

## AN EXACTLY SOLVABLE TWO COMPONENT CLASSICAL COULOMB SYSTEM

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### Abstract

A two component classical Coulomb system is considered, in which particles of charge  $+q$  and  $+2q$  are constrained to lie on a circle and interact *via* the two-dimensional Coulomb potential. At a special value of the coupling constant the correlation functions are calculated exactly and the asymptotic form of the truncated charge-charge correlation is found to obey Jancovici's sum rule.

### 1. Introduction

On the basis of a screening argument, Jancovici [4] has derived a sum rule applicable to  $n$  component classical Coulomb systems with charge species  $q_\alpha$ ,  $\alpha = 1, 2, \dots, n$ , in dimension  $\nu = 2$  or 3 and confined to half space domains. The sum rule states that the asymptotic expansion of the truncated charge-charge correlation function  $C_2^T$ , defined in terms of the usual one and two particle correlations by

$$C_2^T(\mathbf{x}, \mathbf{x}') = \sum_{\alpha=1}^n q_\alpha \rho_\alpha(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') + \sum_{\alpha=1}^n \sum_{\beta=1}^n q_\alpha q_\beta \rho_{\alpha,\beta}^T(\mathbf{x}, \mathbf{x}'), \quad (1.1)$$

for large separations  $|y - y'|$  along a plane wall, must contain the term

$$\frac{f(x, x')}{|y - y'|^\nu}. \quad (1.2)$$

The function  $f(x, x')$  is further specified (see [4]).

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Application of Jancovici’s argument to two dimensional strip domains by Forrester et al. [2] gives a similar sum rule. In particular, in the zero strip width limit the sum rule says the asymptotic expansion of  $C_2^T(y)$  must contain the term

$$-\frac{k_B T}{\pi^2 y^2} \tag{1.3}$$

where  $y$  is now the particle separation on the line.

In [4] and [2] the sum rule has been tested on the few known exactly solvable Coulomb systems—special cases of the two dimensional one component plasma—and found to hold. Further it is expected that above a certain temperature, when the system is in a disordered fluid state, the term given by Jancovici will be the leading term (see [2]).

In this paper, for a special temperature, we evaluate  $C_2^T(y)$  in the case of a two component system interacting *via* the 2-dimensional Coulomb potential and confined to a line. We find the term (1.3) is indeed the leading order term in the asymptotic expansion of  $C_2^T$ . We also verify another sum rule—the perfect screening sum rule—which in physical terms says around each charge in the system a screening cloud of equal and opposite charge is formed.

### 2. Calculation of the partition function

Let there be  $aN$  particles of charge  $+q$  and  $bN$  particles of charge  $+2q$ , labelled  $\theta_1, \theta_2, \dots, \theta_{aN}$  and  $\theta_{aN+1}, \theta_{aN+2}, \dots, \theta_{(a+b)N}$  respectively, interacting on the circle of radius  $R$  *via* the two dimensional Coulomb potential

$$V(\theta_j, \theta_k) = -q_k \log R |\exp(i\theta_j) - \exp(i\theta_k)|. \tag{2.1}$$

To obtain thermodynamic stability we would also require a neutralizing background charge density, but since for circular domains this only contributes a constant term to the Hamiltonian this is irrelevant for the purposes of calculating correlation functions. Thus we may take for the Hamiltonian

$$-q^2 \sum_{1 \leq j < k \leq (a+b)N} \log |\exp(i\theta_j) - \exp(i\theta_k)|^{q_j q_k} \tag{2.2}$$

where  $q_j = 1, 1 \leq j \leq aN$ , and  $q_j = 2, aN + 1 \leq j \leq (a + b)N$ . Further, making the special choice of temperature given by

$$\Gamma \equiv \frac{q^2}{k_B T} = 1, \tag{2.3}$$

and letting  $v_{+1}(\theta)$  and  $v_{+2}(\theta)$  be arbitrary integrable functions, we have the two particle distribution functions given by

$$\rho_{+\alpha, +\beta}(\theta_a, \theta_b) = \lim_{N \rightarrow \infty} \left( \frac{2\pi Q}{(2b + a)N} \right)^2 \frac{\delta^2}{\delta v_{+\alpha}(\theta_a) \delta v_{+\beta}(\theta_b)} \left( \frac{Z(v_{+1}, v_{+2})}{Z(0, 0)} \right) \Bigg|_{v_{+1}=v_{+2}=0}, \tag{2.4}$$

where  $qQ = q(2b + a)N/2\pi R$  is the total charge density, and either  $\alpha = \beta = 1$ ,  $\alpha = \beta = 2$  or  $\alpha = 1, \beta = 2$ . In (2.4)  $\frac{\delta}{\delta v}$  denotes functional differentiation and

$$Z(v_{+1}, v_{+2}) = \prod_{l=1}^{(a+b)N} \int_0^{2\pi} d\theta_l \left[ \prod_{l=1}^{aN} (1 + v_{+1}(\theta_l)) \right] \left[ \prod_{l=aN+1}^{(a+b)N} (1 + v_{+2}(\theta_l)) \right] \times \prod_{1 \leq i < j \leq N(a+b)} |\exp(i\theta_i) - \exp(i\theta_j)|^{q_i q_j}. \tag{2.5}$$

We now transform (2.5) using a method due to Dyson and Mehta [5, Chapter 5]. Since the integrand is symmetric in  $\theta_l, l = 1, 2, \dots, aN$ , we can order those integrations  $0 \leq \theta_1 < \theta_2 < \dots < \theta_{aN} \leq 2\pi$  provided we multiply by  $(aN)!$ . We can now use the identity

$$|\exp(i\theta_i) - \exp(i\theta_j)| = i^{-1}(\exp(i\theta_j) - \exp(i\theta_i)) \exp\left[-\frac{1}{2}i(\theta_i + \theta_j)\right], \quad \theta_j \geq \theta_i, \tag{2.6}$$

to write

$$\begin{aligned} & \prod_{1 \leq i < j \leq (a+b)N} |\exp(i\theta_i) - \exp(i\theta_j)|^{q_i q_j} \\ &= i^{-aN(aN-1)/2} (-1)^{abN} \prod_{l=1}^{aN} \exp(-i\theta_l [(2b + a)N - 1]/2) \\ & \quad \times \prod_{l=aN+1}^{(a+b)N} \exp(-i\theta_l [(2b + a)N - 2]) \\ & \quad \times \prod_{1 \leq i < j \leq (a+b)N} (\exp(i\theta_j) - \exp(i\theta_i))^{q_i q_j}. \end{aligned} \tag{2.7}$$

But we can write

$$\prod_{1 \leq i < j \leq (a+b)N} (\exp(i\theta_j) - \exp(i\theta_i))^{q_i q_j}$$

as a confluent alternate determinant [5, page 208] of dimension  $(2b + a)N$ , with row  $j$ ,  $1 \leq j \leq aN$ , consisting of powers of  $\exp(i\theta_j)$

$$1, \exp(i\theta_j), \exp(2i\theta_j), \dots, \exp([N(2b + a) - 1]i\theta_j),$$

rows  $aN + (2j - 1)$ ,  $1 \leq j \leq bN$  the powers of  $\exp(i\theta_{aN+j})$ , while the rows  $aN + 2j$ ,  $1 \leq j \leq bN$  consist of the derivative with respect to  $\exp(i\theta_{aN+j})$  of the  $(aN + (2j - 1))$ th row. Hence from the definition of a determinant we have

$$\prod_{1 \leq i < j \leq (a+b)N} (\exp(i\theta_j) - \exp(i\theta_i))^{q_{ij}} = \sum_{P=1}^{[(2b+a)N]!} \epsilon(P) \prod_{l=1}^{aN} \exp(i(P(l) - 1)\theta_l) \times \prod_{l=1}^{bN} (P(aN + 2l) - 1)\exp(i\theta_{aN+l}(P(aN + 2l) + P(aN + 2l - 1) - 3)), \tag{2.8}$$

where  $\epsilon(P)$  denotes the parity of the permutation  $P$  of  $\{1, 2, \dots, (2b + a)N\}$ .

Substitution of (2.8) and (2.7) into (2.5) (with the specified order of integration) shows the integrand is now anti-symmetric in  $\theta_1, \theta_2, \dots, \theta_{aN}$ . However if we first integrate over  $\theta_1, \theta_3, \dots, \theta_{aN-1}$  [5, pages 51–52] (it is thus convenient to take  $N$  even) the integrand is symmetric in the integration variables  $\theta_2, \theta_4, \dots, \theta_{aN}$  so we can drop the ordering constraint and divide by  $(aN/2)!$ . Hence

$$\begin{aligned} &Z(v_{+1}, v_{+2}) \\ &= (-i)^{aN/2} \frac{(aN)!}{(aN/2)!} \sum_{P=1}^{[(a+2b)N]!} \epsilon(P) \\ &\times \prod_{l=1}^{aN/2} \int_0^{2\pi} d\theta_{2l} (1 + v_{+1}(\theta_{2l})) \exp(i\theta_{2l}(P(2l) - N(b + a/2)) - 1/2) \\ &\times \int_0^{\theta_{2l}} d\theta_{2l-1} (1 + v_{+1}(\theta_{2l-1})) \exp(i\theta_{2l-1}(P(2l - 1) - N(b + a/2) - 1/2)) \\ &\times \prod_{l=1}^{bN} \int_0^{2\pi} d\theta_{aN+l} (1 + v_{+2}(\theta_{aN+l})) (P(aN + 2l) - 1) \\ &\times \exp(i\theta_{aN+l}[P(aN + 2l) + P(aN + 2l - 1) - 1 - N(2b + a)]). \tag{2.9} \end{aligned}$$

Consider all permutations such that  $P(2l) > P(2l - 1)$  for each  $l = 1, 2, \dots, N(b + a/2)$ . Denote the set of such permutations by  $X$ . From  $X$  we can construct all permutations  $P$  by interchanges, which have the effect of changing the sign of

$\varepsilon(P)$  for each interchange. In fact [5, pages 194–195]

$$\begin{aligned}
 Z(v_{+1}, v_{+2}) &= (-i)^{aN/2} \frac{(aN)!}{(aN/2)!} \sum_X \varepsilon(P) \prod_{l=1}^{aN/2} \psi_{P(2l-1), P(2l)}(v_{+1}) \\
 &\quad \times \prod_{l=1+aN/2}^{(b+a/2)N} \phi_{P(2l-1), P(2l)}(v_{+2})
 \end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
 &\psi_{P(2l-1), P(2l)}(v_{+1}) \\
 &= \int_0^{2\pi} d\theta_{2l} \int_0^{2\pi} d\theta_{2l-1} \operatorname{sgn}(\theta_{2l} - \theta_{2l-1}) (1 + v_{+1}(\theta_{2l-1})) (1 + v_{+1}(\theta_{2l})) \\
 &\quad \times \exp(i\theta_{2l-1}(P(2l-1) - N(b + a/2) - 1/2) \\
 &\quad \quad + i\theta_{2l}(P(2l) - N(b + a/2) - 1/2))
 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
 \phi_{P(2l-1), P(2l)}(v_{+2}) &= (P(aN + 2l) - P(aN + 2l - 1)) \\
 &\quad \times \int_0^{2\pi} d\theta_{aN+l} (1 + v_{+2}(\theta_{aN+l})) \\
 &\quad \times \exp(i\theta_{aN+l}(P(aN + 2l) + P(aN + 2l - 1) - 1 - N(2b + a)))
 \end{aligned} \tag{2.12}$$

We can now evaluate the partition function. Since

$$\begin{aligned}
 &\psi_{P(2l-1), P(2l)}(0) \\
 &= \begin{cases} \frac{2\pi}{i} \left( \frac{1}{P(2l-1) - N(b + a/2) - 1/2} - \frac{1}{P(2l) - N(b + a/2) - 1/2} \right), \\ \quad \text{if } P(2l) + P(2l-1) = N(2b + a) + 1, \\ 0, \quad \text{otherwise,} \end{cases}
 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
 &\phi_{P(2l-1), P(2l)}(0) \\
 &= \begin{cases} 2\pi(P(aN + 2l) - P(aN + 2l - 1)), \\ \quad \text{if } P(aN + 2l) + P(aN + 2l - 1) = N(2b + a) + 1, \\ 0, \quad \text{otherwise,} \end{cases}
 \end{aligned} \tag{2.14}$$

and the only permutations satisfying  $P(2l) + P(2l - 1) = N(2b + a) + 1$  and  $P(2l) > P(2l - 1)$  for each  $l = 1, 2, \dots, (b + a/2)N$  are

$$P(2l - 1) = Q(l), \quad P(2l) = (a + 2b)N + 1 - Q(l), \tag{2.15}$$

where  $Q(l)$  is a permutation of  $\{1, 2, \dots, N(b + a/2)\}$ , we have after straight-forward manipulation

$$Z(0, 0) = \frac{(aN)!(bN)![N(b + a/2)]!}{[N(2b + a)]!} (16\pi)^{aN/2 + bN} \sum_c \prod_{l=1}^{bN} (c(l) - 1/2)^2, \tag{2.16}$$

where the sum is over all combinations of  $\{1, 2, \dots, N(b + a/2)\}$  taken  $bN$  at a time.

### 3. Application of a local limit theorem

When taking the thermodynamic limit we will require an asymptotic formula for the sum in (2.16). This can be achieved by first noting the required sum is the coefficient of  $x^{Na/2}$  in the polynomial

$$(x + (1/2)^2)(x + (3/2)^2) \cdots (x + (N(b + a/2) - 1/2)^2). \tag{3.1}$$

The problem is now analogous to finding the same type of asymptotic formula for the Stirling numbers of the first kind. It was shown by Bender that the asymptotic formula for the Stirling numbers of the first kind can easily be obtained [1, page 105] after first proving a local limit theorem [1, page 100, Theorem 2]. Using the same local limit theorem, and following Bender’s procedure there is no difficulty in showing

$$\sum_c \prod_{l=1}^{bN} (c(l) - 1/2)^2 \sim \frac{|\Gamma(i\alpha + N(b + a/2) + 1/2)|^2 \cosh \pi\alpha}{\pi\alpha^{Na} \sqrt{2\pi\sigma^2}}, \tag{3.2}$$

where if  $\nu$  is the unique positive solution of the equation

$$\frac{a}{2b + a} = \frac{\arctan \nu}{\nu}, \tag{3.3}$$

then

$$\alpha = \frac{(b + a/2)N}{\nu} \tag{3.4}$$

and

$$\sigma^2 = \frac{(b + a/2)N}{2} \left( \frac{\arctan \nu}{\nu} - \frac{1}{1 + \nu^2} \right). \tag{3.5}$$

From the definition (2.4) it remains to take the appropriate functional derivatives of (2.10) before taking the thermodynamic limit to evaluate the correlation functions. We will illustrate the procedure for  $\rho_{+1,+2}(\theta_a, \theta_b)$ . The calculation of  $\rho_{+1,+1}$  and  $\rho_{+2,+2}$  proceeds similarly, so we will omit the details and only present the results.

From the definition of functional differentiation we have

$$\begin{aligned} & \frac{\delta^2}{\delta v_{+1}(\theta_a) \delta v_{+2}(\theta_b)} Z(v_{+1}, v_{+2}) \Big|_{v_{+1}=v_{+2}=0} = (-i)^{aN/2} \frac{(aN)!}{(aN/2)!} \\ & \times \sum_X \epsilon(P) \left[ \sum_{j=1}^{aN/2} \sum_{k=1}^{bN} (P(aN + 2k) - P(aN + 2k - 1)) \right. \\ & \times \left( \frac{1}{P(2j) - N(b + a/2) - 1/2} - \frac{1}{P(2j - 1) - N(b + a/2) - 1/2} \right) \\ & \times \exp(i\theta_a(P(2j) + P(2j - 1) - N(2b + a) - 1)) \\ & \times \exp(i\theta_b(P(aN + 2k) + P(aN + 2k - 1) - N(2b + a) - 1)) \\ & \left. \times \prod_{\substack{l=1 \\ l \neq j}}^{aN/2} \psi_{P(2l-1), P(2l)}(0) \prod_{\substack{l=aN/2+1 \\ l \neq k+aN/2+1}}^{(a/2+b)N} \phi_{P(2l-1), P(2l)}(0) \right]. \tag{3.6} \end{aligned}$$

For non-zero contribution in the sum over permutations, from (2.13) and (2.14) we require condition (2.15) for each  $l = 1, 2, \dots, (a/2 + b)N, l \neq j, aN/2 + k$ , where  $Q$  is now a permutation on  $\{1, 2, \dots, N(b + a/2)\} - \{p, q\}, 1 \leq p, q \leq N(b + a/2)$ . The permutations  $P(2j - 1), P(2j), P(aN + 2k - 1), P(aN + 2k)$  are free to assume the values as given by the following table, subject to the constraint in the column labelled ‘‘comment’’ and with associated parity  $\epsilon(P)$ :

$P(2j - 1)$	$P(2j)$	$P(aN + 2k - 1)$	$P(aN + 2k)$	$\epsilon(P)$	comment
$q$	$N(2b + a) - q$	$p$	$N(2b + a) - p$	+1	$p \neq q$
$p$	$N(2b + a) - q$	$q$	$N(2b + a) - p$	-1	$p \neq q$
$q$	$p$	$N(2b + a) - p$	$N(2b + a) - q$	+1	$p > q$
$N(2b + a) - p$	$N(2b + a) - q$	$q$	$p$	+1	$p > q$

Hence after some straight forward manipulation

$$\begin{aligned} \frac{\delta^2}{\delta v_{+1}(\theta_a)\delta v_{+2}(\theta_b)} Z(v_{+1}, v_{+2}) \Big|_{v_{+1}=v_{+2}=0} &= 2(4\pi)^{-2} \frac{Z(0,0)}{\sum_c \prod_{l=1}^{bN} (c(l) - 1/2)^2} \\ &\times \sum_{p,q=1}^{N(b+a/2)} \left[ (N(2b+a) + 1 - q)^2 (N(2b+a) + 1 - p)^2 \right. \\ &\quad \left. - (N(2b+a) + 1 - p - q)^2 \exp(i\theta_{ab}(p - q)) \right. \\ &\quad \left. - (p - q)^2 \cos \theta_{ab}(N(2b+a) - p - q + 1) \right] \\ &\times \sum_{c'} \prod_{l=1}^{Nb-1} (c'(l) - 1/2)^2, \end{aligned} \tag{3.7}$$

where  $c'$  is the set of all combinations of  $\{1, 2, \dots, (b + a/2)N\} - \{p, q\}$  taken  $(Nb - 1)$  at a time. Again using the local limit theorem [1, page 100, Theorem 2] we can easily show

$$\sum_{c'} \prod_{l=1}^{Nb-1} (c'(l) - 1/2)^2 \sim \frac{\alpha^2 \sum_c \prod_{l=1}^{Nb} (c(l) - 1/2)^2}{(\alpha^2 + p^2)(\alpha^2 + q^2)}. \tag{3.8}$$

Substituting (3.8) in (3.7) and then dividing by  $Z(0,0)$  gives the desired expression for  $\rho_{+1,+2}$  in the finite system. Noting

$$\theta_{ab} = \frac{y}{R} = \frac{2\pi y}{N(a + 2b)} Q, \tag{3.9}$$

$qQ$  denoting the total charge density and  $y$  the particle separation on the line, we observe the sums in (3.7) become Riemann integrals in the thermodynamic limit. Further we can evaluate the first of the integrals resulting from (3.7). We thus have the evaluation

$$\rho_{+1,+2}(y) = \rho_{+1}\rho_{+2} - \frac{1}{8} \left( \frac{Q}{\nu} \right)^2 \int_0^1 dt \int_{-1}^1 ds \frac{(t - s)^2 \cos \pi y Q(s + t)}{(1/\nu^2 + t^2)(1/\nu^2 + s^2)}, \tag{3.10}$$

where  $\rho_{+1}$  is the particle density of the  $+q$  charges and  $\rho_{+2}$  the particle density of the  $+2q$  charges.

Proceeding similarly we can show

$$\rho_{+2,+2}(y) = (\rho_{+2})^2 + \frac{Q^2}{16} \int_0^1 dt \int_{-1}^1 ds \frac{ts(t - s)^2 \cos \pi y Q(t + s)}{(1/\nu^2 + t^2)(1/\nu^2 + s^2)}, \tag{3.11}$$

and

$$\begin{aligned} \rho_{+1,+1}(y) &= (\rho_{+1})^2 + \frac{\pi}{2} \frac{Q^2}{\nu^2} \int_0^1 dt \frac{t \sin \pi Q |y| t}{(1/\nu^2 + t^2)} \\ &+ \frac{Q^2}{4\nu^4} \int_0^1 \int_{-1}^1 dt ds \frac{(t-s)^2 \cos \pi y Q(t+s)}{st(1/\nu^2 + t^2)(1/\nu^2 + s^2)}. \end{aligned} \tag{3.12}$$

We recall  $\nu$  is the unique positive solution of the equation

$$\frac{a}{2b+a} = \frac{\arctan \nu}{\nu} \tag{3.3}$$

and  $\rho_{+1} = aN/(2\pi R)$ ,  $\rho_{+2} = bN/(2\pi R)$ .

#### 4. Verification of the sum rules

The perfect screening sum rule [3] states in the present case

$$\int_{-\infty}^{\infty} dy (\rho_{+1,+1}^T(y) + 2\rho_{+1,+2}^T(y)) = -\rho_{+1} \tag{4.1}$$

and

$$\int_{-\infty}^{\infty} dy (\rho_{+1,+2}^T(y) + 2\rho_{+2,+2}^T(y)) = -2\rho_{+2}. \tag{4.2}$$

To evaluate the above integrals using the expressions (3.10)–(3.12) we interchange the order of integration so that the  $y$ -integration is being performed first. When the integrand is  $\cos \pi y Q(t+s)$  this gives a delta function which reduces the double integrals with respect to  $s$  and  $t$  to a single integral. In the case the integrand is  $\sin \pi Q |y| t$  we introduce a convergence factor  $e^{-\epsilon|y|}$  which allows the integrations to be separated. Following this procedure we readily find the sum rule is obeyed.

To verify Jancovici’s sum rule we require expansions of the truncated two particle correlation function for large  $y$ . To obtain the expansions we note the double integrals in (3.10)–(3.12) can all be separated into products of single integrals. The expansions of the single integrals can be obtained by integration by parts. Thus

$$\rho_{+1,+1}^T(y) = -\frac{1}{(\pi y)^2(\nu^2 + 1)^2} + O(1/y^4), \tag{4.3}$$

$$\rho_{+1,+2}^T(y) = -\frac{\nu^2}{2(\pi y)^2(\nu^2 + 1)^2} + O(1/y^4), \tag{4.4}$$

$$\rho_{+2,+2}^T(y) = -\frac{\nu^4}{4(\pi y)^2(\nu^2 + 1)^2} + O(1/y^4). \quad (4.5)$$

Hence from the definition of the truncated charge-charge correlation function (1.1) and recalling the calculation was performed at the temperature given by (2.3) we have

$$C_2^T(y) \sim \frac{-k_B T}{\pi^2 y^2} \text{ as } y \rightarrow \infty, \quad (4.6)$$

which is precisely (1.3) so Jancovici's sum rule holds.

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