic turbulent velocity in the integral scale of turbulence ℓ_0. For example, in Table 1.1 we give turbulence parameters for various flows, e.g., for laboratory experiments in air flows, industrial flows in a wind tunnel and a diesel engine, atmospheric turbulence in the low troposphere (about 1 or 2 kilometers height from the Earth surface), and astrophysical turbulence, e.g., in the solar convective zone located under the solar surface with the depth about 1/3 of the solar radius and inside a galactic disk with a high concentration of stars. Here

[5] Osborne Reynolds (1842–1912) was an engineer (who was born in Ireland and worked at Owens College in Manchester, now the University of Manchester), well-known for his works in fluid dynamics and heat transfer.

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Table 1.1 *Parameters for engineering, geophysical and astrophysical turbulence*

	ℓ_0 (cm)	u_0 (cm/s)	τ_0 (s)	ν (cm^2/s)	Re
Laboratory experiments	1–10	10–10^2	10^{-2}–1	10^{-1} (air)	10^2–10^4
Diesel engine	0.3	3×10^2	10^{-3}	10^{-2}	10^4
Wind tunnel	(1–3) $\times 10^2$	(1–3) $\times 10^3$	0.03–0.3	10^{-1}	10^6–10^7
Atmospheric turbulence	10^4	10^2	10^2	10^{-1}	10^7
Sun ($r \approx R_\odot$)	3×10^7	10^5	3×10^2	3×10^{-2}	10^{14}
Sun $\left(r \approx \frac{2}{3} R_\odot\right)$	5×10^9	2×10^3	3×10^6	10^{-1}	10^{14}
Galactic disk	10^{20}	10^6	10^{14}	10^{18}	10^8

$\tau_0 = \ell_0/u_0$ is the characteristic turbulent time in the scale ℓ_0, the radius $r = \frac{2}{3} R_\odot$ corresponds to the bottom of the solar convective zone and $R_\odot = 6.96 \times 10^{10}$ cm is the solar radius.

A fully developed turbulence for very large Reynolds numbers can be qualitatively regarded as a sea of eddies, i.e., an ensemble of turbulent eddies of different scales varying from the integral energy-containing scale ℓ_0 to very small viscous scale ℓ_ν. Turbulent eddy can be considered as a blob of vorticity $\boldsymbol{\nabla} \times \boldsymbol{u}$. In the scale ℓ_ν, the viscous dissipation of the turbulent kinetic energy becomes important. The dynamics of the turbulent eddies is as follows. The large eddies are unstable, and they break down into the small eddies. The new small eddies are also unstable and continue to breakdown into the very small eddies. This process is called the Richardson[6] energy cascade and implies the transfer of the turbulent kinetic energy from the integral scale to smaller ones (Richardson, 1920). The energy cascade stops when the size of the small eddies is of the order of the viscous scale of turbulence. At this scale, turbulent kinetic energy is dissipated into thermal energy. The rate of the dissipation of the turbulent kinetic energy ε can be estimated as

$$\varepsilon = \frac{u_0^2}{\tau_0} = \frac{u_0^3}{\ell_0}. \tag{1.8}$$

1.1.2 Kolmogorov Theory of Hydrodynamic Turbulence

In this section, we consider Kolmogorov[7] theory of hydrodynamic turbulence. Let us assume that

[6] Lewis F. Richardson (1881–1953) was an English mathematician, physicist and meteorologist, well-known for his works in turbulence, mathematical physics and mathematical techniques of weather forecasting.

[7] Andrey N. Kolmogorov (1903–1987) was a Russian mathematician, well-known for his works in theory of random processes and probability theory, theory of turbulence, topology, theory of differential equations, functional analysis and information theory.

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- turbulence is homogeneous, i.e., $\nabla \langle \boldsymbol{u}^2 \rangle = 0$;
- turbulence is isotropic, i.e., there is no preferential direction;
- turbulent flow is incompressible, i.e., $\nabla \cdot \boldsymbol{u} = 0$ and the fluid density ρ is constant in time and in space;
- interactions in the turbulence are local, i.e., there are only interactions between turbulent eddies of the same size, and there are no interactions between the eddies of different sizes;
- in a subrange of turbulent scales $\ell_\nu \le \ell \le \ell_0$, the dissipation rate of the turbulent kinetic energy density is constant

$$\boxed{\varepsilon = \frac{u_0^3}{\ell_0} = \frac{u_\ell^3}{\ell} = \ldots = \frac{u_\nu^3}{\ell_\nu} = \text{const},} \tag{1.9}$$

where $u_\ell = \left[\langle \boldsymbol{u}^2 \rangle_\ell\right]^{1/2}$ is the characteristic turbulent velocity at the scale ℓ inside the inertial subrange of turbulence scales $\ell_\nu \le \ell \le \ell_0$ and $u_\nu = \left[\langle \boldsymbol{u}^2 \rangle_{\ell=\ell_\nu}\right]^{1/2}$ is the characteristic velocity at the viscous scale ℓ_ν. For the simplicity we assume here that the constant fluid density is unity. Equation (1.9) allows us to determine turbulent velocities in different scales,

$$u_0 = (\varepsilon\, \ell_0)^{1/3}, \quad u_\ell = (\varepsilon\, \ell)^{1/3}, \quad u_\nu = (\varepsilon\, \ell_\nu)^{1/3}. \tag{1.10}$$

Equation $u_\ell = (\varepsilon\, \ell)^{1/3}$ implies that the scaling for u_ℓ^2 in the inertial subrange of turbulent scales $\ell_\nu \ll \ell \ll \ell_0$ is given by

$$\boxed{u_\ell^2 = \varepsilon^{2/3}\, \ell^{2/3}} \tag{1.11}$$

[see Kolmogorov (1941), and its English translation in Kolmogorov (1991)], and the characteristic time $\tau_\ell = \ell / u_\ell$ in the inertial subrange of scales is

$$\boxed{\tau_\ell = \varepsilon^{-1/3}\, \ell^{2/3}.} \tag{1.12}$$

Using Eq. (1.10), we rewrite the Reynolds number as

$$\text{Re} = \frac{\ell_0\, u_0}{\nu} = \frac{\varepsilon^{1/3}\, \ell_0^{4/3}}{\nu}. \tag{1.13}$$

We introduce the local Reynolds number:

$$\text{Re}_\ell = \frac{\ell\, u_\ell}{\nu} = \frac{\varepsilon^{1/3}\, \ell^{4/3}}{\nu}. \tag{1.14}$$

Equations (1.13)–(1.14) allow us to determine the ratio Re_ℓ/Re as

$$\frac{\text{Re}_\ell}{\text{Re}} = \left(\frac{\ell}{\ell_0}\right)^{4/3}. \tag{1.15}$$

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The viscous scale ℓ_ν (the Kolmogorov scale) is defined as the scale in which the local Reynolds number is 1. This implies that in the Kolmogorov scale, the nonlinear term in the Navier-Stokes equation is of the order of the viscous term. Therefore, Eq. (1.15) with the condition $\mathrm{Re}_{\ell=\ell_\nu}=1$ allow us to relate the Kolmogorov scale ℓ_ν with the integral scale ℓ_0 of turbulence as

$$\boxed{\ell_\nu = \frac{\ell_0}{\mathrm{Re}^{3/4}}\,.} \tag{1.16}$$

Substituting the Kolmogorov scale ℓ_ν given by Eq. (1.16) into Eq. (1.10) for $u_\nu = (\varepsilon\,\ell_\nu)^{1/3}$, we obtain the characteristic velocity in the Kolmogorov scale as $u_\nu = (\varepsilon\,\ell_0)^{1/3}\,\mathrm{Re}^{-1/4}$, so that

$$\boxed{u_\nu = \frac{u_0}{\mathrm{Re}^{1/4}}\,,} \tag{1.17}$$

where $u_0 = (\varepsilon\,\ell_0)^{1/3}$. Therefore, the characteristic viscous time $\tau_\nu = \ell_\nu/u_\nu$ (the Kolmogorov time) is given by

$$\boxed{\tau_\nu = \frac{\tau_0}{\mathrm{Re}^{1/2}}\,.} \tag{1.18}$$

Next, we determine the spectrum of velocity fluctuations in the inertial subrange of scales (the Kolmogorov-Obukhov[8] spectrum). We define the energy spectrum function of the velocity field as

$$u_\ell^2 = \int_{k_0}^{k} E_{\mathrm{u}}(k')\,dk', \tag{1.19}$$

where wave numbers $k_0 = \ell_0^{-1}$ and $k = \ell^{-1}$. Using the dimensional analysis, we rewrite Eq. (1.19) as $u_\ell^2 = E_{\mathrm{u}}(k)\,k$. Therefore, the Kolmogorov-Obukhov spectrum $E_{\mathrm{u}}(k)$ in the inertial subrange of turbulent scales $k_0 \ll k \ll k_\nu$ is given by (Kolmogorov, 1941; Obukhov, 1941)

$$\boxed{E_{\mathrm{u}}(k) = \varepsilon^{2/3} k^{-5/3},} \tag{1.20}$$

where $k_\nu = \ell_\nu^{-1}$ and we take into account that $u_\ell = (\varepsilon/k)^{1/3}$. Equation (1.20) also directly follows from Eq. (1.11) using the relations $\ell = k^{-1}$ and $E_{\mathrm{u}}(k) = u_\ell^2/k$. Since $\varepsilon = u_\ell^2/\tau_\ell = E_{\mathrm{u}}(k)\,k/\tau(k)$, we obtain the scaling for the characteristic time $\tau(k)$ in the inertial subrange of turbulent scales as

$$\boxed{\tau(k) = \varepsilon^{-1/3} k^{-2/3}.} \tag{1.21}$$

[8] Alexander M. Obukhov (1918–1998) was a Russian geophysicist well-known for his works in atmospheric physics, meteorology, turbulence and mathematical statistics.

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Equation (1.21) also directly follows from Eq. (1.12) using the relation $\ell = k^{-1}$. The Kolmogorov-Obukhov spectrum has been detected in many laboratory experiments where turbulence is produced by various sources. This spectrum also has been observed in atmospheric turbulence, space experiments with solar wind, and solar and galactic turbulence. The Kolmogorov-Obukhov spectrum can be considered as a universal spectrum since it is observed in various turbulent systems of different origins.

1.2 Spectra of Temperature Fluctuations: Dimensional Analysis

In this section, we obtain various spectra of temperature fluctuations in a hydrodynamic isotropic homogeneous and incompressible turbulence using the dimensional analysis.

1.2.1 Governing Equations, Averaging and Basic Parameters

The equation for the evolution of fluid temperature field $T(t, \boldsymbol{x})$ in an incompressible fluid velocity field $\boldsymbol{U}(t, \boldsymbol{x})$ reads (Landau and Lifshits, 1987; Batchelor, 1967)

$$\boxed{\frac{\partial T}{\partial t} + (\boldsymbol{U} \cdot \nabla) T = D^{(\theta)} \, \Delta T + I_T,} \tag{1.22}$$

where $D^{(\theta)}$ is the coefficient of the molecular diffusion of the temperature field and I_T is the heat source/sink that for simplicity is neglected below. Equation (1.22) is the convective diffusion equation. The continuity equation for the incompressible fluid velocity field is $\nabla \cdot \boldsymbol{U} = 0$. We apply a mean-field approach, i.e., all quantities are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. For example, the temperature field $T = \overline{T} + \theta$, where $\overline{T} = \langle T \rangle$ is the mean fluid temperature, θ are temperature fluctuations, and $\langle \theta \rangle = 0$. The angular brackets $\langle \dots \rangle$ denote an averaging. Similarly, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity, $\boldsymbol{u}$ are velocity fluctuations and $\langle \boldsymbol{u} \rangle = 0$. There are three main ways of averaging:

- The time averaging (i.e., the averaging over the time):

$$\overline{T} = \frac{1}{t_M} \int_0^{t_M} T(t, \boldsymbol{x}) \, dt, \tag{1.23}$$

  where t_M is the total time of measurements (e.g., in the case of laboratory or field experiments) or the total time of calculations (e.g., in the case of numerical simulations).

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- The spatial (volume) averaging:

$$\overline{T} = \frac{1}{L_x L_y L_z} \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} T(t, \boldsymbol{x})\, dz, \tag{1.24}$$

  where L_x, L_y, L_z are the sizes of the box along x, y, z directions. The plane averaging,

$$\overline{T} = \frac{1}{L_x L_y} \int_0^{L_x} dx \int_0^{L_y} T(t, \boldsymbol{x})\, dy, \tag{1.25}$$

  is used or even the averaging along one direction,

$$\overline{T} = \frac{1}{L_y} \int_0^{L_y} T(t, \boldsymbol{x})\, dy, \tag{1.26}$$

  is also used.
- The ensemble averaging (e.g., averaging over independent spatial distributions $T_n = T(t_n, \boldsymbol{x})$ of temperature fields taken in different times: $t_1, t_2, \ldots, t_N$):

$$\overline{T} = \frac{1}{N} \sum_{n=1}^{N} T_n(\boldsymbol{x}), \tag{1.27}$$

  where t_n are the instants of measurements and N is the total number of data points.

Averaging Eq. (1.22) over an ensemble of turbulent velocity field, we arrive at the mean-field equation for the mean temperature field:

$$\boxed{\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T}\, \overline{\boldsymbol{U}} + \langle \theta\, \boldsymbol{u} \rangle\right) = D^{(\theta)}\, \Delta \overline{T},} \tag{1.28}$$

where $\langle \theta\, \boldsymbol{u} \rangle$ is the turbulent heat flux. In our derivation of Eq. (1.28), we take into account that

- various operators, like the averaging $\langle \ldots \rangle$, the partial derivative over time, the spatial partial derivatives, the operators ∇ and Δ, are linear commutative operators;
- $\langle \boldsymbol{u} \overline{T} \rangle = \overline{T} \langle \boldsymbol{u} \rangle = 0$ and $\langle \overline{\boldsymbol{U}} \theta \rangle = \overline{\boldsymbol{U}} \langle \theta \rangle = 0$.

Let us consider for simplicity the case $\overline{\boldsymbol{U}} = 0$. The obtained results will be the same for the constant mean fluid velocity due to the Galilean[9] invariance.

[9] Galileo Galilei (1564–1642) was an Italian astronomer, physicist, engineer, philosopher and mathematician, well-known for his works in physics, astronomy and applied science.

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The equation for temperature fluctuations $\theta = T - \overline{T}$ is obtained by subtracting Eq. (1.28) from Eq. (1.22):

$$\boxed{\frac{\partial \theta}{\partial t} + \nabla \cdot [\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle] - D^{(\theta)} \Delta\theta = -(\boldsymbol{u} \cdot \nabla)\overline{T}.} \tag{1.29}$$

The second term, $\nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle)$, on the left-hand side of Eq. (1.29) is the nonlinear term, while the first term, $-(\boldsymbol{u} \cdot \nabla)\overline{T}$, on the right-hand side of Eq. (1.29) is the source of temperature fluctuations produced by the tangling of the gradient of the mean temperature $\nabla \overline{T}$ by random velocity fluctuations $\boldsymbol{u}$. The dimensionless ratio of the nonlinear term to the diffusion term in Eq. (1.29) is the Péclet[10] number which is a key parameter in the system:

$$\mathrm{Pe} = \frac{|\nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle)\,|}{|D^{(\theta)} \Delta\theta|}. \tag{1.30}$$

The Péclet number, defined by Eq. (1.30), can be estimated using dimensional analysis as

$$\boxed{\mathrm{Pe} = \frac{\ell_0\, u_0}{D^{(\theta)}}.} \tag{1.31}$$

Using Eq. (1.10) for the turbulent velocity $u_0 = (\varepsilon\, \ell_0)^{1/3}$ at the integral scale, we rewrite the Péclet number as

$$\mathrm{Pe} = \frac{\varepsilon^{1/3} \ell_0^{4/3}}{D^{(\theta)}}. \tag{1.32}$$

Next, we introduce the local Péclet number $\mathrm{Pe}_\ell = \ell\, u_\ell / D^{(\theta)}$ at the scale ℓ and use Eq. (1.10) for the turbulent velocity $u_\ell = (\varepsilon\, \ell)^{1/3}$, so that the local Péclet number is

$$\mathrm{Pe}_\ell = \frac{\ell\, u_\ell}{D^{(\theta)}} = \frac{\varepsilon^{1/3} \ell^{4/3}}{D^{(\theta)}}. \tag{1.33}$$

We determine the ratio $\mathrm{Pe}_\ell / \mathrm{Pe}$ as

$$\frac{\mathrm{Pe}_\ell}{\mathrm{Pe}} = \left(\frac{\ell}{\ell_0} \right)^{4/3}. \tag{1.34}$$

We introduce a diffusion scale ℓ_D defined as the scale in which the local Péclet number is 1. This implies that at the scale ℓ_D, the nonlinear terms in the equation

[10] Jean Claude Eugène Péclet (1793–1857) was a French physicist well-known for his works in fluid dynamics, heat transfer and theory of combustion.

for temperature fluctuations equal the diffusion term. Therefore, Eq. (1.34) yields the diffusion scale ℓ_D as

$$\boxed{\ell_D = \frac{\ell_0}{\mathrm{Pe}^{3/4}}.} \tag{1.35}$$

Let us consider the case when the diffusion scale is inside the inertial subrange of turbulent scales, $\ell_0 \geq \ell_D \geq \ell_\nu$. This implies that $u_D = (\varepsilon\, \ell_D)^{1/3}$ [see Eqs. (1.9)–(1.10)]. Substituting the diffusion scale (1.35) into the equation for u_D, we obtain the characteristic velocity at the diffusion scale as $u_D = (\varepsilon\, \ell_0)^{1/3}\ \mathrm{Pe}^{-1/4}$, so that

$$\boxed{u_D = \frac{u_0}{\mathrm{Pe}^{1/4}},} \tag{1.36}$$

where $u_0 = (\varepsilon\, \ell_0)^{1/3}$. Using Eqs. (1.35) and (1.36), we determine the characteristic diffusion time $\tau_D = \ell_D / u_D$ as

$$\boxed{\tau_D = \frac{\tau_0}{\mathrm{Pe}^{1/2}} \equiv \frac{\ell_D^2}{D^{(\theta)}},} \tag{1.37}$$

where we take into account that $\mathrm{Pe}_{\ell=\ell_D} = 1$, i.e., $u_D \ell_D = D^{(\theta)}$. Let us determine the ratio of the diffusion scale to the viscous scale ℓ_D/ℓ_ν:

$$\frac{\ell_D}{\ell_\nu} = \left(\frac{\mathrm{Re}}{\mathrm{Pe}}\right)^{3/4} = \left(\frac{D^{(\theta)}}{\nu}\right)^{3/4} = \mathrm{Pr}^{-3/4}, \tag{1.38}$$

where

$$\boxed{\mathrm{Pr} = \frac{\nu}{D^{(\theta)}}} \tag{1.39}$$

is the Prandtl[11] number. Small Prandtl numbers $\mathrm{Pr} \ll 1$ implies that $\ell_\nu \ll \ell_D$, i.e., the viscous scale ℓ_ν is the smallest scale. In the opposite case of large Prandtl numbers $\mathrm{Pr} \gg 1$, the diffusion scale $\ell_D \ll \ell_\nu$ is the smallest scale.

1.2.2 Isotropic Temperature Fluctuations

In this section, we consider the case of small Prandtl numbers ($\mathrm{Pr} \ll 1$) and study temperature fluctuations in the inertial subrange of turbulence, $\ell_\nu \ll \ell_D < \ell < \ell_0$. The energy spectrum function of the velocity field is defined as $u_\ell^2 = \int_{k_0}^{k} E_\mathrm{u}(k')\, dk'$, where $k_0 = \ell_0^{-1}$ and $k = \ell^{-1}$. The Kolmogorov-Obukhov spectrum of velocity fluctuations in the inertial subrange of turbulent scales is given by

$$E_\mathrm{u}(k) = \frac{u_\ell^2}{k} = \varepsilon^{2/3}\, k^{-5/3}, \tag{1.40}$$

[11] Ludwig Prandtl (1875–1953) was a German engineer and physicist well-known for his works in fluid dynamics, aerodynamics, shock waves, plasticity, structural mechanics and meteorology.

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and the scaling for the turbulent time $\tau(k)$ is

$$\tau(k) = \frac{\ell}{u_\ell} = \frac{\ell}{(\varepsilon\ell)^{1/3}} = \varepsilon^{-1/3}\, k^{-2/3}. \tag{1.41}$$

Equations (1.40)–(1.41) are only valid when the rate of dissipation of the turbulent kinetic energy density is constant inside the inertial subrange of turbulent scales, i.e.,

$$\varepsilon \equiv \frac{u_\ell^2}{\tau_\ell} = \frac{E_{\rm u}(k)\, k}{\tau(k)} = {\rm const}, \tag{1.42}$$

and $u_\ell = (\varepsilon/k)^{1/3}$.

Spectrum function of temperature fluctuations is defined as

$$\langle\theta^2\rangle_\ell = \int_{k_0}^{k} \tilde{E}_\theta(k')\, dk'. \tag{1.43}$$

Using the dimensional analysis, we rewrite this expression as $\langle\theta^2\rangle_\ell = \tilde{E}_\theta(k)\, k$ and assume that the rate of dissipation of temperature fluctuations is constant inside the subrange of scales $\ell_D < \ell < \ell_0$, i.e.,

$$\varepsilon_\theta \equiv \frac{\left\langle\theta^2\right\rangle_\ell}{\tau_\ell} = \frac{\tilde{E}_\theta(k)\, k}{\tau(k)} = {\rm const}. \tag{1.44}$$

The condition (1.44) for temperature fluctuations is analogous to condition (1.42) for velocity fluctuations in the inertial range of turbulence. Equations (1.42) and (1.44) yield the spectrum of isotropic temperature fluctuations inside the scale-dependent turbulent diffusion range of scales, $\ell_D < \ell < \ell_0$:

$$\boxed{\tilde{E}_\theta(k) \sim E_{\rm u}(k) \sim \varepsilon^{2/3}\, k^{-5/3}.} \tag{1.45}$$

This spectrum was obtained by Obukhov (1949) and Corrsin (1951).[12]

Let us consider the case, $\mathrm{Pr} > 1$, and study temperature fluctuations in the viscous subrange of scales, $\ell_D < \ell < \ell_\nu$. In this range of scales, Eq. (1.44) is valid, but the time $\tau(k)$ does not have a universal scaling. If $\tau(k) = {\rm const}$, the spectrum of temperature fluctuations is

$$\boxed{\tilde{E}_\theta(k) \sim k^{-1}.} \tag{1.46}$$

This spectrum was obtained by Batchelor (1959)[13] and Kraichnan (1968).[14]

[12] Stanley Corrsin (1920–1986) was an American physicist, well-known for his works in experimental and theoretical fluid dynamics, turbulence and turbulent mixing.

[13] George Keith Batchelor (1920–2000) was an applied mathematician (who was born in Australia and worked at the University of Cambridge) well-known for his works in fluid dynamics, theory of turbulence and turbulent transport.

[14] Robert Harry Kraichnan (1928–2008) was an American theoretical physicist well-known for his works in the theory of turbulence, turbulent transport and magnetohydrodynamics.

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1.2.3 Anisotropic Temperature Fluctuations in the Inertial-Diffusion Range

We consider anisotropic temperature fluctuations caused by the tangling of the mean temperature gradient by velocity fluctuations in the inertial-diffusion subrange of scales, $\ell_\nu < \ell < \ell_D$. This subrange of scales corresponds to the Prandtl numbers, $\mathrm{Pr} < 1$. We use dimensional analysis, taking into account that molecular diffusion is a key effect in this subrange of scales. This implies that the molecular diffusion term, $D^{(\theta)} \Delta\theta$, in Eq. (1.29) should be balanced by the source term, $(\boldsymbol{u}\cdot\nabla)\overline{T}$, for temperature fluctuations, i.e.,

$$|D^{(\theta)} \Delta\theta| \sim |(\boldsymbol{u} \cdot \nabla)\overline{T}|, \tag{1.47}$$

which yields

$$\left\langle \theta^2 \right\rangle_\ell \sim u_\ell^2 \left(\frac{\ell^2 \nabla \overline{T}}{D^{(\theta)}} \right)^2 . \tag{1.48}$$

In the $\boldsymbol{k}$ space, Eq. (1.48) implies that

$$\tilde{E}_\theta(k) \sim E_\mathrm{u}(k)\, k^{-4} \left(\frac{\nabla \overline{T}}{D^{(\theta)}} \right)^2 , \tag{1.49}$$

where $E_\mathrm{u}(k)$ is the spectrum function of velocity fluctuations. Since the subrange of scales $\ell_\nu < \ell < \ell_D$ corresponds to the inertial range of scales, velocity fluctuations have the Kolmogorov-Obukhov spectrum (1.40). Therefore, the spectrum of the anisotropic temperature fluctuations in the inertial-diffusion range of scales is

$$\boxed{\tilde{E}_\theta(k) \sim \varepsilon^{2/3}\, k^{-17/3} \left(\frac{\nabla \overline{T}}{D^{(\theta)}} \right)^2 .} \tag{1.50}$$

This spectrum was obtained by G. Batchelor, I. Howells and A. Townsend[15] (Batchelor et al., 1959).

1.2.4 Anisotropic Temperature Fluctuations in the Inertial-Turbulent Diffusion Range

We consider anisotropic temperature fluctuations caused by the tangling of the mean temperature gradient by velocity fluctuations in the inertial-turbulent diffusion range of scales, $\ell_D < \ell < \ell_0$. This subrange of scales corresponds to the small Prandtl numbers, $\mathrm{Pr} < 1$. We take into account that the main effect of turbulence

[15] Albert Alan Townsend (1917–2010) was a physicist (who was born in Australia and worked at the University of Cambridge) well-known for his works in fluid dynamics, experimental study of turbulence and turbulent transport, meteorology and nuclear physics.

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on temperature fluctuations in incompressible flow is the scale-dependent turbulent diffusion that is much larger than the molecular diffusion for large Péclet numbers. Let us average Eq. (1.29) over an ensemble up to the scale ℓ_* that is inside the interval: $\ell_D \ll \ell_* \ll \ell_0$. This yields the renormalized equation for temperature fluctuations:

$$\frac{\partial \theta}{\partial t} - D_{_T}(\ell)\, \Delta\theta = -(\boldsymbol{u} \cdot \nabla)\overline{T}, \tag{1.51}$$

where $D_{_T}(\ell)$ is the scale-dependent turbulent diffusion coefficient that can be estimated as

$$\boxed{D_{_T}(\ell) = \ell\, u_\ell.} \tag{1.52}$$

In the subrange of scales $\ell_D \ll \ell_*$ the turbulent diffusion term $D_{_T}(\ell)\Delta\theta$ in Eq. (1.51) should be balanced by the source term, $(\boldsymbol{u}\cdot\nabla)\overline{T}$, for temperature fluctuations, i.e.,

$$|D_{_T}(\ell)\, \Delta\theta| \sim |(\boldsymbol{u} \cdot \nabla)\overline{T}|. \tag{1.53}$$

This implies that

$$\left\langle \theta^2 \right\rangle_\ell \sim u_\ell^2 \left(\frac{\ell^2\, \nabla\overline{T}}{D_{_T}(\ell)} \right)^2 \sim u_\ell^2 \left(\frac{\ell^2\, \nabla\overline{T}}{\ell\, u_\ell} \right)^2 \sim \left(\ell\, \nabla\overline{T}\right)^2, \tag{1.54}$$

where we used Eq. (1.52). Equation (1.54) written in the $\boldsymbol{k}$ space yields the spectrum of anisotropic temperature fluctuations in the inertial-turbulent diffusion range of scales $\ell_D \ll \ell \ll \ell_0$:

$$\boxed{\tilde{E}_\theta(k) \sim k^{-3} \left(\nabla\overline{T}\right)^2,} \tag{1.55}$$

where we take into account in Eq. (1.54) that according to dimensional analysis, $\langle\theta^2\rangle_\ell = \tilde{E}_\theta(k)\, k$ and $k = \ell^{-1}$. This spectrum is independent of the spectrum of the turbulent velocity field because u_ℓ^2 is canceled in Eq. (1.54). The spectrum (1.55) was obtained by A. Wheelon using the dimensional analysis (Wheelon, 1957) and by T. Elperin, N. Kleeorin and I. Rogachevskii, applying the renormalization approach (Elperin et al., 1996a).

1.3 Turbulent Transport of Temperature Fields: Dimensional Analysis

1.3.1 Governing Equations

We consider incompressible fluid velocity field $\boldsymbol{U}(t, \boldsymbol{x})$ satisfying the continuity equation: $\nabla \cdot \boldsymbol{U} = 0$. Since the velocity field is incompressible, Eq. (1.22) for the fluid temperature field $T(t, \boldsymbol{x})$ can be rewritten in the following form:

$$\boxed{\frac{\partial T}{\partial t} + \nabla \cdot (T\, \boldsymbol{U}) = D^{(\theta)}\, \Delta T.} \tag{1.56}$$

The velocity field is a random turbulent field created, e.g., by external forcing.

1.3.2 Mean-Field Approach

Our goal is to study the long-term evolution of the temperature field in the large scales, i.e., in spatial scales $L_T \gg \ell_0$, and the time scales $t_T \gg \tau_0$, where τ_0 is the characteristic turbulent time in the integral turbulent scale ℓ_0, L_T is the characteristic spatial scale of variations of the mean temperature field and t_T is the characteristic time-scale of variations of the mean temperature field. We use a mean-field approach in which all quantities are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. In particular, the temperature field $T = \overline{T} + \theta$, where $\overline{T} = \langle T \rangle$ is the mean fluid temperature, θ are temperature fluctuations and $\langle \theta \rangle = 0$. The angular brackets $\langle \ldots \rangle$ denote ensemble averaging. In similar fashion, we decompose a velocity field, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity, $\boldsymbol{u}$ are velocity fluctuations and $\langle \boldsymbol{u} \rangle = 0$. This decomposition corresponds to the Reynolds rules. Averaging Eq. (1.56) over an ensemble of turbulent velocity field, we arrive at the mean-field equation (1.28) for the mean temperature field.

1.3.3 Equation for Temperature Fluctuations

Equation (1.28) is not closed because we do not know the turbulent heat flux $\boldsymbol{F}^{(\theta)} = \langle \boldsymbol{u}\theta \rangle$. To determine the turbulent heat flux, we use the equation for temperature fluctuations that is obtained by subtracting Eq. (1.28) from Eq. (1.56):

$$\boxed{\frac{\partial \theta}{\partial t} + \nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle) - D^{(\theta)} \Delta \theta = -(\boldsymbol{u} \cdot \nabla)\overline{T}.} \tag{1.57}$$

The terms, $\nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle)$, on the left-hand side of Eq. (1.57) are the nonlinear terms, while the first term, $-(\boldsymbol{u} \cdot \nabla)\overline{T}$, on the right-hand side of Eq. (1.57) determines the source of temperature fluctuations produced by the tangling of the gradient of the mean temperature, $\nabla \overline{T}$, by random velocity fluctuations, $\boldsymbol{u}$. Since Eq. (1.57) is nonlinear equation for temperature fluctuations, it cannot be solved exactly for the arbitrary range of parameters and arbitrary velocity field. Therefore, we have to use different approximate methods for the solution of Eq. (1.57). First, we consider a one-way coupling, i.e., we take into account the effect of the turbulent velocity on the temperature field, but neglect the feedback effect of the

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temperature field on the turbulent fluid flow. This implies that the temperature field is a passive scalar.

1.3.4 Dimensional Analysis

The first method that we apply here is the dimensional analysis. The dimension of the left-hand side of Eq. (1.57) is the rate of change of temperature fluctuations, i.e., θ/τ_θ, where τ_θ is the characteristic time of temperature fluctuations. We replace the left-hand side of Eq. (1.57) by θ/τ_θ, that yields:

$$\theta = -\tau_\theta \, (\boldsymbol{u} \cdot \nabla)\overline{T}. \tag{1.58}$$

We consider two cases of large and small Péclet numbers:

- *Large Péclet numbers*, $\mathrm{Pe} = u_0 \, \ell_0 / D^{(\theta)} \gg 1$. We also consider the case of large Reynolds numbers, $\mathrm{Re} = u_0 \, \ell_0/\nu \gg 1$. This implies that we consider fully developed turbulence. In this case, the characteristic time of temperature fluctuations, τ_θ, can be identified with the correlation time τ_0 of the turbulent velocity field. Therefore, in the framework of the dimensional analysis, we replace the left-hand side of Eq. (1.57) with θ/τ_0. This yields the expression for temperature fluctuations:

$$\theta = -\tau_0 \, (\boldsymbol{u} \cdot \nabla)\overline{T}. \tag{1.59}$$

- *Small Péclet numbers*, $\mathrm{Pe} \ll 1$. In this case, the nonlinear terms are much smaller than the diffusion terms. This implies that the molecular diffusion for $\mathrm{Pe} \ll 1$ is the main process, which determines the dynamics of temperature fluctuations. Therefore, we assume that the time τ_θ can be identified with the molecular diffusion time $\tau_D = \ell_0^2/D^{(\theta)}$, and the solution of Eq. (1.57) for $\mathrm{Pe} \ll 1$ reads

$$\theta = -\frac{\ell_0^2}{D^{(\theta)}} \, (\boldsymbol{u} \cdot \nabla)\overline{T}. \tag{1.60}$$

1.3.5 Turbulent Heat Flux and Level of Temperature Fluctuations

Large Péclet Numbers

Let us consider the case of large Péclet numbers $\mathrm{Pe} \gg 1$ and determine the turbulent heat flux and level of temperature fluctuations. Multiplying Eq. (1.59) by velocity fluctuations, u_i, and averaging over an ensemble of turbulent velocity field, we obtain the turbulent heat flux:

$$\langle \theta \, u_i \rangle = -\tau_0 \, \langle u_i (\boldsymbol{u} \cdot \nabla)\overline{T} \rangle = -\tau_0 \, \langle u_i u_j \rangle \, \nabla_j \overline{T}, \tag{1.61}$$

where we took into account that $\boldsymbol{u} \cdot \nabla \equiv u_j \, \nabla_j = u_1 \, \nabla_1 + u_2 \, \nabla_2 + u_3 \, \nabla_3 \equiv u_x \, \nabla_x + u_y \, \nabla_y + u_z \, \nabla_z$ (i.e., there is summation in the repeating indexes). In

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isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \, \langle \boldsymbol{u}^2 \rangle / 3$, where δ_{ij} is the Kronecker tensor (or the unit matrix), that is defined as $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. Therefore, for an isotropic turbulence, the turbulent heat flux reads

$$\boxed{\boldsymbol{F}^{(\theta)} \equiv \langle \theta \, \boldsymbol{u} \rangle = -D_{_T} \, \nabla \overline{T},} \tag{1.62}$$

with the coefficient of turbulent diffusion of the temperature field for large Péclet numbers:

$$\boxed{D_{_T} = \frac{1}{3} \, \tau_0 \, \left\langle \boldsymbol{u}^2 \right\rangle.} \tag{1.63}$$

Using Eq. (1.59) for θ^2 and averaging over an ensemble of turbulent velocity field, we determine the level of temperature fluctuations $\langle \theta^2 \rangle$:

$$\left\langle \theta^2 \right\rangle = \tau_0^2 \left\langle \left[(\boldsymbol{u} \cdot \nabla) \overline{T} \, \right]^2 \right\rangle = \tau_0^2 \, \langle u_i u_j \rangle \, \left(\nabla_i \overline{T} \right) \left(\nabla_j \overline{T} \right). \tag{1.64}$$

Therefore, for an isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \, \langle \boldsymbol{u}^2 \rangle / 3$, the level of temperature fluctuations for large Péclet numbers is given by

$$\boxed{\left\langle \theta^2 \right\rangle = \frac{1}{3} \ell_0^2 \, \left(\nabla \overline{T} \right)^2,} \tag{1.65}$$

where $\ell_0 = \tau_0 \, u_0$ and $u_0 \equiv u_{\rm rms} = \sqrt{\langle \boldsymbol{u}^2 \rangle}$ is the r.m.s. velocity fluctuations (characteristic turbulent velocity).

Small Péclet Numbers

Now we consider the case of small Péclet numbers Pe $\ll 1$ and determine the turbulent heat flux and level of temperature fluctuations. Multiplying Eq. (1.60) by u_i and averaging this equation over a statistics of a random velocity field, we obtain

$$\langle \theta \, u_i \rangle = -\frac{\ell_0^2}{D^{(\theta)}} \, \langle u_i u_j \rangle \, (\nabla_j \overline{T}). \tag{1.66}$$

Therefore, for isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \, \langle \boldsymbol{u}^2 \rangle / 3$, the turbulent heat flux $F_i^{(\theta)}$ for small Péclet numbers is given by $\boldsymbol{F}^{(\theta)} \equiv \langle \theta \, \boldsymbol{u} \rangle = -D_{_T} \, \nabla \overline{T}$, which coincides with Eq. (1.62) derived for large Péclet numbers, but with a different coefficient of turbulent diffusion:

$$\boxed{D_T = \frac{u_0 \, \ell_0}{3} \, \mathrm{Pe}.} \tag{1.67}$$

Since Pe $\ll 1$, the coefficient of turbulent diffusion D_T is much smaller than the molecular diffusion coefficient $D^{(\theta)}$. Using Eq. (1.60) for θ^2 and averaging over an ensemble of turbulent velocity field, we determine the level of temperature fluctuations $\langle \theta^2 \rangle$:

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$$\langle\theta^2\rangle = \left(\frac{\ell_0^2}{D^{(\theta)}}\right)^2 \left\langle\left[(\boldsymbol{u}\cdot\nabla)\overline{T}\,\right]^2\right\rangle = \left(\frac{\ell_0^2}{D^{(\theta)}}\right)^2 \langle u_i u_j\rangle\,(\nabla_i\overline{T})(\nabla_j\overline{T}). \quad (1.68)$$

Therefore, for isotropic turbulence the level of temperature fluctuations for small Péclet numbers is given by

$$\boxed{\langle\theta^2\rangle = \frac{1}{3}\mathrm{Pe}^2\ell_0^2\left(\nabla\overline{T}\right)^2.} \quad (1.69)$$

1.3.6 Mean-Field Equation

Substituting the turbulent heat flux (1.62) into Eq. (1.28), and taking into account that for homogeneous turbulence the coefficient of turbulent diffusion is independent of coordinate, so that $\nabla\cdot\left(D_T\,\nabla\overline{T}\right) = D_T\,\Delta\overline{T}$, we obtain the mean-field equation for temperature field for homogeneous, isotropic and incompressible turbulence:

$$\frac{\partial\overline{T}}{\partial t} = \left(D^{(\theta)} + D_T\right)\,\Delta\overline{T}. \quad (1.70)$$

Since the coefficient of turbulent diffusion D_T is positive, Eq. (1.70) implies that *the main effect of turbulence is enhancement of the diffusion of the mean temperature field, i.e., turbulence enhances the mixing*.

1.3.7 Solving the Diffusion Equation

Let us solve the diffusion equation (1.70) with the initial condition $T_0(\boldsymbol{r}) = \overline{T}(t=0,\boldsymbol{r})$. We use the Fourier[16] transform in the $\boldsymbol{k}$ space:

$$\overline{T}(t,\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int \overline{T}(t,\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k}, \quad (1.71)$$

$$\overline{T}(t,\boldsymbol{k}) = \int \overline{T}(t,\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}, \quad (1.72)$$

where $\boldsymbol{k}$ is the wave vector. We seek a solution for Eq. (1.70) in the form given by Eq. (1.71). Substituting solution (1.71) into Eq. (1.70), we obtain

$$\int\left(\frac{d\overline{T}(t,\boldsymbol{k})}{dt} + D_*\boldsymbol{k}^2\overline{T}(t,\boldsymbol{k})\right)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k} = 0, \quad (1.73)$$

[16] Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician well-known for his works in mathematical physics, algebra, etc.

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where $D_* = D^{(\theta)} + D_{_T}$ is the total diffusion coefficient that is independent of the coordinate. Equation (1.73) yields the following ordinary differential equation where $\boldsymbol{k}$ is considered as a parameter, and the time t is a variable:

$$\frac{d\overline{T}(t,\boldsymbol{k})}{dt} = -D_* \boldsymbol{k}^2 \overline{T}(t,\boldsymbol{k}). \tag{1.74}$$

Equation (1.74) with the initial condition $\overline{T}(t = 0, \boldsymbol{k}) = \overline{T}_0(\boldsymbol{k})$ has the following solution:

$$\overline{T}(t,\boldsymbol{k}) = \overline{T}_0(\boldsymbol{k}) \, \exp(-D_* \boldsymbol{k}^2 t). \tag{1.75}$$

The Fourier transform (1.72) for the initial temperature distribution $\overline{T}_0(\boldsymbol{k})$ reads:

$$\overline{T}_0(\boldsymbol{k}) = \int \overline{T}_0(\boldsymbol{r}') \exp(-\mathrm{i}\,\boldsymbol{k} \cdot \boldsymbol{r}') \, d\boldsymbol{r}'. \tag{1.76}$$

Substituting Eq. (1.76) into Eq. (1.75), we obtain

$$\overline{T}(t,\boldsymbol{k}) = \int \overline{T}_0(\boldsymbol{r}') \exp(-\mathrm{i}\,\boldsymbol{k} \cdot \boldsymbol{r}') \, \exp(-D_* \boldsymbol{k}^2 t) \, d\boldsymbol{r}'. \tag{1.77}$$

Now we substitute Eq. (1.77) for $\overline{T}(t,\boldsymbol{k})$ into Eq. (1.71), which yields

$$\overline{T}(t,\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{k} \int \overline{T}_0(\boldsymbol{r}') \exp\left[\mathrm{i}\,\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}') - D_* \boldsymbol{k}^2 t\right] d\boldsymbol{r}'. \tag{1.78}$$

We use the following identity:

$$\begin{aligned} \mathrm{i}\,\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}') - D_* \boldsymbol{k}^2 t &= -D_* t \left[\boldsymbol{k}^2 - \frac{\mathrm{i}\,\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')}{D_* t} + \left(\frac{\mathrm{i}\,(\boldsymbol{r} - \boldsymbol{r}')}{2D_* t}\right)^2\right] - \frac{(\boldsymbol{r} - \boldsymbol{r}')^2}{4D_* t} \\ &= -D_* t \left(\boldsymbol{k} - \frac{\mathrm{i}\,(\boldsymbol{r} - \boldsymbol{r}')}{2D_* t}\right)^2 - \frac{(\boldsymbol{r} - \boldsymbol{r}')^2}{4D_* t}. \end{aligned} \tag{1.79}$$

This identity allows us to rewrite solution (1.78) of the diffusion equation (1.70) in the final form:

$$\boxed{\overline{T}(t,\boldsymbol{r}) = \frac{1}{(4\pi \, D_* t)^{3/2}} \int \overline{T}_0(\boldsymbol{r}') \exp\left[-\frac{(\boldsymbol{r} - \boldsymbol{r}')^2}{4D_* t}\right] d\boldsymbol{r}',} \tag{1.80}$$

where we calculate the following integral,

$$\begin{aligned} I &= \frac{1}{(2\pi)^3} \int \exp\left[-D_* t \left(\boldsymbol{k} - \frac{\mathrm{i}\,(\boldsymbol{r} - \boldsymbol{r}')}{2D_* t}\right)^2\right] d\boldsymbol{k} = \frac{1}{(2\pi)^3} \int \exp(-a^2 \tilde{\boldsymbol{k}}^2) \, d\tilde{\boldsymbol{k}} \\ &= \frac{1}{(2\pi)^3} \int_0^\infty \exp(-a^2 \tilde{k}^2) \, \tilde{k}^2 \, d\tilde{k} \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{1}{(4\pi \, D_* t)^{3/2}}, \end{aligned} \tag{1.81}$$

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by using the following new variables: $a = (D_* t)^{1/2}$ and

$$\tilde{\boldsymbol{k}} = \boldsymbol{k} - \frac{\mathrm{i}\,(\boldsymbol{r} - \boldsymbol{r}')}{2D_* t}. \tag{1.82}$$

When the initial condition is localized at $\boldsymbol{r} = 0$, i.e., $\overline{T}_0(\boldsymbol{r}) = T_* \delta(\boldsymbol{r})$, the solution (1.80) of the diffusion equation reads

$$\boxed{\overline{T}(t, \boldsymbol{r}) = \frac{T_*}{(4\pi\, D_* t)^{3/2}} \exp\left[-\frac{\boldsymbol{r}^2}{4D_* t}\right],} \tag{1.83}$$

where $\delta(\boldsymbol{r})$ is the Dirac delta function defined by the following properties: $\delta(\boldsymbol{r}) = 0$ when $\boldsymbol{r} \neq 0$; $\delta(\boldsymbol{r}) = \infty$ when $\boldsymbol{r} = 0$; the normalization condition for this function is $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$ and $\int f(\boldsymbol{r})\delta(\boldsymbol{r} - \boldsymbol{r}_0)\, d\boldsymbol{r} = f(\boldsymbol{r}_0)$.

When the initial condition is one-dimensional (i.e., it depends only on x), the solution (1.80) of the diffusion equation reads

$$\boxed{\overline{T}(t, x) = \frac{1}{(4\pi\, D_* t)^{1/2}} \int \overline{T}_0(x') \exp\left[-\frac{(x - x')^2}{4D_* t}\right] dx'.} \tag{1.84}$$

When the initial condition is one-dimensional and localized at $x = 0$, i.e., $\overline{T}_0(\boldsymbol{r}) = T_* \delta(x)$, the solution (1.84) of the diffusion equation reads

$$\boxed{\overline{T}(t, x) = \frac{T_*}{(4\pi\, D_* t)^{1/2}} \exp\left[-\frac{x^2}{4D_* t}\right].} \tag{1.85}$$

Equation (1.85) implies that the mean temperature $\overline{T}(t, x = 0)$ at $x = 0$ decreases with time as $\overline{T}(t, x = 0) \sim T_*/\sqrt{t}$ and the characteristic size of the mean temperature variations is $L_T \sim \sqrt{D_* t}$. This implies that the characteristic diffusion time is

$$\boxed{\tau_{\rm diffusion} = \frac{L_T^2}{D_*}.} \tag{1.86}$$

1.4 Multi-Scale Approach

To apply rigorous methods for solving Eq. (1.57) for temperature fluctuations, we consider multi-scale approach (Roberts and Soward, 1975), which allows us to introduce fast and slow variables, and separate small-scale effects corresponding to fluctuations and large-scale effects describing mean fields. In particular, this allows us to study direct effect of small-scale turbulence on the evolution of the mean temperature field.

1.4.1 Multi-Scale Approach: Spatial Scales

In the framework of the mean-field approach, we assume that there is a separation of spatial and temporal scales, i.e., $\ell_0 \ll L_T$ and $\tau_0 \ll t_T$, where L_T and t_T are the characteristic spatial and temporal scales characterizing the variations of the mean temperature field. The mean fields depend on "slow" variables, while fluctuations depend on "fast" variables. Separation into slow and fast variables is widely used in theoretical physics, and all calculations are reduced to Taylor expansions of all functions using small parameters ℓ_0/L_T and τ_0/t_T. The findings are further truncated to leading order terms.

Separation to slow and fast variables is performed by means of a standard multi-scale approach. In the framework of this approach, we consider the two-point second-order correlation function of velocity field:

$$f_{ij}(t, \boldsymbol{x}, \boldsymbol{y}) = \left\langle u_i(t, \boldsymbol{x}) \, u_j(t, \boldsymbol{y}) \right\rangle . \tag{1.87}$$

We use the Fourier transform in the $\boldsymbol{k}$ space:

$$u_i(t, \boldsymbol{x}) = \int u_i(t, \boldsymbol{k}_1) \exp(\mathrm{i}\boldsymbol{k}_1 \cdot \boldsymbol{x}) \, d\boldsymbol{k}_1, \tag{1.88}$$

where hereafter for simplicity of notations we omit the coefficient $(2\pi)^{-3}$, multiplying by the Fourier integral (1.88). We apply the Fourier transform in $\boldsymbol{k}$ for the two-point correlation function as

$$\left\langle u_i(t, \boldsymbol{x}) \, u_j(t, \boldsymbol{y}) \right\rangle = \int \int \left\langle u_i(t, \boldsymbol{k}_1) \, u_j(t, \boldsymbol{k}_2) \right\rangle \exp\left(\mathrm{i}\boldsymbol{k}_1 \cdot \boldsymbol{x} + \mathrm{i}\boldsymbol{k}_2 \cdot \boldsymbol{y}\right) \, d\boldsymbol{k}_1 \, d\boldsymbol{k}_2. \tag{1.89}$$

We use new variables in the physical space, $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ and $\boldsymbol{R} = (\boldsymbol{x} + \boldsymbol{y})/2$, which correspond to small-scale and large-scale spatial variables, respectively. Now we express the old variables $\boldsymbol{x}$ and $\boldsymbol{y}$ in terms of new variables: $\boldsymbol{x} = \boldsymbol{r}/2 + \boldsymbol{R}$ and $\boldsymbol{y} = -\boldsymbol{r}/2 + \boldsymbol{R}$. Using these expressions, we rewrite $\boldsymbol{k}_1 \cdot \boldsymbol{x} + \boldsymbol{k}_2 \cdot \boldsymbol{y}$ in the new variables:

$$\boldsymbol{k}_1 \cdot \boldsymbol{x} + \boldsymbol{k}_2 \cdot \boldsymbol{y} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{r} + (\boldsymbol{k}_1 + \boldsymbol{k}_2) \cdot \boldsymbol{R} = \boldsymbol{k} \cdot \boldsymbol{r} + \boldsymbol{K} \cdot \boldsymbol{R}, \tag{1.90}$$

where we also use new variables in the $\boldsymbol{k}$ space: $\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2$ that corresponds to the small scales, and $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$ that characterizes the large scales. In the integral (1.89) we change variables:

$$\begin{aligned} \left\langle u_i(t, \boldsymbol{x}) \, u_j(t, \boldsymbol{y}) \right\rangle &= \int \int f_{ij}(t, \boldsymbol{k}, \boldsymbol{K}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r} + \mathrm{i}\boldsymbol{K} \cdot \boldsymbol{R}) \, d\boldsymbol{k} \, d\boldsymbol{K} \\ &= \int f_{ij}(t, \boldsymbol{k}, \boldsymbol{R}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \, d\boldsymbol{k}, \end{aligned} \tag{1.91}$$

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where

$$f_{ij}(t,\boldsymbol{k},\boldsymbol{K}) = \left\langle u_i(t,\boldsymbol{k}+\boldsymbol{K}/2)u_j(t,-\boldsymbol{k}+\boldsymbol{K}/2)\right\rangle, \tag{1.92}$$

$$f_{ij}(t,\boldsymbol{k},\boldsymbol{R}) = \int f_{ij}(t,\boldsymbol{k},\boldsymbol{K})\exp(\mathrm{i}\,\boldsymbol{K}\cdot\boldsymbol{R})\,d\boldsymbol{K}, \tag{1.93}$$

and we take into account that the Jacobian of this transformation equals 1. For the one-point correlation function:

$$\begin{aligned}\left\langle u_i(t,\boldsymbol{x})\,u_j(t,\boldsymbol{x})\right\rangle &\equiv \lim_{\boldsymbol{x}\to\boldsymbol{y}}\left\langle u_i(t,\boldsymbol{x})\,u_j(t,\boldsymbol{y})\right\rangle\\ &= \int\left\langle u_i(t,\boldsymbol{k}+\boldsymbol{K}/2)\,u_j(t,-\boldsymbol{k}+\boldsymbol{K}/2)\right\rangle d\boldsymbol{k}\end{aligned} \tag{1.94}$$

[see Eq. (1.91) for $\boldsymbol{r}\to 0$]. For homogeneous turbulence, the correlation function $f_{ij}(t,\boldsymbol{k},\boldsymbol{R})$ is independent of $\boldsymbol{R}$, i.e., $f_{ij}(t,\boldsymbol{k},\boldsymbol{R}) = f_{ij}(t,\boldsymbol{k})$. This implies that for homogeneous turbulence Eq. (1.94) reads

$$\boxed{\left\langle u_i(t,\boldsymbol{x})\,u_j(t,\boldsymbol{x})\right\rangle = \lim_{\boldsymbol{x}\to\boldsymbol{y}}\left\langle u_i(t,\boldsymbol{x})\,u_j(t,\boldsymbol{y})\right\rangle = \int\left\langle u_i(t,\boldsymbol{k})\,u_j(t,-\boldsymbol{k})\right\rangle d\boldsymbol{k}.} \tag{1.95}$$

Trace of the tensor in Eq. (1.95) yields

$$\boxed{\left\langle \boldsymbol{u}^2\right\rangle = \int \langle u_i(t,\boldsymbol{k})\,u_i(t,-\boldsymbol{k})\rangle\;d\boldsymbol{k}.} \tag{1.96}$$

Therefore, such transformation allows us to separate the integration over small-scales, i.e., over turbulence scales, see Eq. (1.91), and over large scales, see Eq. (1.93). This implies that we introduce fast and slow variables and separate the integration over the fast and slow variables. We remember that in this analysis, we assumed that there exists a separation of scales, i.e., this implies that $r \le l_0 \ll R$. Usually, in final results one should check that this assumption is indeed valid.

1.4.2 Multi-Scale Approach: Temporal Scales

Let us consider a one-point non-instantaneous correlation function:

$$f_{ij}(t_1,t_2,\boldsymbol{x}) = \left\langle u_i(t_1,\boldsymbol{x})\,u_j(t_2,\boldsymbol{x})\right\rangle. \tag{1.97}$$

We use the Fourier transform in a spectral ω space,

$$u_i(t,\boldsymbol{x}) = \int u_i(\omega_1,\boldsymbol{x})\exp(\mathrm{i}\,\omega_1 t_1)\,d\omega_1, \tag{1.98}$$

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where ω_1 is the frequency. We use the Fourier transform in the ω space for the non-instantaneous one-point correlation function:

$$\langle u_i(t_1, \boldsymbol{x})\, u_j(t_2, \boldsymbol{x})\rangle = \int\int d\omega_1\, d\omega_2\, \langle u_i(\omega_1, \boldsymbol{x})\, u_j(\omega_2, \boldsymbol{x})\rangle \times \exp\left[\mathrm{i}\,(\omega_1 t_1 + \omega_2 t_2)\right]. \tag{1.99}$$

We use new variables, $\tau = t_1 - t_2$ and $t = (t_1 + t_2)/2$, which correspond to fast and slow variables, respectively. We express the old variables t_1 and t_2 in terms of new variables: $t_1 = \tau/2 + t$ and $t_2 = -\tau/2 + t$. Using these expressions, we rewrite $\omega_1 t_1 + \omega_2 t_2$ in the new variables,

$$\omega_1 t_1 + \omega_2 t_2 = \frac{1}{2}(\omega_1 - \omega_2)\tau + (\omega_1 + \omega_2)t = \omega\,\tau + \Omega\, t, \tag{1.100}$$

where we also use new variables in the spectral space: $\omega = (\omega_1 - \omega_2)/2$, which corresponds to turbulent time scales (the fast variables), and $\Omega = \omega_1 + \omega_2$, which corresponds to the long-term time scales (the slow variables) describing the mean fields. In the integral (1.99), we change variables:

$$\begin{aligned}\langle u_i(t_1, \boldsymbol{x})\, u_j(t_2, \boldsymbol{x})\rangle &= \int\int f_{ij}(\omega, \Omega, \boldsymbol{x}) \exp(\mathrm{i}\,\omega\,\tau + \mathrm{i}\,\Omega\, t)\, d\omega\, d\Omega \\ &= \int f_{ij}(\omega, \Omega, \boldsymbol{x}) \exp(\mathrm{i}\,\omega\,\tau)\, d\omega,\end{aligned} \tag{1.101}$$

where

$$f_{ij}(\omega, \Omega, \boldsymbol{x}) = \langle u_i(\omega + \Omega/2, \boldsymbol{x})\, u_j(-\omega + \Omega/2, \boldsymbol{x})\rangle, \tag{1.102}$$

$$f_{ij}(\omega, t, \boldsymbol{x}) = \int f_{ij}(\omega, \Omega, \boldsymbol{x}) \exp(\mathrm{i}\,\Omega\, t)\, d\Omega, \tag{1.103}$$

and we take into account that the Jacobian of this transformation equals 1. For the one-point correlation function,

$$\boxed{\langle u_i(t, \boldsymbol{x})\, u_j(t, \boldsymbol{x})\rangle \equiv \lim_{t_1 \to t_2} \langle u_i(t_1, \boldsymbol{x})\, u_j(t_2, \boldsymbol{x})\rangle = \int \langle u_i(\omega, \boldsymbol{x})\, u_j(-\omega, \boldsymbol{x})\rangle\, d\omega} \tag{1.104}$$

[see Eq. (1.101) for $\tau \to 0$]. For stationary (in statistical sense) turbulence, the correlation function $f_{ij}(\omega, t, \boldsymbol{x})$ is independent of t, i.e., $f_{ij}(\omega, t, \boldsymbol{x}) = f_{ij}(\omega, \boldsymbol{x})$. Trace of the tensor in Eq. (1.104) yields

$$\boxed{\langle \boldsymbol{u}^2\rangle = \int \langle u_i(\omega, \boldsymbol{x})\, u_i(-\omega, \boldsymbol{x})\rangle\, d\omega.} \tag{1.105}$$

Therefore, such transformation allows us to separate the integration over fast variables, i.e., the integration over turbulence time scales, see Eq. (1.104), and slow variables corresponding to mean fields, see Eq. (1.103). This implies we introduce fast and slow variables and separate the integration over the fast and slow variables. In this analysis, we assume that there is a separation of time scales. In particular, this implies that $\tau_0 \ll t_L$. In the final results, one should check that this assumption is indeed valid.

1.5 Turbulent Transport of Temperature Fields: Quasi-Linear Approach

In this section, we apply the multi-scale approach and the quasi-linear approach to study the turbulent transport of a temperature field in an incompressible homogeneous and isotropic random velocity field. The quasi-linear approach is valid for small Péclet numbers.

1.5.1 Governing Equations

Temperature field $T(t, \mathrm{x})$ is determined by Eq. (1.56). Averaging Eq. (1.56) over an ensemble of turbulent velocity field, we obtain the mean-field equation for the mean temperature $\overline{T}$:

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T}\, \overline{\boldsymbol{U}} + \langle \theta\, \boldsymbol{u} \rangle \right) = D^{(\theta)} \Delta \overline{T}. \tag{1.106}$$

For simplicity, we consider the case $\overline{\boldsymbol{U}} = 0$. To determine the turbulent heat flux $\boldsymbol{F}^{(\theta)} \equiv \langle \theta\, \boldsymbol{u} \rangle$, we use the equation for temperature fluctuations obtained by subtracting Eq. (1.106) from Eq. (1.56):

$$\frac{\partial \theta}{\partial t} + \nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle) - D^{(\theta)} \Delta \theta = -(\boldsymbol{u} \cdot \nabla) \overline{T}. \tag{1.107}$$

In the previous Section, we used the dimensional analysis to solve the nonlinear equation (1.107) and to determine the turbulent heat flux. In this section, we consider the case when the nonlinear terms in Eq. (1.107) are much smaller than the diffusion term, i.e., for small Péclet numbers $\mathrm{Pe} \ll 1$. We also consider a one-way coupling that implies that the temperature field is a passive scalar.

1.5.2 Quasi-Linear Approach

To study temperature fluctuations for small Péclet numbers, we use a quasi-linear approach. In the framework of this approach, we neglect the nonlinear terms but keep the molecular diffusion term in Eq. (1.107). For this reason, this approach is called a quasi-linear or a perturbation approach. In astrophysics, this approach is known as a first-order smoothing approximation or a second-order correlation approximation (Moffatt, 1978; Krause and Rädler, 1980). In scattering theory,

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this approach is similar to the Born approximation (Taylor, 1972). We rewrite Eq. (1.107) in a Fourier space, i.e., we use the Fourier transform in the $\boldsymbol{k} - \omega$ space:

$$\theta(t, \boldsymbol{x}) = \int \int \theta(\omega, \boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\,\omega\, t)\, d\boldsymbol{k}\, d\omega, \tag{1.108}$$

$$u_i(t, \boldsymbol{x}) = \int \int u_i(\omega, \boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\,\omega\, t)\, d\boldsymbol{k}\, d\omega. \tag{1.109}$$

Hereafter in all book, we do not use the factor $(2\pi)^{-4}$ multiplying the Fourier integrals (1.108) and (1.109); particularly, in the three-dimensional $\boldsymbol{k}$ space, the factor is $(2\pi)^{-3}$, and in the ω space, it is $(2\pi)^{-1}$.

We use the following identities:

$$\frac{\partial\theta(t, \boldsymbol{x})}{\partial t} = \int \int (\mathrm{i}\,\omega)\, \theta(\omega, \boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\,\omega\, t)\, d\boldsymbol{k}\, d\omega, \tag{1.110}$$

$$\Delta\theta(t, \boldsymbol{x}) = \int \int (-\boldsymbol{k}^2)\, \theta(\omega, \boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\,\omega\, t)\, d\boldsymbol{k}\, d\omega. \tag{1.111}$$

Substituting Eqs. (1.109)–(1.111) into the linearized Eq. (1.107), we obtain

$$\int \int \Big[\big(D^{(\theta)} k^2 + \mathrm{i}\,\omega\big)\ \theta(\omega, \boldsymbol{k}) + u_i(\omega, \boldsymbol{k}) \nabla_i \overline{T} \Big] \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\,\omega\, t)\, d\boldsymbol{k}\, d\omega = 0. \tag{1.112}$$

The integral in Eq. (1.112) vanishes only if

$$\big(D^{(\theta)} k^2 + \mathrm{i}\,\omega\big)\ \theta(\omega, \boldsymbol{k}) + u_i(\omega, \boldsymbol{k})\, \nabla_i \overline{T} = 0. \tag{1.113}$$

This equation yields the expression for temperature fluctuations $\theta(\omega, \boldsymbol{k})$ in the $\boldsymbol{k} - \omega$ space,

$$\theta(\omega, \boldsymbol{k}) = -G_D(\omega, \boldsymbol{k})\, u_i(\omega, \boldsymbol{k})\, \nabla_i \overline{T}, \tag{1.114}$$

where the function $G_D(\omega, \boldsymbol{k}) = \big(D^{(\theta)} k^2 + \mathrm{i}\,\omega\big)^{-1}$.

Note that Eq. (1.114) is not valid in a general case. However, since (i) we take into account in turbulent heat flux only the effects that are linear in the gradient of the mean temperature $\nabla_j \overline{T}$ and neglect higher-order spatial derivatives of the mean temperature field, and (ii) we do not consider in this section compressibility effects of a turbulent velocity field in the evolution of the mean temperature field (i.e., we consider an incompressible flow), we can use Eq. (1.114) to determine the turbulent heat flux in this case [see Section 2.2.6 for a rigorous derivation of an equation for $\theta(\omega, \boldsymbol{k})$ in a more general case].

We determine the turbulent heat flux using the multi-scale approach:

$$\big\langle \theta(t, \boldsymbol{x})\, u_j(t, \boldsymbol{x}) \big\rangle = \lim_{t_1 \to t_2, \boldsymbol{x} \to \boldsymbol{y}} \big\langle \theta(t_1, \boldsymbol{x})\, u_j(t_2, \boldsymbol{y}) \big\rangle$$

$$= \lim_{\tau\to 0, \boldsymbol{r}\to 0} \int\int \left\langle \theta(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\right\rangle \exp[\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r} + \mathrm{i}\,\omega\,\tau]\, d\omega\, d\boldsymbol{k}$$

$$= \int\int \left\langle \theta(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\right\rangle\, d\omega\, d\boldsymbol{k}, \tag{1.115}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, $\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2$, $\tau = t_1 - t_2$, $\omega = (\omega_1 - \omega_2)/2$, and we took into account that turbulence is homogeneous and stationary in the statistical sense. The latter implies that the correlation function $\langle u_i(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\rangle$ is independent of the large-scale variables $\Omega, \boldsymbol{K}$ and $t, \boldsymbol{R}$. We assume also here that there exists a separation of scales, i.e., the maximum scale of random motions ℓ_0 is much smaller than the characteristic scales of inhomogeneities of the mean temperature. In a similar way, we determine the level of temperature fluctuations $\left\langle\theta^2\right\rangle$:

$$\langle\theta(t,\boldsymbol{x})\,\theta(t,\boldsymbol{x})\rangle = \lim_{t_1\to t_2, \boldsymbol{x}\to\boldsymbol{y}} \langle\theta(t_1,\boldsymbol{x})\,\theta(t_2,\boldsymbol{y})\rangle$$

$$= \lim_{\tau\to 0, \boldsymbol{r}\to 0} \int\int \langle\theta(\omega,\boldsymbol{k})\,\theta(-\omega,-\boldsymbol{k})\rangle \exp[\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r} + \mathrm{i}\,\omega\,\tau]\; d\omega\, d\boldsymbol{k}$$

$$= \int\int \langle\theta(\omega,\boldsymbol{k})\,\theta(-\omega,-\boldsymbol{k})\rangle\; d\omega\, d\boldsymbol{k}. \tag{1.116}$$

Substituting Eq. (1.114) into Eqs. (1.115) and (1.116), we determine the turbulent heat flux $F_i^{(\theta)} = \langle\theta\, u_i\rangle$ and the level of temperature fluctuations $\langle\theta^2\rangle$:

$$F_i^{(\theta)} = -\left(\nabla_j \overline{T}\right) \int\int \left\langle u_i(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\right\rangle G_D\, d\omega\, d\boldsymbol{k}, \tag{1.117}$$

$$\left\langle\theta^2\right\rangle = \left(\nabla_i \overline{T}\right)\left(\nabla_j \overline{T}\right) \int\int \left\langle u_i(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\right\rangle G_D\, G_D^*\, d\omega\, d\boldsymbol{k}, \tag{1.118}$$

where the function G_D^* is given by $G_D^*(\omega,\boldsymbol{k}) = \left(D^{(\theta)} k^2 - \mathrm{i}\,\omega\right)^{-1}$.

1.5.3 Model of Isotropic Homogeneous and Incompressible Random Velocity Field

We use a simple model for the second-order moment $\langle u_i(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\rangle$ of an isotropic, homogeneous and incompressible random velocity field in a Fourier space:

$$\boxed{\left\langle u_i(\omega,\boldsymbol{k})\, u_j(-\omega,-\boldsymbol{k})\right\rangle = \frac{\left\langle\boldsymbol{u}^2\right\rangle E(k)\,\Phi(\omega)}{8\pi k^2}\left(\delta_{ij} - \frac{k_i\, k_j}{k^2}\right),} \tag{1.119}$$

where δ_{ij} is the Kronecker tensor, the energy spectrum function is

$$\boxed{E(k) = (q-1)\, k_0^{-1}\left(\frac{k}{k_0}\right)^{-q},} \tag{1.120}$$

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and the exponent q varies in the interval $1 < q < 3$, the interval in the wave numbers is $k_0 \le k \le k_d$, the wave number $k_0 = 1/\ell_0$, the length ℓ_0 is the maximum scale of random motions and $k_d \gg k_0$. We use the Lorentz profile for the frequency function $\Phi(\omega)$:

$$\boxed{\Phi(\omega) = \frac{1}{\pi \tau_0 \, (\omega^2 + \tau_0^{-2})},} \tag{1.121}$$

where τ_0 is the correlation time of a random velocity field. This model for the frequency function corresponds to the following non-instantaneous correlation function: $\langle u_i(t) u_j(t + \tau)\rangle \propto \exp(-\tau/\tau_0)$. Since the Péclet number is small, we consider the case $\tau_0 \gg (D^{(\theta)} k^2)^{-1}$.

Equation (1.119) is derived using symmetry arguments. In particular, since turbulence is homogeneous, the two-point correlation function of random velocity field is independent of the large-scale variables $\boldsymbol{K}$ and $\boldsymbol{R}$. For homogeneous and isotropic turbulence, there are only two symmetric tensors δ_{ij} and $k_i \, k_j / k^2$. Therefore, we can construct the tensor $\big\langle u_i(\omega, \boldsymbol{k}) \, u_j(-\omega, -\boldsymbol{k})\big\rangle$ as a linear combination of the tensors δ_{ij} and $k_i \, k_j / k^2$:

$$\big\langle u_i(\omega, \boldsymbol{k}) \, u_j(-\omega, -\boldsymbol{k})\big\rangle = A_1 \, \delta_{ij} + A_2 \, \frac{k_i \, k_j}{k^2}. \tag{1.122}$$

The velocity field is incompressible, i.e., div $\boldsymbol{u} = 0$. This condition in a Fourier space implies that $k_i \, u_i(\omega, \boldsymbol{k}) = 0$. Multiplying the latter equation by $u_j(-\omega, -\boldsymbol{k})$ and average over an ensemble of random velocity fluctuations, we arrive at the following condition for the second-order moment of the velocity field:

$$k_i \, \big\langle u_i(\omega, \boldsymbol{k}) \, u_j(-\omega, -\boldsymbol{k})\big\rangle = 0. \tag{1.123}$$

Multiplying Eq. (1.122) by k_i and using Eq. (1.123), we obtain:

$$k_i \, \big\langle u_i(\omega, \boldsymbol{k}) \, u_j(-\omega, -\boldsymbol{k})\big\rangle = A_1 \, k_i \, \delta_{ij} + A_2 \, k_i \, \frac{k_i \, k_j}{k^2} = (A_1 + A_2) k_j = 0. \tag{1.124}$$

Since Eq. (1.124) should be valid for any wave vector k_j, it yields the following relation $A_1 = -A_2$. Next, we can rewrite A_1 in the following form: $A_1 = A \, \Phi(\omega) \, E(k)$, where the normalization conditions for the functions $\Phi(\omega)$ and $E(k)$ are $\int_{-\infty}^{\infty} \Phi(\omega) \, d\omega = 1$ and $\int_{k_0}^{k_d} E(k) \, dk = 1$. This implies that the correlation function $\langle u_i(\omega, \boldsymbol{k}) \, u_j(-\omega, -\boldsymbol{k})\rangle$ is

$$\big\langle u_i(\omega, \boldsymbol{k}) \, u_j(-\omega, -\boldsymbol{k})\big\rangle = A \, \Phi(\omega) \, E(k) \, \left(\delta_{ij} - \frac{k_i \, k_j}{k^2}\right). \tag{1.125}$$

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Let us show that

$$A = \frac{\langle \boldsymbol{u}^2 \rangle}{8\pi k^2}, \tag{1.126}$$

where $\langle \boldsymbol{u}^2 \rangle = \int \int \langle u_i(\omega, \boldsymbol{k})\, u_i(-\omega, -\boldsymbol{k}) \rangle \, d\omega \, d\boldsymbol{k}$. The trace of the tensor $\langle u_i \, u_j \rangle$ in Eq. (1.125) is given by

$$\langle u_i(\omega, \boldsymbol{k})\, u_i(-\omega, -\boldsymbol{k}) \rangle = \frac{\langle \boldsymbol{u}^2 \rangle \, E(k)}{4\pi k^2} \Phi(\omega), \tag{1.127}$$

where we take into account Eq. (1.126). Integrating Eq. (1.127) in the ω space and $\boldsymbol{k}$ space, we obtain

$$\int \int \langle u_i(\omega, \boldsymbol{k})\, u_i(-\omega, -\boldsymbol{k}) \rangle \; d\omega \, d\boldsymbol{k} = \frac{\langle \boldsymbol{u}^2 \rangle}{4\pi} \int_{k_0}^{k_d} E(k) \, dk$$
$$\times \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta \, d\vartheta \int_{-\infty}^{\infty} \Phi(\omega) \, d\omega, \tag{1.128}$$

where we use the spherical coordinates in the $\boldsymbol{k}$ space. Here ϑ, φ are the angles in the spherical coordinates, $0 \le \varphi \le 2\pi$ and $0 \le \vartheta \le \pi$, and $d\boldsymbol{k} = k^2 \, dk \sin\vartheta \, d\vartheta \, d\varphi$. We take into account the following identity:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta \, d\vartheta = 4\pi, \tag{1.129}$$

so that Eq. (1.128) yields

$$\int \int \langle u_i(\omega, \boldsymbol{k})\, u_i(-\omega, -\boldsymbol{k}) \rangle \; d\omega \, d\boldsymbol{k} = \langle \boldsymbol{u}^2 \rangle. \tag{1.130}$$

Equation (1.130) is identity, which shows that the coefficient A is indeed determined by Eq. (1.126).

1.5.4 Turbulent Heat Flux and Level of Temperature Fluctuations

Substituting Eq. (1.119) for the second-order moment $\langle u_i(\omega, \boldsymbol{k})\, u_j(-\omega, -\boldsymbol{k}) \rangle$ into Eq. (1.117) and (1.118), we determine the turbulent heat flux $F_i^{(\theta)}$ and the level of temperature fluctuations $\langle \theta^2 \rangle$ as

$$F_i^{(\theta)} = -\frac{\langle \boldsymbol{u}^2 \rangle}{8\pi} \int_{k_0}^{k_d} E(k) \, dk \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta \, d\vartheta \left[\delta_{ij} - \frac{k_i \, k_j}{k^2} \right]$$
$$\times \int_{-\infty}^{\infty} G_D \, \Phi(\omega) \, d\omega \, \left(\nabla_j \overline{T} \right), \tag{1.131}$$

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$$\left\langle\theta^2\right\rangle = \frac{\left\langle\boldsymbol{u}^2\right\rangle}{8\pi} \int_{k_0}^{k_d} E(k)\, dk \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta \left[\delta_{ij} - \frac{k_i\, k_j}{k^2}\right] \times \int_{-\infty}^{\infty} G_D\, G_D^*\, \Phi(\omega)\, d\omega\, \left(\nabla_i \overline{T}\right) . \tag{1.132}$$

For the integration over ω, we use the following integrals:

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\pm \mathrm{i}\,\omega + D^{(\theta)} k^2)\,(\omega^2 + \tau_0^{-2})} = \frac{\pi\,\tau_0}{\tau_0^{-1} + D^{(\theta)}\,k^2} \approx \frac{\pi\,\tau_0}{D^{(\theta)}\,k^2}, \tag{1.133}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\mathrm{i}\,\omega + D^{(\theta)} k^2)\,(-\mathrm{i}\,\omega + D^{(\theta)} k^2)\,(\omega^2 + \tau_0^{-2})} = \frac{\pi\,\tau_0}{D^{(\theta)}\,k^2\left(\tau_0^{-1} + D^{(\theta)}\,k^2\right)} \approx \frac{\pi\,\tau_0}{(D^{(\theta)}\,k^2)^2}, \tag{1.134}$$

which are determined in the limit when the correlation time $\tau_0 \gg (D^{(\theta)} k^2)^{-1}$. For the integration over angles in the $\boldsymbol{k}$ space, we use the following integral:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta\, \frac{k_i\, k_j}{k^2} = \frac{4\pi}{3}\, \delta_{ij}. \tag{1.135}$$

For the integration over k we use the following integrals:

$$\int_{k_0}^{k_d} \frac{E(k)}{k^2}\, dk = \frac{q-1}{q+1}\, \ell_0^2, \tag{1.136}$$

$$\int_{k_0}^{k_d} \frac{E(k)}{k^4}\, dk = \frac{q-1}{q+3}\, \ell_0^4. \tag{1.137}$$

After integration in the ω space and in $\boldsymbol{k}$ space in Eqs. (1.131) and (1.132), we obtain the final formulae for the turbulent heat flux $F_i^{(\theta)}$ and the level of temperature fluctuations $\langle\theta^2\rangle$:

$$\boxed{\langle\theta\, u_i\rangle = -D_T\, \nabla_i \overline{T},} \tag{1.138}$$

$$\boxed{D_T = \frac{q-1}{3(q+1)}\, u_0\, \ell_0\, \mathrm{Pe},} \tag{1.139}$$

$$\boxed{\left\langle\theta^2\right\rangle = \frac{q-1}{3(q+3)}\, \mathrm{Pe}^2\, \ell_0^2\, \left(\nabla \overline{T}\right)^2,} \tag{1.140}$$

where $u_0 = \sqrt{\langle\boldsymbol{u}^2\rangle}$, $\mathrm{Pe} = \ell_0 u_0 / D^{(\theta)}$ is the Péclet number and D_T is the turbulent diffusion coefficient.

Let us analyze the obtained results determined by Eqs. (1.138)–(1.140). We compare the turbulent heat flux $\langle\theta\, u_i\rangle$, the turbulent diffusion coefficient D_T and the

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second moment $\langle\theta^2\rangle$ derived using the quasi-linear approach for Pe $\ll 1$ with the results obtained by the dimensional analysis for small Péclet numbers. In particular, using the dimensional analysis, we find that the turbulent diffusion coefficient is given by Eq. (1.67), and the level of temperature fluctuations is determined by Eq. (1.69). In both methods, the obtained turbulent heat flux has the same form, $\langle\theta\,\boldsymbol{u}\rangle = -D_T \boldsymbol{\nabla}\overline{T}$; the turbulent diffusion coefficient is linear in the Péclet number, $D_T \propto$ Pe; while the second-order moment for temperature fluctuations is quadratic in the Péclet number, $\langle\theta^2\rangle \propto \mathrm{Pe}^2$. The minor difference in the results obtained by these methods is only in numerical factors. Therefore, Eqs. (1.67) and (1.69) obtained by the dimensional analysis are in agreement with Eqs. (1.138)–(1.140), derived using the quasi-linear approach.

1.6 Turbulent Transport of Temperature Fields: Spectral Tau Approach

In this section, we study temperature fluctuations in the case when nonlinear terms in Eq. (1.107) for θ are much larger than the molecular diffusion term, i.e., for large Péclet numbers. We also consider the case of large Reynolds numbers, so that turbulence is fully developed. In this case, the Strouhal number Sr, which is the ratio of the correlation time τ_0 of velocity fluctuations at the integral scale to the turn over time ℓ_0/u_0 of turbulent eddies, is of the order of unity, and the turbulent correlation time is scale-dependent in the inertial range. We apply the Fourier transform only in the $\boldsymbol{k}$ space but not in the ω space, because in a fully developed Kolmogorov-type turbulence, the turbulent time is universally related to spatial scales. We take into account the nonlinear terms in equations for velocity and temperature fluctuations and apply the spectral τ approach (Orszag, 1970; Pouquet et al., 1976; Kleeorin et al., 1990).

The spectral τ approach is a universal tool in turbulent transport for strongly nonlinear systems that allows us to obtain closed results and compare them with the results of laboratory experiments, observations and numerical simulations. The spectral τ approximation reproduces many well-known phenomena found by other methods in turbulent transport of particles and temperature field, in turbulent convection and stably stratified turbulent flows for large Reynolds and Péclet numbers.

1.6.1 Equations for Second Moments

To determine the turbulent heat flux and the level of temperature fluctuations, we use the Navier-Stokes equation for velocity fluctuations and Eq. (1.107) for temperature fluctuations. Equation (1.107) can be rewritten as

$$\frac{\partial\theta}{\partial t} = -Q - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\overline{T}, \tag{1.141}$$

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where $Q = \nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle) - D^{(\theta)} \Delta\theta$. We consider a one-way coupling, when turbulence affects the temperature field, while the feedback of the temperature field on the turbulence is ignored. In such treatment, the temperature field is a passive scalar. Using Eq. (1.141), written in a Fourier space, we derive equations for the following second-order moments: $F_i^{(\theta)}(t, \boldsymbol{k}) = \langle u_i(t, \boldsymbol{k})\, \theta(t, -\boldsymbol{k}) \rangle$ and $E^{(\theta)}(t, \boldsymbol{k}) = \langle \theta(t, \boldsymbol{k})\, \theta(t, -\boldsymbol{k}) \rangle$:

$$\frac{\partial F_i^{(\theta)}}{\partial t} = - \big\langle u_i(t, \boldsymbol{k})\, u_j(t, -\boldsymbol{k}) \big\rangle\, \nabla_j \overline{T} + \hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k}), \tag{1.142}$$

$$\frac{\partial E^{(\theta)}}{\partial t} = - \Big[\big\langle u_j(t, \boldsymbol{k})\, \theta(t, -\boldsymbol{k}) \big\rangle + \big\langle \theta(t, \boldsymbol{k})\, u_j(t, -\boldsymbol{k}) \big\rangle \Big]\, \nabla_j \overline{T} + \hat{\mathcal{M}} E_\theta^{(III)}(\boldsymbol{k}), \tag{1.143}$$

where

$$\hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k}) = - \langle u_i(t, \boldsymbol{k})\, Q(t, -\boldsymbol{k}) \rangle + \left\langle \frac{\partial u_i(t, \boldsymbol{k})}{\partial t}\, \theta(t, -\boldsymbol{k}) \right\rangle, \tag{1.144}$$

$$\hat{\mathcal{M}} E_\theta^{(III)}(\boldsymbol{k}) = - \langle \theta(t, \boldsymbol{k})\, Q(t, -\boldsymbol{k}) \rangle - \langle Q(t, \boldsymbol{k})\, \theta(t, -\boldsymbol{k}) \rangle \tag{1.145}$$

are the third-order moments appearing due to the nonlinear terms in equations for velocity and temperature fluctuations. Small dissipative terms are also included in $\hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k})$ and $\hat{\mathcal{M}} E_\theta^{(III)}(\boldsymbol{k})$.

1.6.2 Spectral Tau Approach

Due to the nonlinearity of the governing equation for the temperature and velocity fields, it follows that equations for the second-order moments involve third-order moments. The equation for the third-order moments brings in fourth-order moments and so on. This leads to the famous closure problem of turbulence and turbulent transport – how to close the system to retain only a finite number of terms (Monin and Yaglom, 1971, 2013; McComb, 1990). In particular, the equations for the second-order moments include the first-order spatial differential operators $\hat{\mathcal{M}}$ applied to the third-order moments $F^{(III)}$. A problem arises how to close the system, i.e., how to express the third-order moments $\hat{\mathcal{M}} F^{(III)}$ through the lower-order moments $F^{(II)}$.

We apply here the spectral τ approach (or the spectral relaxation approach) that is valid only for large Péclet and Reynolds numbers. Let us first formulate the main assumption in this method which is a sort of a turbulence closure. The nature of the spectral tau approach and ranges of validity of this method are discussed in Section 1.7. The spectral τ approach postulates that the deviations of the third-order moments, $\hat{\mathcal{M}} F^{(III)}(\boldsymbol{k})$, from the contributions to these terms afforded by the background turbulence, $\hat{\mathcal{M}} F^{(III,0)}(\boldsymbol{k})$, can be expressed through the similar deviations of the second-order moments, $F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})$:

$$\boxed{\hat{\mathcal{M}} F^{(III)}(\boldsymbol{k}) - \hat{\mathcal{M}} F^{(III,0)}(\boldsymbol{k}) = -\frac{1}{\tau_r(k)} \left[F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k}) \right]} \tag{1.146}$$

(Orszag, 1970; Pouquet et al., 1976; Kleeorin et al., 1990), where $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large Reynolds and Péclet numbers. The functions with the superscript (0) correspond to the background turbulence with a zero gradient of the mean temperature. Note that for incompressible turbulence, the contributions of the terms with the superscript (0) vanish because when the gradient of the mean temperature is zero, the turbulent heat flux vanishes. Consequently, Eq. (1.146) for $\hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k})$ reduces to $\hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k}) = -F_i(\boldsymbol{k})/\tau(k)$ and for $\hat{\mathcal{M}} E_\theta^{(III)}(\boldsymbol{k}) = -E^{(\theta)}(\boldsymbol{k})/\tau(k)$. Validations of the τ approach for different situations have been performed in various direct numerical simulations (see Sections 1.8, 2.6 and 3.8).

1.6.3 Turbulent Heat Flux and Level of Temperature Fluctuations

We assume that the characteristic time of variation of the second moments $F_i(\boldsymbol{k})$ and $E_\theta(\boldsymbol{k})$ are substantially larger than the correlation time $\tau(k)$ for all turbulence scales. This allows us to use a steady-state solution of Eqs. (1.142) and (1.143). Therefore, applying the spectral τ approximation discussed in the previous Section and using the steady-state solution of Eqs. (1.142) and (1.143), we obtain the following expressions for the turbulent heat flux $F_i^{(\theta)}$ and the level of temperature fluctuations:

$$F_i^{(\theta)} = -\left(\nabla_j \overline{T}\right) \int \tau(k) \left\langle u_i(t, \boldsymbol{k})\, u_j(t, -\boldsymbol{k}) \right\rangle d\boldsymbol{k}, \tag{1.147}$$

$$\left\langle \theta^2 \right\rangle = -\left(\nabla_j \overline{T}\right) \int \tau(k) \left[\left\langle u_j(t, \boldsymbol{k})\, \theta(t, -\boldsymbol{k}) \right\rangle + \left\langle \theta(t, \boldsymbol{k})\, u_j(t, -\boldsymbol{k}) \right\rangle \right] d\boldsymbol{k}. \tag{1.148}$$

Equation (1.147) implies that

$$\langle u_i(t, \boldsymbol{k})\, \theta(t, -\boldsymbol{k}) \rangle = -\tau(k) \left\langle u_i(t, \boldsymbol{k})\, u_j(t, -\boldsymbol{k}) \right\rangle \nabla_j \overline{T}. \tag{1.149}$$

Substituting Eq. (1.149) into Eq. (1.148), we obtain:

$$\left\langle \theta^2 \right\rangle = 2 \left(\nabla_i \overline{T}\right) \left(\nabla_j \overline{T}\right) \int \tau^2(k) \left\langle u_i(t, \boldsymbol{k})\, u_j(t, -\boldsymbol{k}) \right\rangle d\boldsymbol{k}. \tag{1.150}$$

We use the following model for the second moments of a turbulent velocity field, $\langle u_i(t,\boldsymbol{k})\, u_j(t,-\boldsymbol{k})\rangle$, of an isotropic, homogeneous and incompressible turbulence in a Fourier space:

$$\boxed{\left\langle u_i(t,\boldsymbol{k})\, u_j(t,-\boldsymbol{k})\right\rangle = \frac{\left\langle \boldsymbol{u}^2\right\rangle E(k)}{8\pi k^2}\left(\delta_{ij} - \frac{k_i\, k_j}{k^2}\right).} \tag{1.151}$$

Equation (1.151) is different from Eq. (1.119), because Eq. (1.151) is independent of ω. Since the fluid Reynolds number is very large, i.e., the background turbulence is fully developed, we can use the spectrum function of the turbulent kinetic energy density as $E(k) = k_0^{-1}\,(2/3)\,(k/k_0)^{-5/3}$, where the wave number k varies within the interval: $k_0 \le k \le k_\nu$, the smallest wave number $k_0 = 1/\ell_0$ with the length ℓ_0 being the maximum scale of random motions, the largest wave number $k_\nu = \ell_\nu^{-1}$ with the length $\ell_\nu = \ell_0\,\mathrm{Re}^{-3/4}$ being the Kolmogorov (viscous) scale. In the Kolmogorov turbulence, the expression for the turbulent correlation time is $\tau(k) = 2\,\tau_0\,(k/k_0)^{-2/3}$, where $\tau_0 = \ell_0/u_0$ is the characteristic turbulent time at the integral scale. Substituting Eq. (1.151) into Eqs. (1.147) and (1.150), we determine the turbulent heat flux $F_i^{(\theta)}$ and the level of temperature fluctuations $\langle\theta^2\rangle$:

$$F_i^{(\theta)} = -\frac{\left\langle \boldsymbol{u}^2\right\rangle}{8\pi}\left(\nabla_j \overline{T}\right)\int_{k_0}^{k_\nu} \tau(k)\, E(k)\, dk \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta \left[\delta_{ij} - \frac{k_i\, k_j}{k^2}\right], \tag{1.152}$$

$$\left\langle\theta^2\right\rangle = \frac{\left\langle \boldsymbol{u}^2\right\rangle}{4\pi}\left(\nabla_i \overline{T}\right)\left(\nabla_j \overline{T}\right)\int_{k_0}^{k_\nu} \tau^2(k)\, E(k)\, dk \times \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta \left[\delta_{ij} - \frac{k_i\, k_j}{k^2}\right]. \tag{1.153}$$

For the integration over angels in the $\boldsymbol{k}$ space, we use identities (1.129) and (1.135), and for the integration over k, we use the following integrals for large Reynolds numbers, Re$= u_0\ell_0/\nu \gg 1$:

$$\int_{k_0}^{k_\nu} \tau(k)\, E(k)\, dk = \tau_0, \tag{1.154}$$

$$\int_{k_0}^{k_\nu} \tau^2(k)\, E(k)\, dk = \frac{4}{3}\tau_0^2. \tag{1.155}$$

After integration in the $\boldsymbol{k}$ space in Eqs. (1.152) and (1.153), we arrive at the final expressions for the turbulent heat flux $\langle\theta\, u_i\rangle$, and the level of temperature fluctuations, $\langle\theta^2\rangle$, in an isotropic homogeneous and incompressible turbulence for large Péclet and Reynolds numbers:

$$\boxed{\langle\theta\, \boldsymbol{u}\rangle = -D_T\, \nabla\overline{T},} \tag{1.156}$$

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$$\boxed{D_T = \frac{1}{3}\, u_0\, \ell_0,} \tag{1.157}$$

$$\boxed{\langle\theta^2\rangle = \frac{8}{9}\, \ell_0^2\, \left(\nabla\overline{T}\right)^2,} \tag{1.158}$$

where $u_0 = \sqrt{\langle \boldsymbol{u}^2\rangle}$. Note that Eqs. (1.156)–(1.158), obtained using the tau approach, are in agreement with Eqs. (1.62)–(1.63) and (1.65), obtained by the dimensional analysis for large Péclet and Reynolds numbers. The form of the turbulent heat flux is similar for small and large Péclet numbers. However, for large Péclet numbers, the turbulent diffusion coefficient D_T and the level of temperature fluctuations $\langle\theta^2\rangle$ are independent of the Péclet number, while for small Péclet numbers, $D_T \propto$ Pe and $\langle\theta^2\rangle \propto \mathrm{Pe}^2$.

1.7 Ranges of Validity of Various Analytical Methods

In this section, we discuss ranges of validity for the various analytical methods applied in this book. The dimensional analysis, the quasi-linear approach and the spectral tau approach are applied in Chapters 1–3. The analysis based on the budget equations is applied in Chapter 4, and the path-integral approach is applied in Chapter 5.

In Table 1.2, we present a summary of the ranges of validity of various analytical methods applied in this book, as well as one-way and two-way couplings in the system. Here $\mathrm{Pe} = u_0\ell_0/D^{(\theta)}$ is the Péclet number used for temperature field, $\mathrm{Pe}^{(n)} = u_0\ell_0/D^{(n)}$ is the Péclet number for particles, $\mathrm{Re} = u_0\ell_0/\nu$ is the fluid Reynolds number, $\mathrm{Rm} = u_0\ell_0/\eta$ is the magnetic Reynolds number, $D^{(n)}$ is the coefficient of Brownian diffusion of particles, η is the coefficient of magnetic diffusion due to the electrical conductivity of the fluid and Sr is the Strouhal number that is the ratio of the correlation time τ_0 of velocity fluctuations at the integral scale to the turn over time ℓ_0/u_0 of turbulent eddies.

Both one-way and two-way couplings between turbulence and temperature, or particles, or magnetic field can be taken into account only in the spectral tau approach. The one-way coupling implies only the direct effect of turbulence on temperature, or particles, or magnetic field, but the feedback effect of temperature, or particles, or magnetic field on turbulence is ignored. On the other hand, the two-way coupling implies that both ways of coupling between turbulence and temperature, or particles, or magnetic field exist. The two-way coupling is taking into account when the dynamic equation for the Reynolds stress (including both diagonal and off-diagonal terms) is used explicitly, e.g., to determine nonlinear turbulent electromotive force (Rogachevskii and Kleeorin, 2000, 2001, 2006; Kleeorin and

Table 1.2 *Ranges of validity of analytical methods*

Method	Range of validity	Coupling
Dimensional analysis	Pe $\ll$ 1 or Pe $\gg$ 1 $\mathrm{Pe}^{(n)} \ll 1$ or $\mathrm{Pe}^{(n)} \gg 1$ Rm $\ll$ 1 or Rm $\gg$ 1	one-way
Quasi-linear approach	Pe $\ll$ 1 or Pe $\gg$ 1 but Sr $\ll$ 1 $\mathrm{Pe}^{(n)} \ll 1$ or $\mathrm{Pe}^{(n)} \gg 1$ but Sr $\ll$ 1 Rm $\ll$ 1 or Rm $\gg$ 1 but Sr $\ll$ 1	one-way
Spectral tau approach	Re $\gg$ 1 Pe $\gg$ 1, $\mathrm{Pe}^{(n)} \gg 1$ Rm $\gg$ 1	one-way (or two-way)
Budget equations approach	Re $\gg$ 1 Pe $\gg$ 1, $\mathrm{Pe}^{(n)} \gg 1$ Rm $\gg$ 1	two-way
Path-integral approach	Short correlation time of turbulent velocity field	one-way

Rogachevskii, 2007), or in the theory of formation of inhomogeneous magnetic structures (Kleeorin et al., 1990; Kleeorin and Rogachevskii, 1994a; Kleeorin et al., 1996; Rogachevskii and Kleeorin, 2007).

The analysis based on the budget equations takes into account the two-way coupling, where velocity fluctuations affect the turbulent transport of temperature, while temperature fluctuations and turbulent heat flux affect velocity fluctuations (Monin and Yaglom, 1971, 2013). This approach is a kind of energetic analysis, and it is effective for the study of turbulent transport in homogeneous turbulence when the divergence of third-order moments caused by nonlinear terms vanishes.

The quasi-linear approach (Krause and Rädler, 1980; Moffatt, 1978) and the path-integral approach (Feynman and Hibbs, 1965; Kac, 1959; Zeldovich et al., 1988, 1990; Piterbarg and Ostrovskii, 2013) are exact methods and can be considered as perturbation approaches. These two methods are applied for study of kinematic problems, i.e., for investigations of the evolution of passive or vector fields in a given prescribed random or turbulent velocity field. This implies that formal conditions on the fluid Reynolds number are not required for validity of these approaches. However, for dynamical problems when the velocity field is determined by the Navier-Stokes equation, an additional condition for validity of the quasi-linear approach is small fluid Reynolds numbers. A condition of very short correlation time of random velocity field in comparison with the characteristic time of the mean-field variations is necessary for validity of the path-integral approach.

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The tau approach can be applied in physical space, referred to as the minimal tau approach (Field et al., 1999; Blackman and Field, 2003), as well as in the Fourier space referred to as the spectral tau approach (Orszag, 1970; Pouquet et al., 1976; Kleeorin et al., 1990). The spectral tau approach allows one to take into account the spectrum of turbulence and the scale-dependence of the turbulent correlation time. The spectral tau approach is a sort of high-order closure and in general is similar to Eddy Damped Quasi Normal Markovian (EDQNM) approximation. However, some principle differences exist between these two approaches (Orszag, 1970). The EDQNM closures do not relax to equilibrium (the background turbulence), and the EDQNM approach does not describe properly the motions in the equilibrium state in contrast to the spectral τ approach. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached. In the spectral tau approach, the relaxation time for small departures from equilibrium is determined by the turbulent motions in the equilibrium state, i.e., it is determined by the background turbulence, rather than by the departure from the equilibrium. As follows from the analysis (Orszag, 1970), the spectral tau approach describes the relaxation to the background turbulence much more accurately than the EDQNM approach.

Despite the fact that the spectral tau approach is a form of closure, this method works well enough if there is a small parameter. For instance, in turbulent transport of particles or temperature field, the small parameter can be the ratio of the integral turbulence scale to the characteristic scale of the mean particle number density variations (or the ratio of the integral turbulence scale to the mean fluid temperature variation scale). In the case of turbulent magnetohydrodynamics, the small parameter can be the ratio of the energy of the mean magnetic field to turbulent kinetic energy.

The spectral tau approach works well enough for developing the mean-field theories. This approach allows one to determine one-point second-order correlation functions, e.g., the turbulent heat flux, the turbulent flux of particles, the turbulent electromotive force. However, the spectral tau approach cannot be used for the study of intermittency of passive or vector fields, which require one to determine two-point or multi-point non-instantaneous second-order and high-order moments (Zeldovich et al., 1990; Frisch, 1995; Falkovich et al., 2001).

There are also other methods applied for large Péclet and Reynolds numbers. One of these methods is the renormalization procedure (Moffatt, 1981, 1983; Kleeorin and Rogachevskii, 1994a; Elperin et al., 1996a), which is applied for investigation of passive scalar fluctuations and magnetic fluctuations. The averaging in the renormalization procedure is performed over fluctuations of scales from the viscous scale ℓ_ν to scale ℓ_* inside the inertial range of turbulence $\ell_\nu < \ell_* < \ell_0$.

After this averaging, the range of scales $\ell > \ell_*$ in the renormalization procedure corresponds to "mean" fields, whereas fluctuations are considered in the range $\ell < \ell_*$. The effect of fluctuations on the "mean" fields is described by the turbulent transport coefficients, e.g., the turbulent viscosity and turbulent diffusivity which are functions of the scale of the averaging ℓ_*.

The next step of the renormalization procedure comprises a step-by-step increase of the scale of the averaging. This procedure allows one to derive equations for the turbulent transport coefficients. In the framework of the renormalization procedure, an interaction between the weakly inhomogeneous mean fields and background turbulence (with zero gradients of the mean passive scalar field) is studied. In particular, this procedure considers small deviations from the background turbulence under the influence of weak gradients of mean fields, while the spectrum and statistical properties of the background turbulence are assumed to be known.

This renormalization procedure is essentially different from the renormalization group method (RNG) (Yakhot and Orszag, 1986; McComb, 1990), which provides a closure for the dynamical problem of turbulence. Common in the renormalization procedure and in the renormalization group method is a step-by-step averaging. However, a renormalization of the background hydrodynamic turbulence as it is done in the RNG method is not performed in the renormalization procedure. In the renormalization procedure, a parameter of nonlinear interaction is not used contrary to the RNG method (Yakhot and Orszag, 1986; McComb, 1990).

1.8 Further Reading

Turbulence, and the associated transport of scalar and vector fields, is one of the classical problems of physics. Although it has been investigated systematically for more than a hundred years in theoretical, experimental and numerical studies, fundamental questions remain. This is particularly true in applications such as astrophysics, where the governing parameter values are too extreme to be modeled either experimentally or numerically. The large variety of interesting phenomena related to the nature of turbulent flows have been discussed in various books (Batchelor, 1953; Hinze, 1959; Monin and Yaglom, 1971, 2013; Bradshaw, 1971; Tennekes and Lumley, 1972; Townsend, 1980; Kraichnan and Montgomery, 1980; McComb, 1990; Chorin, 1994; Frisch, 1995; Libby, 1996; Pope, 2000; Mathieu and Scott, 2000; Bernard and Wallace, 2002; Biskamp, 2003; Jovanovic, 2004; Lumley, 2007; Lesieur, 2008; Sagaut and Cambon, 2008; Tsinober, 2009, 2014; Nazarenko, 2011; Zakharov et al., 2012; Piquet, 2013; Davidson, 2013, 2015; Beresnyak and Lazarian, 2019; Landau and Lifshits, 1987).

Turbulent transports of passive scalar and vector fields have various applications in geophysical, astrophysical and industrial flows. Different aspects of passive scalar transport in turbulent flows have been investigated theoretically, experimentally and using numerical simulations in a number of publications (Csanady, 2012; Pasquill and Smith, 1983; Piterbarg and Ostrovskii, 2013; Blackadar, 2012; Majda and Kramer, 1999; Warhaft, 2000). Interesting effects in turbulent transport include anomalous turbulent diffusion, intermittency and anomalous scalings (the violation of the dimensional analysis predictions for the scaling laws), fractal structure of a concentration field and formation of large-scale coherent structures in a small-scale turbulence (Kolmogorov, 1962; Zeldovich et al., 1988, 1990; Kraichnan, 1994; Frisch, 1995; Gawedzki and Kupiainen, 1995; Shraiman and Siggia, 1995, 2000; Chertkov and Falkovich, 1996; Falkovich et al., 2001; Cardy et al., 2008).

Compressibility of turbulent flow plays an essential role in a passive scalar dynamics and causes qualitative changes in the properties of both the mean passive scalar field and fluctuations (Elperin et al., 1995, 1997b; Rädler et al., 2011; Rogachevskii et al., 2018; Rogachevskii and Kleeorin, 2021). Compressibility of a surrounding fluid flow results in a slow scale-dependent turbulent diffusion of small-scale passive scalar fluctuations for large Pèclet numbers (Elperin et al., 1995). In addition, the level of passive scalar fluctuations in the presence of a gradient of the mean passive scalar field in compressible flow of fluid can be fairly strong. On the other hand, passive scalar transport in a density-stratified turbulent fluid flow [i.e., $\operatorname{div} \boldsymbol{u} = -(\boldsymbol{u} \cdot \nabla \overline{\rho})/\overline{\rho} \neq 0$] is accompanied by formation of large-scale structures due to instability of the mean passive scalar field in inhomogeneous turbulent velocity field (Elperin et al., 1995).

Compressibility of a fluid flow also reduces the turbulent diffusivity of the mean temperature field (Rogachevskii and Kleeorin, 2021) similarly to that for particle number density and magnetic field (Rädler et al., 2011; Rogachevskii et al., 2018). However, expression for turbulent diffusion coefficient for the mean temperature field in a compressible turbulence is different from that for particle number density and magnetic field. The combined effect of compressibility and inhomogeneity of turbulence for small Péclet numbers causes an increase of the mean temperature in the regions with more intense velocity fluctuations due to turbulent pumping. Formally, this effect is similar to a phenomenon of compressible turbophoresis found for non-inertial particles or gaseous admixtures (Rogachevskii et al., 2018). The gradient of the mean fluid pressure results in an additional turbulent pumping of the mean temperature field. The later effect is similar to turbulent barodiffusion of particles and gaseous admixtures (Elperin et al., 1997c; Rogachevskii et al., 2018). Compressibility of a fluid flow also causes a turbulent cooling of the surrounding fluid due to an additional sink term in the equation for the mean temperature field. There is no analog of this effect for particles (Rogachevskii and Kleeorin, 2021).

Another interesting feature for a stratified turbulence is that turbulent flux of entropy in compressible temperature stratified turbulence is different from turbulent convective flux of fluid internal energy (Braginsky and Roberts, 1995; Rogachevskii and Kleeorin, 2015). In particular, turbulent flux of entropy is given by $\boldsymbol{F}_s = \overline{\rho}\,\langle s'\,\boldsymbol{u}\rangle$, where $\overline{\rho}$ is the mean fluid density and s' are fluctuations of entropy. On the other hand, the turbulent convective flux of fluid internal energy is $\boldsymbol{F}_c = \overline{T}\,\overline{\rho}\,\langle s'\,\boldsymbol{u}\rangle$, where $\overline{T}$ is the mean fluid temperature. This turbulent convective flux is well-known in the astrophysical and geophysical literature, and it cannot be used as a turbulent flux in the equation for the mean entropy. This is the exact result for low-Mach-number temperature stratified turbulence and is independent of the turbulence model used. The velocity-entropy correlation is given by $\langle s'\,\boldsymbol{u}\rangle = D_T \boldsymbol{\nabla}\overline{s}$, where the turbulent diffusion coefficient D_T of the mean entropy depends on the Péclet number, i.e., $D_T \propto u_0\,\ell_0\,\mathrm{Pe}$ for small Péclet numbers, while $D_T \propto u_0\,\ell_0$ for large Péclet numbers (Rogachevskii and Kleeorin, 2015), where $\overline{s}$ is the mean entropy.

Numerous experiments demonstrate that the mean temperature distribution in turbulent convection is strongly inhomogeneous and anisotropic due to the large-scale circulations (Ahlers et al., 2009; Bukai et al., 2009; Barel et al., 2020; Ching, 2014), and the mean temperature gradient in the horizontal direction inside the large-scale circulations can be significantly larger than in the vertical direction, i.e., the vertical turbulent heat flux inside the large-scale circulations is very small. Moreover, even in convective turbulence, stably stratified regions can be formed, where the mean temperature gradient in the flow core inside the large-scale circulation is directed upward (Barel et al., 2020; Toppaladoddi et al., 2017).

Temperature fluctuations in turbulent flow can be passive in the case of a one-way coupling when there is no feedback of temperature fluctuations on turbulent flow. In this case, there is only an effect of turbulence on the temperature field. Such regime has been studied experimentally and theoretically in unstably stratified turbulent flow in externally forced Rayleigh-Bénard convection without large-scale circulation (Bukai et al., 2011). In these experiments, an additional source of turbulence has been produced by two oscillating grids located nearby the side walls of the chamber. When the frequency of the grid oscillations is larger than 2 Hz, the large-scale circulation in turbulent convection is destructed, and it is accompanied by a strong change in the mean temperature distribution. However, in all regimes of the unstably stratified turbulent flow the ratio $\sigma_* \equiv \left[(\ell_x \nabla_x \overline{T})^2 + (\ell_y \nabla_y \overline{T})^2 + (\ell_z \nabla_z \overline{T})^2\right]/\langle\theta^2\rangle$ varies slightly. This result is valid even in the range of parameters whereby the behavior of the temperature field is different from that of the passive scalar. Here ℓ_i are the integral scales of turbulence along x, y, z directions, $\overline{T}$ and θ are the mean and fluctuating parts of the fluid temperature. The ratio σ_* is also found to be constant (independent of the frequency of the grid oscillations) in the experiments with stably stratified forced turbulence in air flow produced by two oscillating grids, and it has the same magnitude for both stably and unstably stratified turbulent flows (Eidelman et al., 2013).

Many studies of turbulent transport of a passive scalar use a model of linear velocity field (Chertkov and Falkovich, 1996; Falkovich et al., 2001). Employment of this simple model allows one to perform calculations in a closed form for a passive scalar. A model of a linear velocity field can be viewed as: (i) an expansion in Taylor series of the velocity field in a local frame of reference moving with a fluid element and (ii) a real flow field in an infinite space (e.g., like the Hubble flow in cosmology). Some peculiar aspects of a transport of a passive scalar in a linear velocity field have been discussed by Elperin et al. (2001b), where the Cauchy problem for a passive scalar in both laminar and random linear velocity fields has been considered. It is shown that a spatial distribution of a passive scalar evolves either into a thin infinite "pancake" or into an infinite "rope" structures. The properties of a passive scalar are similar for laminar and random incompressible linear velocity fields. A compressibility of a fluid flow results in a non-zero flux of particles from infinity. For strong compressibility a spatial distribution of a passive scalar evolves into a ball (or ellipsoid) with a very small radius that is of the order of a diffusion scale. The higher statistical moments can grow exponentially in spite of a decay of a typical realization of a passive scalar. Such "strange" behavior of a passive scalar demonstrates that transport of a passive scalar in linear velocity field cannot be considered as a general and universal phenomenon. Remarkably, the dynamics of a passive scalar and a magnetic field (Zeldovich et al., 1984; Moffatt, 1963; Gvaramadze et al., 1988) in linear velocity fields are similar.