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OPTIMAL STOPPING WITH VARIABLE ATTENTION

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Abstract

We consider an optimal stopping problem of a linear diffusion under Poisson constraint where the agent can adjust the arrival rate of new stopping opportunities. We assume that the agent may switch the rate of the Poisson process between two values. Maintaining the lower rate incurs no cost, whereas the higher rate requires effort that is captured by a cost function c. We study a broad class of payoff functions, cost functions and diffusion dynamics, for which we explicitly characterize the solution to the constrained stopping problem. We also characterize the case where switching to the higher rate is always suboptimal. The results are illustrated with two examples.

Keywords: Optimal stopping; Diffusion process; Resolvent operator; Poisson process

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1. Introduction

Consider an agent with an option to invest in the style of [30]. That is, the agent can make an irreversible investment in a project, where the returns follow a continuous-time stochastic process. In standard models of this style it is often assumed that the option can be exercised at any time. This reflects the idea that the agent is attentive to the relevant state variable at all times. In practice, this type of constant attention can be very costly, even prohibitively so. The issue of costly information acquisition and/or limited information processing capacity in economic decision making is addressed in the economics literature under the title 'rational inattention'. Rational inattention can be described so that when there is uncertainty about future events regarding a decision making problem and obtaining and processing information comes with a cost (at least a utility or an opportunity cost), it is rational for agents to update their information and strategies only sporadically. We refer the reader to the recent survey by Maćkowiak, Matějka and Wiederholt [29] for a comprehensive overview on the rational inattention literature.

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Mathematical models of rational inattention can be roughly divided into two main categories: structural models and models of reduced type. In structural models the information constraint is modelled directly and its effects on the resulting decision making problem are studied; see, e.g., [36, 37]. In these papers the observations on the system are acquired through a communication channel with a finite information processing capacity. As a result, rational agents are not able to utilize all the available information instantaneously and, hence, cannot optimally update their strategies in continuous time.

In models of reduced type one introduces a simplified proxy for attention into the model to study its implications on the optimal strategies. The proxy is usually chosen to be a deterministic or random discrete set of admissible decision making times. Papers of this type include, e.g., [17, 28, 32, 33]. In [32], estimation of an Ornstein–Uhlenbeck process with discrete periodic noisy observations is considered. Expectation formation by consumers is studied in [33] in the case where consumers update their plans only sporadically and remain inattentive otherwise. A similar idea is adapted in [17], where portfolio selection with only discrete periodic news updates is considered. Executive option valuation is considered in [28], where the proxy for attention is an independent Poisson process and the option can be exercised only at the jump times of this process.

Our study is related to the models of reduced type. Consider an agent observing the value of a dynamic stochastic variable at discrete irregularly spaced time points. The agent is facing an optimal stopping problem, where the objective is to maximize the expected present value of the exercise payoff contingent on the observations. The irregular observations stem from a linear diffusion process, which is observed at the arrival times of two independent Poisson processes with rates $\lambda_1 < \lambda_2$. The observation mechanism is set up as follows. At the initial time, the agent may stop or choose one of the Poisson processes and acquire the next observation of the diffusion at the first arrival of the chosen Poisson process. If the agent chooses the Poisson process with higher rate, she has to pay a potentially state-dependent cost c. At the arrival of the chosen Poisson process, the agent can choose between the following: she can either stop, continue following the same Poisson process or switch to the other Poisson process. The procedure is then repeated perpetually. The rate parameter of each Poisson process is understood as the mean number of observations per unit time contingent on the agent's decision. We call these parameters *attention rates* since they refer to the agent's level of attention. Lower attention can be maintained for free, whereas higher attention requires additional effort that is modelled by the cost c.

The decision rule of our agent is seen to be two fold. It is a combination of the stopping time and the sequence of decisions modulating the attention rate. Thus, our problem may be cast as a problem of stochastic control with an additional optimal stopping component. These types of problems are also known as stochastic control problems with discretionary stopping and there exists a wealth of literature on them. We refer the interested reader to [7] for a recent literature review. In most papers that have been written on the topic (such as [9, 18, 20]), the controlling aspect is formulated as a control that directly affects either the state or the dynamics of the underlying stochastic process. A notable exception is the cooperative game set-up of [19] in which controlling is understood as choosing a probability measure under which the expected payoff is calculated. Our model conforms to the latter framework since the attention rate control influences only the probability distribution of the next controlling/stopping opportunity.

As an application of our model, consider the optimal liquidation of a reserve of a renewable natural resource. The standard models (see, e.g., [4, 5, 38]) usually assume that the level of reserve is observable at all times. In practice, this quantity can be difficult to monitor due to different physical constraints. For instance, the reserve can be distributed spatially over a large area and it takes time and effort to quantify its level for decision making purposes. This setting can be captured by the proposed models. The reserve level is no longer observable continuously in time but rather for a given number of times per unit time on average. The rate λ_1 is the average number of observations per time unit that can be maintained without additional costs, whereas maintaining a higher average number of observations λ_2 incurs an extra cost c.

The proposed model is related to the recent paper by Hobson and Zeng [15]. In [15] the agent observes the diffusion and has the capacity to control the rate of the Poisson process continuously in time. Moreover, maintaining a rate will incur an instantaneous cost that is a function of the maintained rate; maintaining a higher rate is more costly than maintaining a lower rate. We consider an irregularly spaced discrete time version of this problem, where observations and rate adjustments are made only at the arrival times. However, in many cases studied in [15] the optimal adjustment rule is to switch between two rates, from 'low' to 'maximal'.

In this view, our restriction to modulate between two rates is reasonable. In [15] the authors consider examples with Brownian motion and geometric Brownian motion (GBM), and note that their results could be generalized to more general diffusions. The main contribution of the study at hand is to present a broad class of models with easily verifiable conditions for which the optimal solution can be characterized in terms of the minimal r-excessive functions of the diffusion process. This complements the existing literature. We also characterize the case where switching to the higher rate is always suboptimal.

Furthermore, models in which the underlying evolves in continuous time but the admissible control times are restricted have been studied in the literature over the past two decades. They are often called Poisson constrained optimal stopping problems. These problems were introduced by [10], where the diffusion is a GBM and the exercise payoff is of an American call option type. The results of [10] were extended in [23] to cover a broad class of payoff functions and linear diffusion dynamics. Menaldi and Robin [31] provide further generalizations, going so far as to have a not necessarily time-homogeneous Markov process with a locally compact state space and independent and identically distributed intervention times. However, at this level of generality not much can be said about the optimal stopping times besides the usual characterization and well-posedness results. Other papers in this vein include [6, 11–16, 21, 24–27, 34]. Arai and Takenaka [6] study Poisson constrained optimal stopping with a regime switching GBM. In [12], optimal stopping of the maximum of the GBM at Poisson arrivals is considered. The shape properties of the value function for a class of Poisson stopping problems with diffusion dynamics are analysed in [14]. In [21], optimal stopping with Poisson constraint is considered for a multidimensional state variable. Optimal stopping games with Poisson constraints under various degrees of generality are addressed in [11, 16, 24–26]. Lastly, Liang and Wei [27] study optimal switching at Poisson arrivals and Rogers and Zane [34] consider portfolio optimization under the Poisson constraint. There is also a wide variety of Poisson-constrained control problems that are not purely stopping problems. For a comprehensive and recent review on this topic, see [35, Section 3.4]. We point out that in all these papers, the rate of the Poisson process is not a decision variable.

The structure of the paper is as follows. In Section 2 we formulate the main stopping problems. In Section 3 we derive the candidate value functions and associated policies. The candidate values are proved to be the actual values in Section 4. Section 5 contains a brief discussion on the asymptotic properties of the main stopping problem. The findings are illustrated in Section 6.

2. Problem formulation

2.1. The dynamics

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space supporting a diffusion X with a state space I that is an interval in $\mathbb{R}$ such that $\bar{I} = [\mathfrak{l}, \mathfrak{r}]$, $-\infty \leq \mathfrak{l} < \mathfrak{r} \leq \infty$. We denote by $\mathbb{P}_x$ the probability measure $\mathbb{P}$ conditioned on the event that the diffusion is started from the state $x \in I$. The corresponding expectation is denoted by $\mathbb{E}_x[\,\cdot\,]$. We assume that the boundaries $\mathfrak{l}$, $\mathfrak{r}$ are natural. Thus, the lifetime of X

$$\zeta = \inf\{t \geq 0 \colon X_t \notin I\}$$

satisfies $\zeta = \infty$, $\mathbb{P}$-almost surely. As usual, we denote by m and S the speed measure and the scale function of X, which we assume to be absolutely continuous with respect to the Lebesgue measure on I and twice continuously differentiable. The infinitesimal generator $\mathcal{A} \colon C^2(I) \to C_b(I)$ of X can be written as

$$\mathcal{A} = \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx},$$

and the infinitesimal parameters $\mu \colon I \to \mathbb{R}$, $\sigma^2 \colon I \to \mathbb{R}_+$ are defined by

$$m'(x) = \frac{2}{\sigma^2(x)}\mathrm{e}^{B(x)}, \qquad S'(x) = \mathrm{e}^{-B(x)},$$

where $B(x) = -\int^x (2\mu(y)/\sigma^2(y))\,\mathrm{d}y$. Note that we use the convention $\mathbb{R}_+ = (0, \infty)$ here. Let $r > 0$. We denote by $\boldsymbol{\psi}_r$ ($\boldsymbol{\varphi}_r$) the unique increasing (decreasing) fundamental solution to the ordinary differential equation $\mathcal{A}u = ru$. These fundamental solutions can be identified as minimal r-excessive mappings for the diffusion X. Let $L_r^1(I)$ be the set of measurable functions $f \colon I \to \mathbb{R}$ such that

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-rs}|f(X_s)|\,\mathrm{d}s\right] < \infty$$

and, for any $f \in L_r^1(I)$, define the resolvent $\mathbf{R}_r$ as

$$(\mathbf{R}_r f)(x) = \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-rs}f(X_s)\,\mathrm{d}s\right]. \tag{1}$$

By combining the relevant identities from Sections I.2.7, II.1.4 and II.1.11 in [8] and making use of Fubini's theorem (which holds for the integral $(R_r f)$ when $f \in L_r^1(I)$), we see that the resolvent $\mathbf{R}_r$ can be expressed in terms of the minimal r-excessive functions $\boldsymbol{\psi}_r$ and $\boldsymbol{\varphi}_r$ as

$$(\mathbf{R}_r f)(x) = B_r^{-1}\left(\boldsymbol{\varphi}_r(x)(\boldsymbol{\Psi}_r f)(x) + \boldsymbol{\psi}_r(x)(\boldsymbol{\Phi}_r f)(x)\right). \tag{2}$$

Here, B_r is the (constant) Wronskian

$$B_r = \frac{\boldsymbol{\psi}_r'(x)\boldsymbol{\varphi}_r(x) - \boldsymbol{\psi}_r(x)\boldsymbol{\varphi}_r'(x)}{S'(x)} \tag{3}$$

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and the functionals Ψ_r and Φ_r are defined as

$$(\mathbf{\Psi}_r f)(x) = \int_{\mathfrak{l}}^{x} \boldsymbol{\psi}_r(z) f(z) m'(z)\,\mathrm{d}z,$$
$$(\mathbf{\Phi}_r f)(x) = \int_{x}^{\mathfrak{r}} \boldsymbol{\varphi}_r(z) f(z) m'(z)\,\mathrm{d}z.$$

We also recall the resolvent equation (see, e.g., [8, p. 4])

$$\mathbf{R}_q \mathbf{R}_r = \frac{\mathbf{R}_r - \mathbf{R}_q}{q - r}, \tag{4}$$

where $q > r > 0$.

Lastly, we make use of the formulae (see, e.g., [2, Corollary 3.2])

$$\frac{f'(x)\boldsymbol{\psi}_r(x) - \boldsymbol{\psi}_r'(x) f(x)}{S'(x)} = \int_{\mathfrak{l}}^{x} \boldsymbol{\psi}_r(z)((\mathcal{A} - r)f)(z) m'(z)\,\mathrm{d}z, \tag{5}$$

$$\frac{\boldsymbol{\varphi}_r'(x) h(x) - h'(x)\boldsymbol{\varphi}_r(x)}{S'(x)} = \int_{x}^{\mathfrak{r}} \boldsymbol{\varphi}_r(z)((\mathcal{A} - r)h)(z) m'(z)\,\mathrm{d}z, \tag{6}$$

which hold for functions $f,\, h \in C^2(I)$ such that $\lim_{x\downarrow\mathfrak{l}} f(x)/\boldsymbol{\varphi}_r(x) = \lim_{x\uparrow\mathfrak{r}} h(x)/\boldsymbol{\psi}_r(x) = 0$ and $(\mathcal{A} - r)f,\ (\mathcal{A} - r)h \in L_r^1(I)$. Note that the last two conditions are equivalent to the integrability conditions of [2, Cor. 3.2]. On the other hand, the lower boundary $\mathfrak{l}$ was assumed to be natural so $\lim_{x\downarrow\mathfrak{l}} \boldsymbol{\varphi}_r(x) = \infty$ and $\lim_{x\downarrow\mathfrak{l}} f(x)/\boldsymbol{\varphi}_r(x) = 0$ may be used in the proof of [2, Corollary 3.2] instead of the condition $|f(0+)| < \infty$.

2.2. The stopping problem

In this section we set up the optimal stopping problem. We start by recalling the Poisson stopping problem discussed in [23]. Let N be a Poisson process with rate $\lambda > 0$ and assume that the filtration $\mathbb{F}$ carries information on both X and N. Define the discrete-time filtration $\mathbb{G}_0 = (\mathcal{G}_n)_{n\geq 0}$ as $\mathcal{G}_n = \mathcal{F}_{T_n}$, where $T_0 = 0$ and $(T_n - T_{n-1}) \sim \mathrm{Exp}(\lambda)$ for all $n \in \mathbb{N}$. We call the times T_n arrival times because they model the arrival of new stopping opportunities. Define the admissible stopping times $\mathcal{S}_0$ for a Poisson stopping problem as mappings $\tau :\ \Omega \to \{T_n :\ n \geq 0\}$ such that

(i) for all $x \in I$, $\mathbb{E}_x[\tau] < \infty$;

(ii) for all $n \geq 0$, $\{\tau = T_n\} \in \mathcal{G}_n$.

Then the value function of the Poisson stopping problem is defined as

$$V_0^{\lambda}(x) = \sup\nolimits_{\tau\in\mathcal{S}_0} \mathbb{E}_x\left[\mathrm{e}^{-r\tau} g(X_\tau)\right], \tag{7}$$

where the function g is the payoff. This Poisson stopping problem can be solved under mild assumptions and the value function (7) can be written in a semi-explicit form in terms of the minimal r-excessive functions; see [23].

We extend the idea above to accommodate rate controls. Let $0 < \lambda_1 < \lambda_2$ and denote by N^a, where $a = 1, 2$, a Poisson process with rate λ_i. Denote the arrival times of N^a as T_n^a, $a = 1, 2$.

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We assume that the filtration $\mathbb{F}$ is rich enough to carry the processes X, N^1 and N^2, and that X, N^1 and N^2 are independent. Now the agent has two arrival processes at her disposal and can switch between these over time. The switching decisions are expressed as sequences of random variables $A = (A_n)_{n\geq 0}$; we call such A attention sequences. We define attention sequences inductively as follows. Let A_0 be $\mathcal{F}_0$-measurable such that $A_0(\omega) \in \{\lambda_1, \lambda_2\}$ for all $\omega \in \Omega$. The value of A_0 indicates which arrival process the agent is following at time 0. Let $T_0^A = 0$ and define

$$T_n^A = T_{n-1}^A + U^{A_{n-1}}, \ \ (U^{A_{n-1}} | A_{n-1} = \lambda_i) \sim \mathrm{Exp}(\lambda_i) \quad \text{for } n \geq 1.$$

The variables U^{A_m} model the time remaining until the next arrival, conditional on the agent's mth choice. Using this, we define A_n as a $\mathcal{F}_{T_n^A -}$-measurable random variable taking values in $\{\lambda_1, \lambda_2\}$ for $n \geq 1$. These variables indicate which arrival process the agent chooses to follow at the nth time of choosing. If $A_n = \lambda_a$ for a given n, we say that the agent in maintaining λ_a attention, $a = 1, 2$.

Define the discrete time filtration $\mathbb{G}_0^A = (\mathcal{G}_n^A)_{n\geq 0}$ recursively by

$$\mathcal{G}_n^A = \mathcal{G}_{n-1}^A \bigvee \mathcal{F}_{T_{n-1}^A + U^{A_{n-1}}}, \qquad \mathcal{G}_0^A = \mathcal{F}_0.$$

For a fixed sequence A, we define the admissible stopping times as mappings $\tau \colon \Omega \to \{T_n^A \colon n \geq 0\}$ such that

(i) for all $x \in I$, $\mathbb{E}_x[\tau] < \infty$;

(ii) for all $n \geq 0$, $\{\tau = T_n^A\} \in \mathcal{G}_n^A$.

Let $\mathcal{S}_0^A$ be the class of stopping times that are admissible with respect to the attention sequence A. We define an auxiliary stopping problem as

$$V_0^A(x) = \sup\nolimits_{\tau \in \mathcal{S}_0^A} \mathbb{E}_x \left[\mathrm{e}^{-r\tau} g(X_\tau) - \sum_{n=0}^{\infty} \mathrm{e}^{-rT_n^A} c(X_{T_n^A}) \mathbf{1}(A_n = \lambda_2) \mathbf{1}(T_n^A < \tau) \right],$$

where g is the exercise payoff and c is the period cost function. The function c is the cost the agent must pay in order to wait for a single arrival with the higher rate λ_2. Now our main stopping problem reads as

$$V_0(x) = \sup\nolimits_A V_0^A(x). \tag{8}$$

We also set up constrained stopping problems, where immediate stopping is not possible; this corresponds to the situation where immediate observation of X is not available but the future observations occur according to our model. To this end, we define two sets of stopping times as follows. Let $\mathcal{S}_a^A$, where $a = 1, 2$, be the set of mappings $\tau \colon \Omega \to \{T_n^A \colon n \geq 0\}$ such that

(i) for all $x \in I$, $\mathbb{E}_x[\tau] < \infty$;

(ii) for all $n \geq 1$, $\{\tau = T_n^A\} \in \mathcal{G}_n^A$;

(iii) $\mathbb{P}(\tau = T_0^A) = 0, \ A_0 = \lambda_a$.

We define, for a fixed A with $A_0 = \lambda_a$, the auxiliary stopping problem

$$V_a^A(x) = \sup\nolimits_{\tau \in \mathcal{S}_a^A} \mathbb{E}_x \left[\mathrm{e}^{-r\tau} g(X_\tau) - \sum_{n=0}^{\infty} \mathrm{e}^{-rT_n^A} c(X_{T_n^A}) \mathbf{1}(A_n = \lambda_2) \mathbf{1}(T_n^A < \tau) \right].$$

Thus, the version of the main stopping problem where immediate stopping is not allowed reads as

$$V_a(x) = \sup_A V_a^A(x), \ \ a \in \{1, 2\}. \tag{9}$$

The function (9) can be interpreted as the value of the future stopping potentiality contingent on the level of attention and the state of diffusion X. This function is related to the value of the stopping problem in between the arrival times. In our proposed model we assume that the state of X is not observed between the arrivals. However, this does not prohibit us from devising this function and using it in our analysis. In the following, we refer to this function as the value of future stopping potentiality.

We study the problems (8) and (9) under the following assumptions.

Assumption 1. *Let*

$$p(x) = \lambda_2(\mathbf{R}_r c)(x) + c(x), \qquad P(x) = g(x) + p(x),$$
$$y^{\lambda_2, c} = \inf\left\{ y \in I \colon \frac{P(y)}{\boldsymbol{\psi}(y)} > \frac{(\boldsymbol{\Phi}_{r+\lambda_2} P)(y)}{(\boldsymbol{\Phi}_{r+\lambda_2}\boldsymbol{\psi}_r)(y)} \right\},$$
$$l_g = \inf\{x \in I \colon g(x) > 0\}.$$

We assume the following assertions hold.

(i) *The period cost c is non-negative, non-decreasing and r excessive. Furthermore, $c \in L_r^1(I) \cap C^2(I)$, $(r - \mathcal{A})c \in L_r^1(I)$, $\lim_{x\downarrow \mathfrak{l}} p(x)/\boldsymbol{\varphi}_r(x) = \lim_{x\uparrow \mathfrak{r}} p(x)/\boldsymbol{\psi}_r(x) = 0$ and $p/\boldsymbol{\psi}_r$ is strictly decreasing.*

(ii) *The payoff g is non-negative, non-decreasing and piecewise C^2. The left and right derivatives $d^{\pm}g(x)/dx$, $d^{2\pm}g(x)/dx^2$ are assumed to be finite for all $x \in I$. Furthermore, $g \in L_r^1(I) \cap C^0(I)$, $(r - \mathcal{A})g \in L_r^1(I)$, $\lim_{x\downarrow \mathfrak{l}} g(x)/\boldsymbol{\varphi}_r(x) = \lim_{x\uparrow \mathfrak{r}} g(x)/\boldsymbol{\psi}_r(x) = 0$ and there exists a unique state*

$$y^{\infty} = \operatorname{argmax}_{x\in(l_g, \mathfrak{r})} \left\{ g(x)/\boldsymbol{\psi}_r(x) \right\}$$

such that $g/\boldsymbol{\psi}_r$ is non-decreasing on (l_g, y^{∞}) and non-increasing on $(y^{\infty}, \mathfrak{r})$.

(iii) *The function $(g - \lambda_2(\mathbf{R}_{r+\lambda_2} g) + c)/\boldsymbol{\varphi}_{r+\lambda_2}$ is strictly increasing on $(y^{\lambda_2, c}, \mathfrak{r})$. Moreover, the function $(g - \lambda_1(\mathbf{R}_{r+\lambda_1} g))/\boldsymbol{\varphi}_{r+\lambda_1}$ is strictly increasing on $(y^{\lambda_1}, \mathfrak{r})$ and*

$$\frac{P(x)}{\boldsymbol{\psi}_r(x)} < \frac{(\boldsymbol{\Phi}_{r+\lambda_2} P)(x)}{(\boldsymbol{\Phi}_{r+\lambda_2}\boldsymbol{\psi}_r)(x)} \quad \text{for } x \in (l_g, y^{\lambda_2, c}).$$

Remark 1. Assumption 1(i) and 1(ii) contain relevant regularity and monotonicity assumptions on the period cost and the payoff. It is interesting to note that r excessivity of the cost function c follows from the other assumptions for the examples we consider in Section 6. In particular, positive constants are trivially r excessive and if $c(x) = kx$ for some $k > 0$, the diffusion is a GBM and the payoff is $g(x) = x^{\theta} - \eta$ with $0 < \theta < 1 < \eta$, then r excessivity of c follows from $(r - \mathcal{A})g \in L_r^1(I)$. The specific reason we require the cost to be r excessive is that in certain points of the proofs in Appendix A we encounter integrals that need to be of a certain sign. These signs behave as they should if $(\mathcal{A} - r)c(x) \le 0$ for all $x \in I$, i.e. if c is r excessive.

On the other hand, in Assumption 1(iii) the monotonicity assumptions are related to the classical stopping problem studied, for example, in [1]. These are typical assumptions that

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guarantee that a unique point splits the state space to the continuation region and the stopping region. Together with the limit assumptions for $g/\boldsymbol{\psi}_r$, these correspond to the assumptions made in [23] for the Poisson stopping problem.

Assumption 1 (iii) is connected to two Poisson stopping problems: one with the attention rate λ_1 and one with the attention rate λ_2 and period cost c. From a technical point of view it is needed to prove uniqueness of the optimal solution in our case. We also observe that Assumption 1(iii) is closely connected to typical superharmonicity assumptions for twice continuously differentiable payoffs. More precisely, assume for a moment that $g \in C^2(I)$ and that there exists a unique $\hat{x} \in I$ such that $((\mathcal{A}-r)g)(x) \lesseqqgtr 0$, when $x \gtreqqless \hat{x}$ (see [3, 22]). These type of conditions imply the monotonicity conditions in Assumption 1(ii) and are easily verified. In our case, making use of (5) and (6), we find that

$$\begin{aligned}&\frac{P(x)-\lambda_2(\mathbf{R}_{r+\lambda_2}P)(x)}{\boldsymbol{\varphi}_{r+\lambda_2}(x)}\\&\qquad = B^{-1}_{r+\lambda_2}\left((\boldsymbol{\Psi}_{r+\lambda_2}(r-\mathcal{A})P)(x)+\frac{\boldsymbol{\varphi}_{r+\lambda_2}(x)}{\boldsymbol{\psi}_{r+\lambda_2}(x)}(\boldsymbol{\Phi}_{r+\lambda_2}(r-\mathcal{A})P)(x)\right),\end{aligned}$$

which is strictly increasing if and only if $(\boldsymbol{\Phi}_{r+\lambda_2}(\mathcal{A}-r)P)(x)<0$. Also,

$$\frac{P(y)}{\boldsymbol{\psi}_r(y)} < \frac{(\boldsymbol{\Phi}_{r+\lambda_2}P)(y)}{(\boldsymbol{\Phi}_{r+\lambda_2}\boldsymbol{\psi}_r)(y)}$$

if and only if

$$\frac{\boldsymbol{\psi}_r(y)}{\boldsymbol{\varphi}_{r+\lambda_2}(y)}(\boldsymbol{\Phi}_{r+\lambda_2}(\mathcal{A}-r)P)(y)+(\boldsymbol{\Psi}_r(\mathcal{A}-r)P)(y)>0.$$

Hence, it is evident that, for twice continuously differentiable payoffs, Assumption 1(iii) is closely connected to the sign of $(\mathcal{A}-r)g$.

We use the standard Poisson stopping problem (7) as a point of reference in our study. Indeed, we know from [23] and Remark 1 that under Assumption 1, the value function in (7) reads as

$$V_0^\lambda(x)=\begin{cases} g(x), & x\geq y^\lambda,\\ \dfrac{g(y^\lambda)}{\boldsymbol{\psi}_r(y^\lambda)}\boldsymbol{\psi}_r(x), & x\leq y^\lambda,\end{cases} \tag{10}$$

where the level $y^\lambda < y^\infty$ is characterized uniquely by the condition

$$g(y^\lambda)(\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\psi}_r)(y^\lambda)=\boldsymbol{\psi}_r(y^\lambda)(\boldsymbol{\Phi}_{r+\lambda}g)(y^\lambda).$$

We close this section with certain technical calculations that will be useful in the later sections.

Lemma 1. *Let* $\lambda>0$*, and* $x\in I$*. Then*

1.
$$\begin{aligned}&\frac{1}{B_r}\left((\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\varphi}_r)(x)(\boldsymbol{\Psi}_r(r-\mathcal{A})p)(x)+(\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\psi}_r)(x)(\boldsymbol{\Phi}_r(r-\mathcal{A})p)(x)\right)\\&\qquad=(\boldsymbol{\Psi}_{r+\lambda}p)(x)+\lambda^{-1}(\boldsymbol{\Psi}_{r+\lambda}(r-\mathcal{A})p)(x),\\&\frac{1}{B_r}\left((\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\varphi}_r)(x)(\boldsymbol{\Psi}_r(r-\mathcal{A})p)(x)+(\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\psi}_r)(x)(\boldsymbol{\Phi}_r(r-\mathcal{A})p)(x)\right)\\&\qquad=(\boldsymbol{\Phi}_{r+\lambda}p)(x)+\lambda^{-1}(\boldsymbol{\Phi}_{r+\lambda}(r-\mathcal{A})p)(x);\end{aligned}$$

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2.
$$\begin{vmatrix} p & \boldsymbol{\psi}_r & \boldsymbol{\varphi}_r \\ (\boldsymbol{\Phi}_{r+\lambda}p) & (\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\psi}_r) & (\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\varphi}_r) \\ (\boldsymbol{\Psi}_{r+\lambda}p) & (\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\psi}_r) & (\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\varphi}_r) \end{vmatrix}(x) = \frac{B_r B_{r+\lambda}}{\lambda^2}\left(c(x) + (\lambda_2 - \lambda)(\mathbf{R}_r c)(x)\right).$$

Proof. See Appendix A.

3. Necessary conditions

For brevity, we use from here on the following notation for the relevant resolvents, fundamental solutions and functionals:

$$R_a := \mathbf{R}_{r+\lambda_a}, \quad R_0 := \mathbf{R}_r, \quad \psi_a := \boldsymbol{\psi}_{r+\lambda_a}, \quad \varphi_a := \boldsymbol{\varphi}_{r+\lambda_a}, \quad \psi_0 := \boldsymbol{\psi}_r, \quad \varphi_0 := \boldsymbol{\varphi}_r,$$
$$\Psi_a := \boldsymbol{\Psi}_{r+\lambda_a}, \quad \Phi_a := \boldsymbol{\Phi}_{r+\lambda_a}, \quad \Psi_0 := \boldsymbol{\Psi}_r, \quad \Phi_0 := \boldsymbol{\Phi}_r,$$

for $a = 1, 2$.

The purpose of this section is to derive a solution candidate for the problem (8) under Assumption 1. Since the diffusion X is time homogeneous and the functions g and c are non-decreasing, we make the working assumption that the optimal policy is a threshold rule with two constant thresholds x^* and y^*. The policy is as follows.

(A1) Assume that the agent is maintaining λ_1 attention. Let $\hat{T}^1$ be the next N^1-arrival time. If $X_{\hat{T}^1} < x^*$ then the agent maintains the λ_1 rate. If $X_{\hat{T}^1} \geq x^*$ then the agent either switches to rate λ_2 or stops. Stopping occurs if $X_{\hat{T}^1} \geq y^*$.

(A2) Assume that the agent is maintaining λ_2 attention. Let $\hat{T}^2$ be the next N^2-arrival time. If $X_{\hat{T}^2} < x^*$ then the agent switches to the λ_1 rate. If $X_{\hat{T}^2} \geq x^*$ then the agent either maintains the λ_2 rate or stops. Stopping occurs if $X_{\hat{T}^2} \geq y^*$.

If $x^* > y^*$ then the main problem (8) reduces to the Poisson stopping problem (7). We assume in the following that $x^* < y^*$ unless otherwise stated.

Next we construct the value function candidate associated with the strategy described in (A1) and (A2). Denote as G_1 the associated candidate value of future stopping potentiality while maintaining λ_1 attention; function G_2 has an analogous definition. Moreover, we denote as G_0 the associated candidate value at the arrival times. We let $A^* = (A_n^*)_{n\geq 0}$ and τ^* denote the attention sequence and the stopping time associated with the strategy described in (A1) and (A2) when the problem is started at an arrival time. Note that A_n^* and τ^* are measurable since their values are determined by collections of sets of the form $\{X_{T_m} \in B_m\} \subseteq \Omega$, where $m \geq 0$ and B_m are measurable subsets of the state space I. Looking at the right-hand side of (8), we get an explicit expression for G_0:

$$G_0(x) = \mathbb{E}_x\left[e^{-r\tau^*} g(X_{\tau^*}) - \sum_{n=0}^{\infty} e^{-rT_n^{A^*}} c(X_{T_n^{A^*}}) \mathbf{1}(A_n^* = \lambda_2) \mathbf{1}(T_n^{A^*} < \tau^*)\right]. \tag{11}$$

Expressions for G_1 and G_2 may be obtained analogously. Supposing further that the dynamic programming principle (DPP) holds for G_0 and making use of the memoryless property of the exponential distribution, we get

$$\begin{aligned} G_0(x) &= \max\{g(x), G_1(x), G_2(x)\}, \\ G_1(x) &= \mathbb{E}_x\left[e^{-rU_1} G_0(X_{U_1})\right], \\ G_2(x) &= \mathbb{E}_x\left[e^{-rU_2} G_0(X_{U_2})\right] - c(x), \end{aligned} \tag{12}$$

where independent random variables $U_a \sim \mathrm{Exp}(\lambda_a)$ for $a = 1, 2$. It should be noted that in this section the DPP (12) is used to derive certain necessary conditions (described by the pair of equations (22)), that an optimal strategy conforming to (A1) and (A2) must satisfy. In Proposition 1 we prove that there exist three uniquely determined functions satisfying the DPP (12) and the mentioned necessary conditions. In Section 4 we prove that these three functions coincide with the three value functions defined in (8) and (9), verifying the optimality of the strategy outlined in (A1) and (A2).

We may write the candidates G_1, G_2 alternatively as

$$\begin{aligned} G_1(x) &= G_0(x) + \mathbb{E}_x\left[\mathrm{e}^{-rU_1} G_0(X_{U_1})\right] - G_0(x), \\ G_2(x) &= G_0(x) + \mathbb{E}_x\left[\mathrm{e}^{-rU_2} G_0(X_{U_2})\right] - c(x) - G_0(x). \end{aligned} \tag{13}$$

Equations (13) are mathematically trivial in relation to (12), but their significance is to highlight the relations between the candidates conceptually rather than technically. Indeed, keeping in mind the DPP of (12), condition (13) states that, for $i = 1, 2$, the candidate G_i is obtained from G_0 by subtracting the loss incurred from being forced to initially wait for an arrival with attention rate λ_i. We also have the following equivalence result for the smoothness properties of G_0, G_1 and G_2.

Lemma 2. *Suppose that Assumption 1 holds and the infinitesimal parameter σ^2 satisfies $\sigma^2 \in C^0(I)$. Then the conditions $G_1 \in C^2(I)$ and $G_2 \in C^2(I)$ are equivalent to $G_0 \in C^0(I)$.*

Proof. Recall that $c \in C^2(I)$ by our assumptions and note that it follows from (1) and (12) that

$$\begin{aligned} G_1(x) &= \mathbb{E}_x\left[\mathrm{e}^{-rU_1} G_0(X_{U_1})\right] = \lambda_1 (R_1 G_0)(x), \\ G_2(x) &= \mathbb{E}_x\left[\mathrm{e}^{-rU_2} G_0(X_{U_2})\right] - c(x) = \lambda_2 (R_2 G_0)(x) - c(x). \end{aligned}$$

Representation (2) implies that the mappings $(R_i G_0)$, $i \in \{1, 2\}$, are continuously differentiable so that $G_1, G_2 \in C^1(I)$. Calculating the second derivatives gives

$$\begin{aligned} G_1''(x) &= \frac{\lambda_1}{B_1}\left(\varphi_1''(x)\,(\Psi_1 G_0)\,(x) + \psi_1''(x)\,(\Phi_1 G_0)\,(x)\right) - \frac{2\lambda_1}{\sigma^2(x)} G_0(x), \\ G_2''(x) &= \frac{\lambda_2}{B_2}\left(\varphi_2''(x)\,(\Psi_2 G_0)\,(x) + \psi_2''(x)\,(\Phi_2 G_0)\,(x)\right) - \frac{2\lambda_2}{\sigma^2(x)} G_0(x) - c''(x), \end{aligned}$$

proving the claim.

The rest of this section is devoted to finding an analytical rather than probabilistic expression for G_0 and deriving necessary optimality conditions for x^* and y^*. By (A1) and (12), we find that $G_0(x) = G_1(x) = \lambda_1 (R_1 G_0)(x)$ for $x < x^*$. Now [23, Lemma 2.1] implies that G_1 is r-harmonic for $x < x^*$. We find similarly that the function $x \mapsto G_0(x) + p(x)$ is r harmonic on $[x^*, y^*)$. Since we are looking for a candidate that is finite at the lower boundary $\mathfrak{l}$, we find that $G_0(x) = C_1 \psi_0(x)$ for $x < x^*$ and some fixed $C_1 \in \mathbb{R}_+$. By (12) we have

$$\lambda_1 C_1 (R_1 \psi_0)(x) = C_1 \psi_0(x) = \lambda_1 C_1 (R_1 G_0)(x) \quad \text{for } x < x^*.$$

Hence,

$$C_1 (\Phi_1 \psi_0)(x^*) = (\Phi_1 G_0)(x^*). \tag{14}$$

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Let $x \in [x^*, y^*)$. Then $G_0(x) + p(x) = C_2\psi_0(x) + C_3\varphi_0(x)$ for some $C_2, C_3 \in \mathbb{R}_+$. We can now expand condition (14) as

$$C_1(\Phi_1\psi_0)(x^*) = \int_{x^*}^{y^*} \varphi_1(z)(C_2\psi_0(z) + C_3\varphi_0(z) - p(z))m'(z)\,\mathrm{d}z + (\Phi_1 g)(y^*).$$

Conversely, the condition

$$\begin{aligned}\lambda_2(R_2(C_2\psi_0 + C_3\varphi_0))(x) &= C_2\psi_0(x) + C_3\varphi_0(x)\\ &= G_0(x) + p(x)\\ &= \lambda_2(R_2(G_0 + p))(x)\end{aligned}$$

holds and it can be rewritten as

$$\begin{aligned}&\varphi_2(x)(\Psi_2(C_2\psi_0 + C_3\varphi_0))(x^*) + \psi_2(x)(\Phi_2(C_2\psi_0 + C_3\varphi_0))(y^*)\\ &\quad = \varphi_2(x)(\Psi_2(C_1\psi_0 + p))(x^*) + \psi_2(x)(\Phi_2(P))(y^*).\end{aligned}$$

Since the functions ψ_2 and φ_2 are linearly independent on open intervals, the previous equality yields the pair of equations

$$\begin{aligned}(\Phi_2(C_2\psi_0 + C_3\varphi_0)(y^*) &= (\Phi_2 P)(y^*),\\ (\Psi_2(C_2\psi_0 + C_3\varphi_0)(x^*) &= (\Psi_2(C_1\psi_0 + p))(x^*).\end{aligned} \tag{15}$$

We see from the considerations above that the candidate G_0 contains a total of five unknown variables $(C_1, C_2, C_3, x^*, y^*)$ that need to be determined. Conditions (14) and (15) contain three equations and the last two are given by the boundary conditions imposed by the continuity of G_0:

$$\begin{aligned}C_2\psi_0(x^*) + C_3\varphi_0(x^*) - p(x^*) &= C_1\psi_0(x^*),\\ C_2\psi_0(y^*) + C_3\varphi_0(y^*) &= P(y^*).\end{aligned} \tag{16}$$

The system of equations consisting of (14), (15) and (16) can be rewritten as

$$\begin{aligned}&C_1(\Phi_1\psi_0)(x^*) + C_2((\Phi_1\psi_0)(y^*) - (\Phi_1\psi_0)(x^*))\\ &\quad + C_3((\Phi_1\varphi_0)(y^*) - (\Phi_1\varphi_0)(x^*)) = (\Phi_1 P)(y^*) - (\Phi_1 p)(x^*),\end{aligned} \tag{17a}$$

$$C_2(\Phi_2\psi_0)(y^*) + C_3(\Phi_2\varphi_0)(y^*) = (\Phi_{2P})(y^*), \tag{17b}$$

$$(C_2 - C_1)(\Phi_2\psi_0)(x^*) + C_3(\Phi_2\varphi_0)(x^*) = (\Phi_2 p)(x^*), \tag{17c}$$

$$C_2\psi_0(x^*) + C_3\varphi_0(x^*) - p(x^*) = C_1\psi_0(x^*), \tag{17d}$$

$$C_2\psi_0(y^*) + C_3\varphi_0(y^*) = P(y^*). \tag{17e}$$

Next we reduce the system (17) to a pair of equations where the only variables are x^* and y^*. The constants C_2 and C_3 can be solved from (17b) and (17e), yielding

$$\begin{aligned}C_3 &= \frac{(\Phi_2\psi_0)(y^*)P(y^*) - (\Phi_2 P)(y^*)\psi_0(y^*)}{(\Phi_2\psi_0)(y^*)\varphi_0(y^*) - (\Phi_2\varphi_0)(y^*)\psi_0(y^*)},\\ C_2 &= \frac{(\Phi_2 P)(y^*) - (\Phi_2\varphi_0)(y^*)C_3}{(\Phi_2\psi_0)(y^*)}.\end{aligned} \tag{18}$$

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Solving for C_1 in (17d) gives

$$C_1 = C_2 + \frac{\varphi_0(x^*)}{\psi_0(x^*)} C_3 - \frac{p(x^*)}{\psi_0(x^*)}, \tag{19}$$

and lastly solving C_3 from (17a) and (17c) yields the condition

$$H_1(x^*) = H_2(y^*) = f(x^*, y^*), \tag{20}$$

where

$$H_1(x) = \frac{p(x)(\Psi_2\psi_0)(x) - \psi_0(x)(\Psi_2 p)(x)}{\varphi_0(x)(\Psi_2\psi_0)(x) - \psi_0(x)(\Psi_2\varphi_0)(x)},$$

$$H_2(x) = \frac{P(x)(\Phi_2\psi_0)(x) - \psi_0(x)(\Phi_2 P)(x)}{\varphi_0(x)(\Phi_2\psi_0)(x) - \psi_0(x)(\Phi_2\varphi_0)(x)},$$

$$f(x, y) = \frac{((\Phi_2 P)(y)/(\Phi_2\psi_0)(y))(\Phi_1\psi_0)(y) - (\Phi_1 P)(y) - (p(x)/\psi_0(x))(\Phi_1\psi_0)(x) + (\Phi_1 p)(x)}{((\Phi_2\varphi_0)(y)/(\Phi_2\psi_0)(y))(\Phi_1\psi_0)(y) - (\Phi_1\varphi_0)(y) - (\varphi_0(x)/\psi_0(x))(\Phi_1\psi_0)(x) + (\Phi_1\varphi_0)(x)}.$$

Even though the pair of equations (20) is nonlinear, it can be solved under our assumptions. We begin by deriving an additional constraint on the pair (20) that rules out degenerate cases, where the candidate G_0 reduces to the value function of the Poisson stopping problem (10).

Lemma 3. *If* $H_1(y^{\lambda_1}) \leq H_2(y^{\lambda_1})$ *then* $G_0 = V_0^{y^{\lambda_1}}$ *and if* $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$ *then the thresholds* x^* *and* y^* *satisfy* $x^* < y^{\lambda_1} < y^*$.

Proof. See Appendix A.

In light of Lemma 3, we now assume that

$$H_1(y^{\lambda_1}) > H_2(y^{\lambda_1}) \tag{21}$$

for the remainder of this section. Despite its technical appearance, condition (21) is closely connected to the structure of the optimal strategy. We prove in Theorem 2 that choosing to wait for the next arrival time with λ_2 attention is suboptimal for all initial states $x \in I$ precisely when $H_1(y^{\lambda_1}) \leq H_2(y^{\lambda_1})$. This is in line with the implication of Lemma 3 stating that the value function candidate G_0 coincides with the value function of the Poisson stopping problem (10) precisely when $H_1(y^{\lambda_1}) \leq H_2(y^{\lambda_1})$.

To further simplify the pair (20), we can use the condition $H_1(x^*) = H_2(y^*)$ to separate the parts depending on x^* and y^* in the other equation. Then we end up with the equivalent necessary conditions

$$\begin{aligned} H_1(x^*) &= H_2(y^*), \\ K_1(x^*) &= K_2(y^*), \end{aligned} \tag{22}$$

where

$$K_1(x) = \frac{p(x)}{\psi_0(x)}(\Phi_1\psi_0)(x) - (\Phi_1 p)(x) - H_1(x)\left(\frac{\varphi_0(x)}{\psi_0(x)}(\Phi_1\psi_0)(x) - (\Phi_1\varphi_0)(x)\right),$$

$$K_2(x) = \frac{(\Phi_2 P)(x)}{(\Phi_2\psi_0)(x)}(\Phi_1\psi_0)(x) - (\Phi_1 P)(x) - H_2(x)\left(\frac{(\Phi_2\varphi_0)(x)}{(\Phi_2\psi_0)(x)}(\Phi_1\psi_0)(x) - (\Phi_1\varphi_0)(x)\right).$$

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Remark 2. If $g \in C^2(I)$, we can simplify the functions H_i, K_i further by expressing them in terms of the generator $\mathcal{A}$. Defining

$$F_2(x) = \frac{\varphi_0(x)(\Phi_2 P)(x) - P(x)(\Phi_2\varphi_0)(x)}{\varphi_0(x)(\Phi_2\psi_0)(x) - \psi_0(x)(\Phi_2\varphi_0)(x)}$$

we have the relations

$$P(x) = H_2(x)\varphi_0(x) + F_2(x)\psi_0(x),$$
$$(\Phi_2 P)(x) = H_2(x)(\Phi_2\varphi_0)(x) + F_2(x), (\Phi_2\psi_0)(x),$$

and, thus,

$$K_2(x) = \frac{P(x)}{\psi_0(x)}(\Phi_1\psi_0)(x) - (\Phi_1 P)(x) - H_2(x)\left(\frac{\varphi_0(x)}{\psi_0(x)}(\Phi_1\psi_0)(x) - (\Phi_1\varphi_0)(x)\right).$$

Now we can use (5) and (6) to obtain

$$H_1(x) = \frac{1}{B_0}\left(-\frac{\psi_0(x)}{\psi_2(x)}(\Psi_2(r-\mathcal{A})p)(x) + (\Psi_0(r-\mathcal{A})p)(x)\right),$$
$$H_2(x) = \frac{1}{B_0}\left(\frac{\psi_0(x)}{\varphi_2(x)}(\Phi_2(r-\mathcal{A})P)(x) + (\Psi_0(r-\mathcal{A})P)(x)\right),$$
$$K_1(x) = \frac{1}{\lambda_1}\left(\frac{\varphi_1(x)}{\psi_2(x)}(\Psi_2(r-\mathcal{A})p)(x) + (\Phi_1(r-\mathcal{A})p)(x)\right),$$
$$K_2(x) = \frac{1}{\lambda_1}\left(-\frac{\varphi_1(x)}{\varphi_2(x)}(\Phi_2(r-\mathcal{A})P)(x) + (\Phi_1(r-\mathcal{A})P)(x)\right).$$

Recalling that $p(x) = c(x) + \lambda_2(R_r c)(x)$, we see that in this case the monotonicity properties of H_i and K_i can be easily determined by studying the behaviour of the functions $(r-\mathcal{A})p(x) = (r+\lambda_2-\mathcal{A})c(x)$ and $(r-\mathcal{A})g(x)$.

We now prove some auxiliary results regarding the functions H_1, H_2, K_1 and K_2 to study the solvability of the equation pair (22).

Lemma 4. *The functions H_i and K_i have opposite signs of derivative everywhere for $i = 1, 2$ and the curves $H_1(x)$ and $H_2(x)$ intersect at least once in the interval $(y^{\lambda_1} \vee y^{\lambda_2,c}, y^{\lambda_2})$. We denote the smallest such intersection point by $\hat{y}$. Furthermore,*

(i) $H_1(x) > 0$, $H_1'(x) > 0$, $\lim_{x\downarrow\mathfrak{l}} H_1(x) = 0$;

(ii) $K_1(x) > 0$, $K_1'(x) < 0$, $\lim_{x\downarrow\mathfrak{l}} K_1(x) = \infty$;

(iii) *H_2 is positive and strictly increasing on $(y^{\lambda_2,c}, \mathfrak{r})$;*

(iv) *K_2 is positive and strictly decreasing on $(y^{\lambda_2,c}, \hat{y})$;*

(v) $K_2(\hat{y}) > K_1(\hat{y})$.

Proof. See Appendix A.

The reader is advised to refer to Figure 1 in order to discern how the functions H_i, K_i will look under our assumptions in a typical example.

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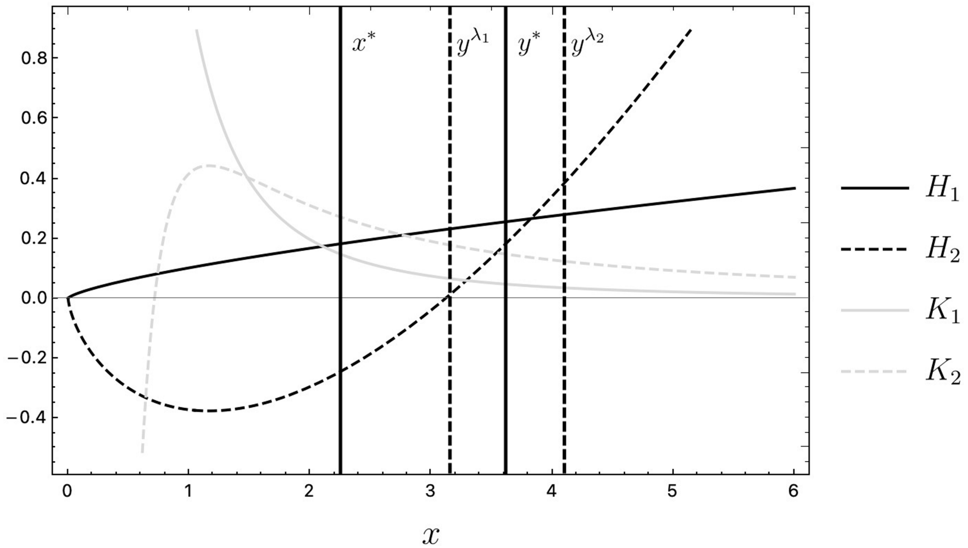

FIGURE 1. A prototype plot of the functions H_1, H_2, K_1, K_2 and the thresholds $x^*, y^*, y^{\lambda_1}, y^{\lambda_2}$.

We can now prove our main proposition about the existence and uniqueness of solutions to the pair of equations (22).

Proposition 1. *Let Assumption 1 hold and suppose that $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$. Then the pair (22) has a unique solution (x^*, y^*) such that $x^* \in (\mathfrak{l}, \hat{y})$, $y^* \in (y^{\lambda_2,c}, \hat{y})$, and $x^* < y^{\lambda_1} < y^*$.*

Proof. See Appendix A.

To close the section, we collect our findings on the solution candidate to the next proposition. The purpose of Proposition 2 is to show that there exist three uniquely determined functions that satisfy the necessary conditions described by the equation pair (22) and a DPP of the form (12). In Section 4 we prove that the three functions given by Proposition 2 coincide with the three value functions defined in (8) and (9).

Proposition 2. *Let Assumption 1 hold and suppose that $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$. Then the functions*

$$\begin{aligned}\mathcal{K}_0(x) &= \begin{cases} g(x), & x \geq y^*, \\ C_2\psi_0(x) + C_3\varphi_0(x) - c(x) - \lambda_2(R_0c)(x), & x^* \leq x < y^*, \\ C_1\psi_0(x), & x < x^*, \end{cases} \\ \mathcal{K}_1(x) &= \lambda_1(R_1\mathcal{K}_0)(x), \\ \mathcal{K}_2(x) &= \lambda_2(R_2\mathcal{K}_0)(x) - c(x),\end{aligned} \tag{23}$$

where the constants C_2, C_3 are given by (18), the constant C_1 is given by (19), and the thresholds x^* and y^* are given by Proposition 1, satisfy the DPP

$$\begin{aligned}\mathcal{K}_0(x) &= \max\{g(x), \mathcal{K}_1(x), \mathcal{K}_2(x)\}, \\ \mathcal{K}_1(x) &= \mathbb{E}_x\left[\mathrm{e}^{-rU_1}\mathcal{K}_0(X_{U_1})\right], \\ \mathcal{K}_2(x) &= \mathbb{E}_x\left[\mathrm{e}^{-rU_2}\mathcal{K}_0(X_{U_2})\right] - c(x),\end{aligned} \tag{24}$$

where independent random variables $U_a \sim \mathrm{Exp}(\lambda_a)$ for $a = 1, 2$. Moreover, the function $\mathcal{K}_0$ is continuous and the functions $\mathcal{K}_1$ and $\mathcal{K}_2$ are in $C^2(I)$ if and only if $\sigma^2 \in C^0(I)$.

Proof. See Appendix A.

4. Sufficient conditions

The purpose of this section is to prove the following theorem, which is our main result on the stopping problem (8).

Theorem 1. *Let Assumption 1 hold and suppose that $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$. Let (x^*, y^*) be the solution to (22) given by Proposition 1. Then the values (8) and (9) read as $V_0 = \mathcal{K}_0$, $V_1 = \mathcal{K}_1$ and $V_2 = \mathcal{K}_2$, where the functions $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$ are given in Proposition 2. The optimal policy can be described as follows.*

(a) *Assume that the agent is maintaining λ_1 attention If the process is in the region $[y^*, \mathfrak{r})$ at the next λ_1 arrival, stop. If the process is in the region $[x^*, y^*)$ at the next λ_1 arrival, switch to λ_2 attention. If the process is in the region $(\mathfrak{l}, x^*)$ at the next λ_1 arrival, maintain λ_1-attention.*

(b) *Assume that the agent is maintaining λ_2 attention. If the process is in the region $[y^*, \mathfrak{r})$ at the next λ_2 arrival, stop. If the process is in the region $[x^*, y^*)$ at the next λ_2 arrival, maintain λ_2 attention. If the process is in the region $(\mathfrak{l}, x^*)$ at the next λ_2 arrival, switch to λ_1 attention.*

Remark 3. Recall that $V_0^{\lambda_2}$ is the value of the Poisson stopping problem (7) with arrival rate λ_2 and the representation (10) holds under Assumption 1. The cost c is a nonnegative function so it follows from the first paragraph in the proof of Lemma 3 that (using the proof's notation)

$$\mathcal{K}_0(x) = V_0(x) \leq \sup_A \sup_{\tau \in \mathcal{S}_0^A} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right] = G_{0,0}(x) = V_0^{\lambda_2}(x).$$

We need the following result on uniform integrability to prove Theorem 1.

Lemma 5. *For a fixed attention sequence A, the process*

$$S^A = (S_n^A; \mathcal{G}_n^A)_{n \geq 0},$$
$$S_n^A = \mathrm{e}^{-rT_n^A} \mathcal{K}_0(X_{T_n^A}) - \sum_{n=0}^{\infty} \mathrm{e}^{-rT_n^A} c(X_{T_n^A}) \mathbf{1}(A_n = \lambda_2) \mathbf{1}(T_n^A < \tau)$$

is a uniformly integrable supermartingale.

Proof. Fix the attention sequence A and let $U_a \sim \mathrm{Exp}(\lambda_a)$, $a \in \{1, 2\}$. Then

$$\begin{aligned} \mathcal{K}_0(x) &\geq \max\{\mathcal{K}_1(x), \mathcal{K}_2(x)\} \\ &= \mathcal{K}_1(x)\mathbf{1}(A_0 = 1) + \mathcal{K}_2(x)\mathbf{1}(A_0 = 2) \\ &= \mathbb{E}_x \left[\mathrm{e}^{-rU_1} \mathcal{K}_0(X_{U_1}) \mathbf{1}(A_0 = 1) + \left(\mathrm{e}^{-rU_2} \mathcal{K}_0(X_{U_2}) - c(x) \right) \mathbf{1}(A_0 = 2) \right]. \end{aligned}$$

Hence, S^A has the claimed supermartingale property.

To prove uniform integrability, define the process

$$L^A = \left(L_n^A; \mathcal{G}_n^A \right)_{n \geq 0}, \qquad L_n^A = \mathrm{e}^{-rT_n^A} \frac{\psi_0(X_{T_n^A})}{\psi_0(x)}$$

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Since the boundary $\mathfrak{r}$ is natural, we find that

$$\mathbb{E}_x[L_1^A] = \frac{1}{\psi_0(x)}\mathbb{E}_x\left[\mathrm{e}^{-rU_2}\psi_0(X_{U_2})\mathbf{1}(A_0=2)+\mathrm{e}^{-rU_1}\psi_0(X_{U_1})\mathbf{1}(A_0=1)\right]=1.$$

This implies that $\mathbb{E}_x[L_n^A]=1$ for all n. Define now the measure

$$\mathbb{P}_{x,n}^A(B)=\mathbb{E}_x[L_n^A\mathbf{1}(B)] \quad \text{for } B\in\mathcal{F}.$$

By Remark 3 we find that

$$\begin{aligned}
0 &\le \frac{\mathbb{E}_x[S_n^A\mathbf{1}(B)]}{\psi_0(x)}\\
&\le \mathbb{E}_x\left[\frac{\mathcal{K}_0(X_{T_n^A})}{\psi_0(X_{T_n^A})}L_n^A\mathbf{1}(B)\right]\\
&\le \mathbb{E}_x\left[\frac{g(y^{\lambda_2})}{\psi_0(y^{\lambda_2})}L_n^A\mathbf{1}(B)\mathbf{1}(X_{T_n^A}\le y^{\lambda_2})\right]+\mathbb{E}_x\left[\frac{g(X_{T_n^A})}{\psi_0(X_{T_n^A})}L_n^A\mathbf{1}(B)\mathbf{1}(X_{T_n^A}\ge y^{\lambda_2})\right]\\
&\le \frac{g(y^\infty)}{\psi_0(y^\infty)}\mathbb{P}_{x,n}^A(B)
\end{aligned} \tag{25}$$

for all $B\in\mathcal{F}$. By taking $B=\Omega$, we find that $\sup_n\mathbb{E}_x[S_n^A]<\infty$. On the other hand, we observe that if $\mathbb{P}_x(B)\to 0$ then $\mathbb{P}_{x,n}^A(B)\to 0$ for all $n\ge 0$. Thus, inequality (25) implies that $\sup_n\mathbb{E}_x[S_n^A\mathbf{1}(B)]\to 0$ as $\mathbb{P}_x(B)\to 0$. This yields the claimed uniform integrability.

Proof of Theorem 1. Fix the attention sequence A and let $U_a\sim\mathrm{Exp}(\lambda_a)$, $a\in\{1,2\}$. Then optional sampling implies that

$$\begin{aligned}
\mathcal{K}_0(x) &\ge \mathbb{E}_x\left[\mathrm{e}^{-r\tau}\mathcal{K}_0(X_\tau)-\sum_{n=0}^\infty \mathrm{e}^{-rT_n^A}c(X_{T_n^A})\mathbf{1}(A_n=\lambda_2)\mathbf{1}(T_n^A<\tau)\right]\\
&\ge \mathbb{E}_x\left[\mathrm{e}^{-r\tau}g(X_\tau)-\sum_{n=0}^\infty \mathrm{e}^{-rT_n^A}c(X_{T_n^A})\mathbf{1}(A_n=\lambda_2)\mathbf{1}(T_n^A<\tau)\right] \quad \text{for all } \tau\in S_0^A.
\end{aligned}$$

This implies that $\mathcal{K}_0\ge V_0^A$ for all A and, consequently, $\mathcal{K}_0\ge V_0$.

To prove the opposite inequality, we consider the admissible stopping/switching rule that produced the candidate $\mathcal{K}_0$. Denote the associated (random) attention sequence as A^* and the stopping time as $\tau^*\in\mathcal{S}_0^{A^*}$. To prove the results, we show that the stopped process

$$\begin{aligned}
&Q=(Q_n;\mathcal{G}_n^{A^*})_{n\ge 0},\\
&Q_n=\mathrm{e}^{-r(\tau^*\wedge T_n)}\mathcal{K}_0(X_{\tau^*\wedge T_n})-\sum_{n=0}^\infty \mathrm{e}^{-rT_n^{A^*}}c(X_{T_n^{A^*}})\mathbf{1}(A_n^*=\lambda_2)\mathbf{1}(T_n^{A^*}<\tau^*\wedge T_n)
\end{aligned}$$

is a martingale. To this end, let $n\ge 1$. Then

$$\mathbb{E}_x\left[Q_n\mid\mathcal{G}_{n-1}^{A^*}\right]=\mathbb{E}_x\left[S_n^{A^*}\mathbf{1}(\tau^*\ge T_n^{A^*})\mid\mathcal{G}_{n-1}^{A^*}\right]+\sum_{k=0}^{n-1}S_k^{A^*}\mathbf{1}(\tau^*=T_k^{A^*}).$$

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Consider the first term on right-hand side: by the strong Markov property and the dynamic programming equation (24), we find that

$$\begin{aligned}
&\mathbb{E}_x\left[S_n^{A^*}\mathbf{1}(\tau^*\geq T_n^{A^*})\mid \mathcal{G}_{n-1}^{A^*}\right]\\
&=\Bigg\{\mathrm{e}^{-rT_{n-1}^{A^*}}\mathbb{E}_{X_{T_{n-1}^{A^*}}}\Big[\mathrm{e}^{-rU_1}\mathcal{K}_0(X_{U_1})\mathbf{1}(A_{n-1}^*=1)\\
&\qquad+\left(\mathrm{e}^{-rU_2}\mathcal{K}_0(X_{U_2})-c(X_{T_{n-1}^{A^*}})\right)\mathbf{1}(A_{n-1}^*=2)\Big]\\
&\qquad\qquad-\sum_{k=0}^{n-1}\mathrm{e}^{-rT_k^{A^*}}c(X_{T_k^{A^*}})\mathbf{1}(A_k^*=\lambda_2)\Bigg\}\mathbf{1}(\tau^*\geq T_n^{A^*})\\
&=\Bigg\{\mathrm{e}^{-rT_{n-1}^{A^*}}\left(\mathcal{K}_1(X_{T_{n-1}^{A^*}})\mathbf{1}(A_{n-1}^*=1)+\mathcal{K}_2(X_{T_{n-1}^{A^*}})\mathbf{1}(A_{n-1}^*=2)\right)\\
&\qquad-\sum_{k=0}^{n-1}\mathrm{e}^{-rT_k^{A^*}}c(X_{T_k^{A^*}})\mathbf{1}(A_k^*=\lambda_2)\Bigg\}\mathbf{1}(\tau^*\geq T_n^{A^*}).
\end{aligned}$$

Thus, we obtain

$$\mathbb{E}_x[Q_n|\mathcal{G}_{n-1}^{A^*}]=S_{n-1}^{A^*}\mathbf{1}(\tau^*\geq T_n^{A^*})+\sum_{k=0}^{n-1}S_k^{A^*}\mathbf{1}(\tau^*=T_k^{A^*})=Q_{n-1}$$

and optional sampling on Q with respect to τ^* gives

$$\begin{aligned}
\mathcal{K}_0(x)&=\mathbb{E}_x\left[\mathrm{e}^{-r\tau^*}\mathcal{K}_0(X_{\tau^*})-\sum_{n=0}^{\infty}\mathrm{e}^{-rT_n^{A^*}}c(X_{T_n^{A^*}})\mathbf{1}(A_n^*=\lambda_2)\mathbf{1}(T_n^{A^*}<\tau^*)\right]\\
&=\mathbb{E}_x\left[\mathrm{e}^{-r\tau^*}g(X_{\tau^*})-\sum_{n=0}^{\infty}\mathrm{e}^{-rT_n^{A^*}}c(X_{T_n^{A^*}})\mathbf{1}(A_n^*=\lambda_2)\mathbf{1}(T_n^{A^*}<\tau^*)\right].
\end{aligned}$$

On the other hand, A^* is an admissible attention sequence and $\tau^*\in\mathcal{S}_0^{A^*}$ so we have the inequality $\mathcal{K}_0\leq\sup_A V_0^A=V_0$, proving that $\mathcal{K}_0=V_0$.

Next we prove $\mathcal{K}_1=V_1$ and $\mathcal{K}_2=V_2$. Conditions (24) and optional sampling imply that, for a fixed A,

$$\begin{aligned}
\mathcal{K}_1(x)&=\lambda_1(R_1\mathcal{K}_0)(x)\\
&\geq\mathbb{E}_x\left[\mathrm{e}^{-r\tau}g(X_\tau)-\sum_{n=0}^{\infty}\mathrm{e}^{-rT_n^A}c(X_{T_n^A})\mathbf{1}(A_n=\lambda_2)\mathbf{1}(T_n^A<\tau)\right]\quad\text{for all }\tau\in\mathcal{S}_1^A.
\end{aligned}$$

Consequently $\mathcal{K}_1\geq V_1$. To prove the opposite inequality, let τ^* and A^* be as before. Recall that if τ and A are an admissible stopping time and an attention sequence for V_0, then $U_1+\tau\circ\theta_{U_1}$ and $(1,A)$ are admissible for V_1. Here θ denotes the usual shift operator that acts on the process X as $\theta_{U_1}(X_t)=X_{U_1+t}$. Let $\tau^1=U_1+\tau^*\circ\theta_{U_1}$ and $A^1=(1,A^*)$. Conditioning on the

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first arrival time and using the strong Markov property yields

$$\mathcal{K}_1(x) = \mathbb{E}_x\left[\mathrm{e}^{-rU_1}\mathbb{E}_{X_{U_1}}\left[\mathrm{e}^{-r\tau^*}g(X_{\tau^*}) - \sum_{n=0}^{\infty}\mathrm{e}^{-rT_n^{A^*}}c(X_{T_n^{A^*}})\mathbf{1}(A_n^* = \lambda_2)\mathbf{1}(T_n^{A^*} < \tau^*)\right]\right]$$
$$= \mathbb{E}_x\left[\mathrm{e}^{-r\tau^1}g(X_{\tau^1}) - \sum_{n=0}^{\infty}\mathrm{e}^{-rT_n^{A^1}}c(X_{T_n^{A^1}})\mathbf{1}(A_n^1 = \lambda_2)\mathbf{1}(T_n^{A^1} < \tau^1)\right],$$

and since the stopping time and attention sequence associated with $\mathcal{K}_1$ are admissible for V_1, we have $\mathcal{K}_1 \le \sup_A V_1^A = V_1$ and, hence, $\mathcal{K}_1 = V_1$. The proof of $\mathcal{K}_2 = V_2$ is completely analogous.

The following theorem presents an easily verifiable condition determining whether the main problem (8) reduces to the Poisson stopping problem (10). It also gives further clarification for the assumption $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$ made in Section 3. There the condition was shown to be connected to the order of the thresholds x^* and y^*. Here the proof of Theorem 2 indicates that $H_1(y^{\lambda_1}) \le H_2(y^{\lambda_1})$ if and only if waiting with the higher attention rate is suboptimal everywhere in the state space I.

Theorem 2. *The function $V_0 = V_0^{\lambda_1}$ if and only if $\lambda_2(R_2(V_0^{\lambda_1} + p))(y^{\lambda_1}) \le V_0^{\lambda_1}(y^{\lambda_1}) + p(y^{\lambda_1})$ and the inequality is equivalent to $H_1(y^{\lambda_1}) \le H_2(y^{\lambda_1})$, where H_i are given by (22).*

Proof. Let $V_0 = V_0^{\lambda_1}$. Waiting with λ_2 attention is always suboptimal so

$$\lambda_2(R_2(V_0^{\lambda_1} + p))(x) \le V_0^{\lambda_1}(x) + p(x) \quad \text{for all } x \in I.$$

Now suppose that the inequality holds for $x = y^{\lambda_1}$. Noting that $a_2(x)\varphi_2(x) = -b(x)\psi_2(x)$ for all $x \in I$, we get

$$\begin{aligned}
&\lambda_2(R_2(V_0^{\lambda_1} + p))(y^{\lambda_1}) \le V_0^{\lambda_1}(y^{\lambda_1}) + p(y^{\lambda_1}) \\
&\quad \Leftrightarrow \frac{\lambda_2}{B_2}\left(\varphi_2\left(\Psi_2\left(\frac{g(y^{\lambda_1})}{\psi_0(y^{\lambda_1})} + p\right)\right) + \psi_2(\Phi_2 P)\right)(y^{\lambda_1}) \le P(y^{\lambda_1}) \\
&\quad \Leftrightarrow (\varphi_2(\Psi_2 p) + \psi_2(\Phi_2 P))\,(y^{\lambda_1}) \le \left(\frac{B_2}{\lambda_2}P - \frac{g}{\psi_0}(\Psi_2\psi_0)\varphi_2\right)(y^{\lambda_1}) \\
&\quad \Leftrightarrow \left(\frac{p}{\psi_0}(\Psi_2\psi_r) - (\Psi_2 p)\right)(y^{\lambda_1})\varphi_2(y^{\lambda_1}) \ge \left((\Phi_2 P) - \frac{P}{\psi_0}(\Phi_2\psi_0)\right)(y^{\lambda_1})\psi_2(y^{\lambda_1}) \\
&\quad \Leftrightarrow H_1(y^{\lambda_1}) \le H_2(y^{\lambda_1}).
\end{aligned}$$

Now Lemma 3 implies that $V_0^{\lambda_1}$ satisfies the DPP

$$V_0^{\lambda_1} = \max\{g, \lambda_1(R_1 V_0^{\lambda_1}), \lambda_2(R_2(V_0^{\lambda_1} + p)) - p\}.$$

We can employ the same arguments as in the proof of Lemma 5 and Theorem 1 to show that $V_0^{\lambda_1} \ge V_0^A$ for any fixed attention sequence A and, consequently, that $V_0^{\lambda_1} \ge V_0$. But the optimal stopping rule corresponding to $V_0^{\lambda_1}$ is admissible for V_0 as well, so $V_0^{\lambda_1} \le V_0$ and, thus, $V_0^{\lambda_1} = V_0$.

5. A note on asymptotics

In this section we briefly discuss various asymptotic properties of the stopping problem (8). We recall the proof of Lemma 3, where it was stated that if c is the cost function of the main problem (8) and $\alpha \geq 0$, then αc is the cost function of another similar problem. It was also proven that in the vanishing period cost limit, i.e. when $\alpha = 0$ (or c is identically zero), we have $V_0 = V_0^{\lambda_2}$ meaning that the problem (8) reduces to the usual Poisson stopping problem (7) with attention rate λ_2. On the other hand, if c is not identically zero then there exists a critical value α_c for the scaling coefficient α, such that the problem (8) reduces to the Poisson stopping problem (7) with attention rate λ_1. A natural interpretation for these reduction results is that if choosing the attention rate λ_2 results in a nonzero period cost, then it is optimal to choose the smaller but free attention rate λ_1 far away from the stopping region. In this case removing the period cost outweighs the opportunity cost that results from choosing the smaller attention rate. On the other hand, if the period cost is sufficiently large, the cost of choosing the higher attention rate λ_2 will outweigh its benefits.

It is also interesting to study the asymptotic properties of the problem with respect to the attention rates. Recalling the analysis of Sections 3 and 4, we have by monotone convergence $\lim_{\lambda_1 \downarrow 0} \mathcal{K}_1 = 0$ and $\lim_{\lambda_2 \downarrow 0} \mathcal{K}_2 = -c$. Thus, $\lim_{\lambda_1 \downarrow 0} V_0 = V_0^{\lambda_2, c}$ and $\lim_{\lambda_2 \downarrow 0} V_0 = V_0^{\lambda_1}$, where $V_0^{\lambda_2, c} = \max\{g, \mathcal{K}_2\}$. Here $V_0^{\lambda_2, c}$ may also be shown to be the value function of a Poisson stopping problem with an attention rate λ_2 and a cost function c by utilizing the verification arguments of Section 4. We also have $\lim_{\lambda_1 \uparrow \lambda_2} V_0 = V_0^{\lambda_2}$ since at the limit the agent can choose to stop or to continue with either a free or a costly attention rate λ_2, the last option being always suboptimal.

In order to determine the limit $\lim_{\lambda_1 \uparrow \infty} V_0$ we have to remove the assumption $\lambda_1 < \lambda_2$. Let $\tilde{V}$ be the value function of the optimal stopping problem in continuous time, i.e.

$$\tilde{V}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathrm{e}^{-r\tau} g(X_\tau) 1_{\{\tau < \infty\}} \right], \tag{26}$$

where $x \in I$ and $\mathcal{T}$ is the set of stopping times of X. Here, $V_0 \geq \max\{V_0^{\lambda_1}, V_0^{\lambda_2, c}\}$ and Assumption 1, Remark 1 and Proposition 2.6 in [23] imply that $\lim_{\lambda_1 \uparrow \infty} V_0^{\lambda_1} = \tilde{V}$ and $V_0^{\lambda_2, c} \leq V_0^{\lambda_2} \leq \tilde{V}$, so $\lim_{\lambda_1 \uparrow \infty} V_0 \geq \tilde{V}$. But $\tilde{V}$ is an r-excessive majorant of g, so $V_0 \leq \tilde{V}$ for all positive λ_1. Thus, $\lim_{\lambda_1 \uparrow \infty} V_0 = \tilde{V}$. The result is intuitively clear, since letting $\lambda_1 \uparrow \infty$ allows the agent to get rid of the period cost and the Poisson constraint at the same time.

Determining the last limit $\lim_{\lambda_2 \uparrow \infty} V_0$ is not so straightforward. At the limit, an agent may either choose to stop, continue with free Poisson arrivals having an attention rate λ_1, or continue in continuous time while paying a running cost c. This is a completely different type of problem since the optimal stopping strategy may be a mixture of Poisson and continuous-time stopping strategies. As such, the problem lies beyond the scope of the present paper and is therefore left for future research.

6. Illustration

We illustrate the general results obtained in the previous sections with two examples. In the first example the underlying is taken to be a GBM and the period cost is assumed to be proportional to the state of the process. In the second example we study a logistic diffusion process with a fixed period cost.

6.1. Geometric Brownian motion and proportional period cost

In this example we assume that the underlying is a GBM, the payoff is $g(x) = x^\theta - \eta$, where $0 < \theta < 1 < \eta$, and the period cost is $c(x) = kx$ where $k > 0$. The infinitesimal generator of the diffusion reads as

$$\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx},$$

where $\mu \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+$ are given constants. we make the additional assumption that $r > \mu$. Furthermore, the scale density and the density of the speed measure read as

$$S'(x) = x^{-2\mu/\sigma^2}, \qquad m'(x) = \frac{2}{\sigma^2} x^{(2\mu/\sigma^2)-2}.$$

Denote

$$\beta_i = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda_i)}{\sigma^2}},$$
$$\alpha_i = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda_i)}{\sigma^2}}.$$

Then the minimal r-excessive functions for X read as

$$\psi_i(x) = x^{\beta_i}, \qquad \varphi_i(x) = x^{\alpha_i}.$$

It is worth emphasizing that now $\beta_i > 1 > \theta > 0 > \alpha_i$ and it is a straightforward exercise to show that the conditions of Assumption 1 are satisfied.

Since the payoff is twice continuously differentiable everywhere, we can write the pair of equations as in Remark 2. The auxiliary functionals

$$(\Phi_i(r+\lambda_2-\mathcal{A})c)(x) = \frac{2k(r+\lambda_2-\mu)}{\sigma^2(\beta_i-1)} x^{1-\beta_i},$$
$$(\Psi_i(r+\lambda_2-\mathcal{A})c)(x) = \frac{2k(r+\lambda_2-\mu)}{\sigma^2(1-\alpha_i)} x^{1-\alpha_i},$$
$$(\Phi_i(r-\mathcal{A})g)(x) = \frac{2}{\sigma^2}\left(\frac{r-\mu\theta-\frac{\sigma^2}{2}\theta(\theta-1)}{\beta_i-\theta} x^{\theta-\beta_i} - \frac{r\eta}{\beta_i-1} x^{-\beta_i}\right),$$
$$(\Psi_i(r-\mathcal{A})g)(x) = \frac{2}{\sigma^2}\left(\frac{r-\mu\theta-\frac{\sigma^2}{2}\theta(\theta-1)}{\theta-\alpha} x^{\theta-\alpha_i} - \frac{r\eta}{1-\alpha_i} x^{-\alpha_i}\right)$$

can be used to write an explicit form for the equation pair.

Next we investigate the condition $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$ that guarantees that our solution does not reduce to a Poisson stopping problem with rate λ_1. We know from Theorem 2 that this condition is equivalent to $\lambda_2(R_2(V_0^{\lambda_1} + p)(y^{\lambda_1}) \geq g(y^{\lambda_1}) + p(y^{\lambda_1})$. Expanding the resolvent with the help of (2) and using the resolvent identity (4) on the period cost terms, we see that the stated condition is equivalent to

$$\frac{\lambda_2}{B_2}\psi_0(y^{\lambda_1})\left(\Phi_2\left(g - \frac{g(y^{\lambda_1})}{\psi_0(y^{\lambda_1})}\psi_0\right)\right)(y^{\lambda_1}) \geq c(y^{\lambda_1}),$$

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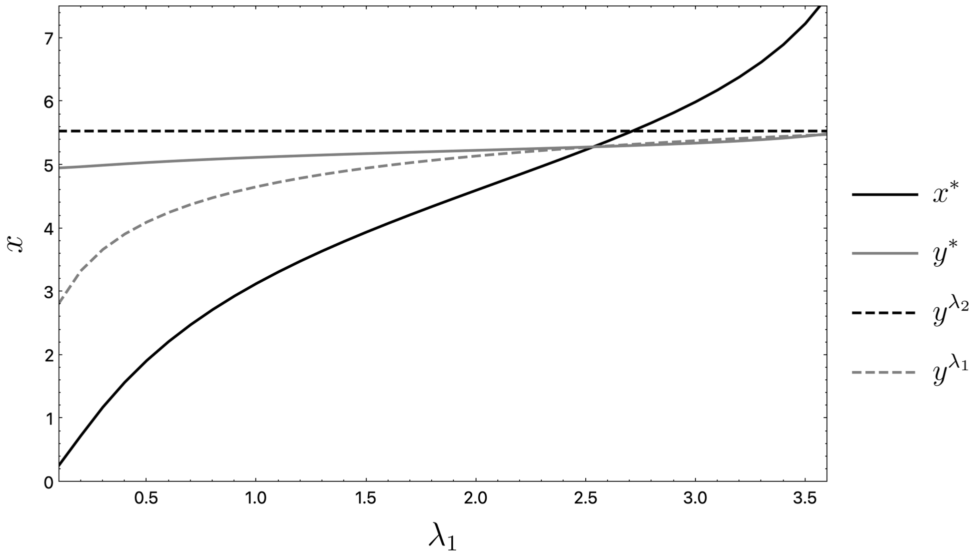

FIGURE 2. Optimal thresholds as a function of attention rate λ_1. The other parameters are chosen to be $\mu = 0.02$, $\sigma = \sqrt{0.3}$, $r = 0.05$, $k = 0.001$, $\lambda_2 = 4.0$, $\theta = 0.67$, $\eta = 1.5$.

where y^{λ_1} is the unique root of the equation $\psi_0(y^{\lambda_1})(\Phi_1 g)(y^{\lambda_1}) = g(y^{\lambda_1})(\Phi_1\psi_0)(y^{\lambda_1})$ and it can be calculated to be

$$y^{\lambda_1} = \left(\frac{\beta_0(\beta_1 - \theta)}{\beta_1(\beta_0 - \theta)}\eta\right)^{1/\theta}.$$

If the proportional period cost parameter k exceeds the critical value

$$\begin{aligned} k^* &= \frac{\lambda_2}{B_2 y^{\lambda_1}}\psi_0(y^{\lambda_1})\left(\Phi_2\left(g - \frac{g(y^{\lambda_1})}{\psi_0(y^{\lambda_1})}\psi_0\right)\right)(y^{\lambda_1}) \\ &= \frac{2\lambda_2\eta\beta_0(\beta_2 - \beta_1)}{B_2\sigma^2 y^{\lambda_1}\beta_1\beta_2(\beta_2 - \beta_0)(\beta_2 - \theta)}, \end{aligned}$$

then $V_0 = V_0^{\lambda_1}$ and the problem reduces to the standard Poisson stopping problem. The value k^* has a natural interpretation in terms of the cost parameters η and k. Since $\theta \in (0, 1)$, $\eta > 1$ and k^* is always positive when $\lambda_1 < \lambda_2$, we find that increasing η or k enough will reduce the problem to Poisson stopping when other parameters are fixed. Indeed, increasing either cost parameter results in a smaller realized payoff when the process is stopped.

Regarding the attention rates, we find that if $\lambda_1 = \lambda_2$ then $k^* = 0$ and the problem reduces to the standard Poisson stopping problem. Likewise, because y^{λ_1} is increasing in λ_1, we see that increasing λ_1 will decrease k^* and make the agent less likely to swap to λ_2. This is again in line with the intuition that it is never optimal to swap to λ_2 and pay the cost $c(x)$ if the rate λ_1 is high enough. These observations are further illustrated in Figures 2 and 3. In Figure 2 we see that there exists a critical threshold $\hat{\lambda}_1$ for the attention rate λ_1, where eventually the optimal thresholds x^*, y^* and y_1^{λ} are the same (in this case about 2.6). When $\lambda_1 < \hat{\lambda}_1$, the optimal strategy is to switch to higher attention rate, but when $\lambda_1 > \hat{\lambda}_1$, switching is always suboptimal. In Figure 3 we see a rather similar situation. In this case if λ_2 is below the critical threshold $\hat{\lambda}_2$ (about 1.15 in the figure), it is suboptimal to switch, but when λ_2 is above the threshold, we see that switching is optimal.

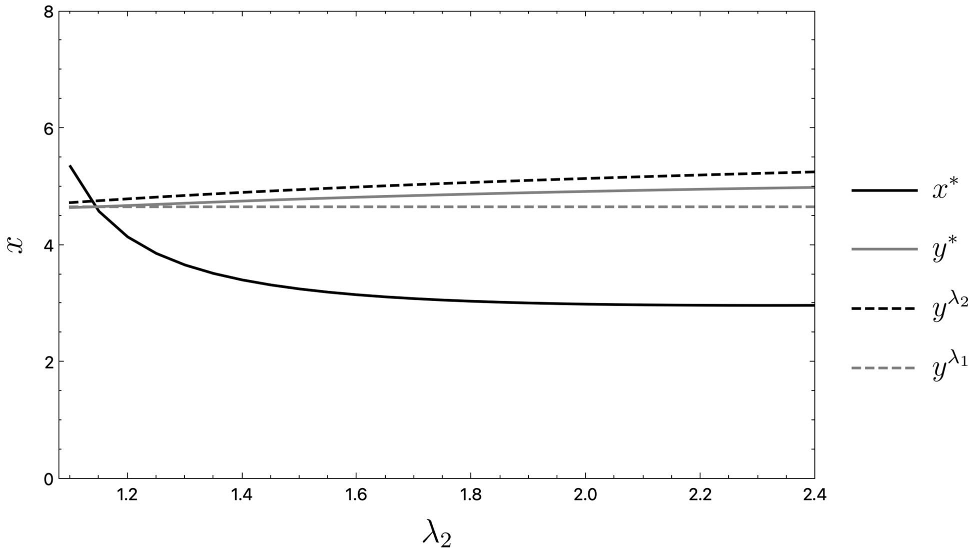

FIGURE 3. Optimal thresholds as a function of attention rate λ_2. The other parameters are chosen to be $\mu = 0.02$, $\sigma = \sqrt{0.3}$, $r = 0.05$, $k = 0.001$, $\lambda_1 = 1.0$, $\theta = 0.67$, $\eta = 1.5$.

The effects of r, μ and σ are not as direct and straightforward to see, but differentiations show that their effects are not linear, and instead usually for high and low values of these parameters the problem tends to reduce to Poisson stopping with attention rate λ_1.

6.2. Logistic diffusion

Let the underlying follow a standard logistic diffusion driven by the stochastic differential equation

$$dX_t = \mu X_t(1 - \gamma X_t)dt + \sigma X_t dW_t, \qquad X_0 = x \in \mathbb{R}_+,$$

where $\mu, \gamma, \sigma > 0$. The scale density and the density of the speed measure read as

$$S'(x) = x^{-2\mu/\sigma^2} e^{2\mu\gamma x/\sigma^2}, \qquad m'(x) = \frac{2}{\sigma^2} x^{(2\mu/\sigma^2)-2} e^{-2\mu\gamma x/\sigma^2}.$$

The minimal r-excessive functions are

$$\varphi_r(x) = x^\alpha U\Big(\alpha, 1 + \alpha - \beta, \frac{2\mu\gamma}{\sigma^2} x\Big), \qquad \psi_r(x) = x^\alpha M\Big(\alpha, 1 + \alpha - \beta, \frac{2\mu\gamma}{\sigma^2} x\Big),$$

where M and U denote, respectively, the confluent hypergeometric functions of first and second kind and

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}},$$
$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

TABLE 1. Optimal thresholds with various attention rates. The other parameters are chosen to be $\mu = 0.01$, $\sigma = \sqrt{0.1}$, $r = 0.05$, $\gamma = 0.5$, $k = 0.01$ and $\eta = 2.0$.

	$\lambda_1 = 0.05$	$\lambda_1 = 0.1$	$\lambda_1 = 0.5$
$\lambda_2 = 1.0$	$x^* = 0.87,\ y^* = 3.59$	$x^* = 1.01,\ y^* = 3.60$	$x^* = 1.81,\ y^* = 3.65$
$\lambda_2 = 2.0$	$x^* = 0.95,\ y^* = 3.62$	$x^* = 1.10,\ y^* = 3.65$	$x^* = 1.78,\ y^* = 3.75$

TABLE 2. Optimal thresholds with various costs for higher attention rate. The other parameters are chosen to be $\mu = 0.01$, $\sigma = \sqrt{0.1}$, $r = 0.05$, $\gamma = 0.5$, $\lambda_1 = 0.1$, $\lambda_2 = 1.0$ and $\eta = 2.0$.

$k = 0.001$	$k = 0.01$	$k = 0.1$
$x^* = 0.88,\ y^* = 3.80$	$x^* = 1.01,\ y^* = 3.60$	$x^* = 2.21,\ y^* = 2.67$

We choose the payoff to be linear $g(x) = x - \eta,\ \eta \in \mathbb{R}^+$ and the cost to be constant $c(x) = k \in \mathbb{R}^+$.

Unfortunately, due to the complicated nature of the minimal r-excessive functions in this example, the integral functionals such as $(\Psi_{r+\lambda_l}\psi_0)(x)$ and $(\Phi_{r+\lambda_l}P)(x)$ cannot be calculated explicitly. Consequently, the pair of equations for the optimal thresholds cannot be simplified from their original integral forms in any meaningful way, and are thus left unstated. Hence, we demonstrate the results numerically.

In Table 1 we demonstrate how the optimal thresholds behave when the attention rates λ_1 and λ_2 are changed. We observe that increasing the rate λ_2 increases the stopping threshold y^*, but interestingly the switching threshold x^* can increase or decrease. In Table 2 the effect of increasing the constant cost of higher attention rate is studied. As is intuitively clear, increasing the cost postpones the switching decision by increasing threshold x^*.

Appendix A. Proofs of technical results

A.1. Proof of Lemma 1

(i) We present a proof for the first condition, the second one is completely analogous. Using (5) and (6) yields

$$\frac{1}{B_r}\big((\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\varphi}_r)(x)(\boldsymbol{\Psi}_r(r-\mathcal{A})p)(x) + (\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\psi}_r)(x)(\boldsymbol{\Phi}_r(r-\mathcal{A})p)(x)\big)$$
$$= \frac{1}{\lambda B_r S'(x)^2}(\boldsymbol{\psi}'_r(x)\boldsymbol{\varphi}_r(x) - \boldsymbol{\psi}_r(x)\boldsymbol{\varphi}'_r(x))(\boldsymbol{\psi}'_{r+\lambda}(x)p(x) - \boldsymbol{\psi}_{r+\lambda}(x)p'(x))$$
$$= (\boldsymbol{\Psi}_{r+\lambda}p)(x) + \lambda^{-1}(\boldsymbol{\Psi}_{r+\lambda}(r-\mathcal{A})p)(x).$$

(ii) A straightforward calculation yields

$$\begin{vmatrix} p & \boldsymbol{\psi}_r & \boldsymbol{\varphi}_r \\ (\boldsymbol{\Phi}_{r+\lambda}p) & (\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\psi}_r) & (\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\varphi}_r) \\ (\boldsymbol{\Psi}_{r+\lambda}p) & (\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\psi}_r) & (\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\varphi}_r) \end{vmatrix}(x)$$
$$= \frac{p(x)}{\lambda^2 S'(x)^2}\left((\boldsymbol{\psi}'_r\boldsymbol{\varphi}_r - \boldsymbol{\psi}_r\boldsymbol{\varphi}'_r)(\boldsymbol{\psi}'_{r+\lambda}\boldsymbol{\varphi}_{r+\lambda} - \boldsymbol{\psi}_{r+\lambda}\boldsymbol{\varphi}'_{r+\lambda})\right)(x)$$

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$$
\begin{aligned}
&- \left(\left(\frac{(\boldsymbol{\Phi}_{r+\lambda} p)}{\lambda S'} \boldsymbol{\psi}_{r+\lambda} + \frac{(\boldsymbol{\Psi}_{r+\lambda} p)}{\lambda S'} \boldsymbol{\psi}_{r+\lambda} \right) (\boldsymbol{\psi}'_r \boldsymbol{\varphi}_r - \boldsymbol{\psi}_r \boldsymbol{\varphi}'_r) \right) (x) \\
&= \frac{B_r B_{r+\lambda}}{\lambda^2} \left(p(x) - \lambda (\mathbf{R}_{r+\lambda} p)(x) \right) \\
&= \frac{B_r B_{r+\lambda}}{\lambda^2} \left(c(x) + \lambda_2 (\mathbf{R}_r c)(x) - \lambda (\mathbf{R}_{r+\lambda} c)(x) \right) \\
&- \frac{B_r B_{r+\lambda}}{\lambda^2} \lambda \lambda_2 (\mathbf{R}_{r+\lambda} \mathbf{R}_r c)(x) \\
&= \frac{B_r B_{r+\lambda}}{\lambda^2} \left(c(x) + (\lambda_2 - \lambda)(\mathbf{R}_r c)(x) \right),
\end{aligned}
$$

where the first equality follows by using (5) and (6), and the second equality is obtained by recalling the representations (2) and (3) for the resolvent and the Wronskian. The third equality follows from the definition of p and the last step is achieved by applying the resolvent equation (4). In particular, we may use (5) and (6) on $(\boldsymbol{\Psi}_{r+\lambda}\boldsymbol{\psi}_r)$ and $(\boldsymbol{\Phi}_{r+\lambda}\boldsymbol{\varphi}_r)$ because

$$
\lim_{x \downarrow \mathfrak{l}} \frac{\boldsymbol{\varphi}_r(x)}{\boldsymbol{\varphi}_{r+\lambda}(x)} = \lim_{x \uparrow \mathfrak{r}} \frac{\boldsymbol{\psi}_r(x)}{\boldsymbol{\psi}_{r+\lambda}(x)} = 0
$$

by the proof of Lemma 2.1 in [23]. The said proof contains the result $\lim_{x\uparrow\mathfrak{r}} \boldsymbol{\psi}_r(x)/\boldsymbol{\psi}_{r+\lambda}(x) = 0$ for the choice $\mathfrak{r} = \infty$, but proving the condition for some other $\mathfrak{r} \in \mathbb{R}$ or the condition $\lim_{x\downarrow\mathfrak{l}} \boldsymbol{\varphi}_r(x)/\boldsymbol{\varphi}_{r+\lambda}(x) = 0$ is completely analogous.

A.2. Proof of Lemma 3

We introduce a nonnegative scaling coefficient on the cost function in order to prove the result. Let $\alpha \geq 0$. Since $c \in L^1_r(I)$, we also have that $\alpha c \in L^1_r(I)$. Let

$$
\begin{aligned}
G_{2,\alpha}(x) &= \mathbb{E}_x \left[\mathrm{e}^{-rU^{\lambda_2}} G_{0,\alpha}(X_{U^{\lambda_2}}) \right] - \alpha c(x), \\
G_{0,\alpha}(x) &= \max\{g(x), G_1(x), G_{2,\alpha}(x)\},
\end{aligned}
$$

and let x^*_α, y^*_α be the corresponding thresholds. Here $G_{0,\alpha}$ is continuous with respect to α so x^*_α and y^*_α are continuous functions of α as well. We naturally have

$$
G_{0,0}(x) = \max\{\max\{g(x), \lambda_1(R_1 G_{0,0})(x)\}, \max\{g(x), \lambda_2(R_2 G_{0,0})(x)\}\} \quad \text{for all } x \in I.
$$

Under Assumption 1, the Poisson stopping problem (10) satisfies the DPP $V_0^{\lambda_1} = \max\{g, \lambda_1(R_1 G_{0,0})\}$ (see Remark 1 and the proof of [23, Theorem 1.1]). Thus, $G_{0,0} = \max\{V_0^{\lambda_1}, V_0^{\lambda_2}\}$. On the other hand, we assumed that $\lambda_1 < \lambda_2$ so that $g(y^{\lambda_1})/\psi_0(y^{\lambda_1}) < g(y^{\lambda_2})/\psi_0(y^{\lambda_2})$ and, consequently, $V_0^{\lambda_1} \leq V_0^{\lambda_2}$. Thus, $G_{0,0} = V_0^{\lambda_2}$, meaning that $\lim_{\alpha\downarrow 0} x^*_\alpha = l_g$ and $y^*_0 = y^{\lambda_2}$.

Next we show that there exists a critical value $\alpha_c > 0$ for the scale such that $G_{0,\alpha} = V_0^{\lambda_1}$ for $\alpha \geq \alpha_c$ and $x^*_\alpha < y^{\lambda_1} < y^*_\alpha$ for $\alpha < \alpha_c$. To see this, note that, for all $x \in I$, $G_{2,\alpha}(x)$ is strictly decreasing with respect to α with $\lim_{\alpha\uparrow\infty} G_{2,\alpha}(x) = -\infty$. Consequently, x^*_α is strictly increasing and y^*_α is strictly decreasing with respect to α and $\lim_{\alpha\uparrow\infty} G_{0,\alpha} = V_0^{\lambda_1}$.

Consider now a critically scaled (i.e. the cost function is $\alpha_c c$) version of the pair (20). We know that $x^*_{\alpha_c} = y^{\lambda_1} = y^*_{\alpha_c}$ and it is an easy exercise to show that $H_2(y^{\lambda_1}) = f(y^{\lambda_1}, y^{\lambda_1})$. The

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remaining equation $H_1(y^{\lambda_1}) = H_2(y^{\lambda_1})$ yields an expression for the critical scale,

$$\alpha_c = \frac{\dfrac{g(y^{\lambda_1})(\Phi_2\psi_0)(y^{\lambda_1}) - \psi_0(y^{\lambda_1})(\Phi_2 g)(y^{\lambda_1})}{\varphi_0(y^{\lambda_1})(\Phi_2\psi_0)(y^{\lambda_1}) - \psi_0(y^{\lambda_1})(\Phi_2\varphi_0)(y^{\lambda_1})}}{H_1(y^{\lambda_1}) - \dfrac{p(y^{\lambda_1})(\Phi_2\psi_0)(y^{\lambda_1}) - \psi_0(y^{\lambda_1})(\Phi_2 p)(y^{\lambda_1})}{\varphi_0(y^{\lambda_1})(\Phi_2\psi_0)(y^{\lambda_1}) - \psi_0(y^{\lambda_1})(\Phi_2\varphi_0)(y^{\lambda_1})}},$$

from which we can conclude that

$$\alpha_c \gtrless 1 \Leftrightarrow H_1(y^{\lambda_1}) \gtrless H_2(y^{\lambda_1}).$$

The result follows since $G_0 = G_{0,1}$, $x^* = x_1^*$ and $y^* = y_1^*$.

A.3. Proof of Lemma 4

We may write the condition (21) as $H_1(y^{\lambda_1}) > H_2(y^{\lambda_1})$ and if $y^{\lambda_1} < y^{\lambda_2,c}$ then $H_2(y^{\lambda_2,c}) = 0 < H_1(y^{\lambda_2,c})$. On the other hand, a direct calculation shows that $H_1(y^{\lambda_2}) - H_2(y^{\lambda_2}) < 0$ if and only if

$$\begin{vmatrix} p & \psi_0 & \varphi_0 \\ (\Phi_2 p) & (\Phi_2\psi_0) & (\Phi_2\varphi_0) \\ (\Psi_2 p) & (\Psi_2\psi_0) & (\Psi_2\varphi_0) \end{vmatrix} (y^{\lambda_2}) > 0, \tag{27}$$

which follows from Lemma 1(ii). Recalling that $y^{\lambda_2,c} \le y^{\lambda_2}$, we see that H_i $(i = 1, 2)$ have opposite order at $y^{\lambda_1} \vee y^{\lambda_2,c}$ and y^{λ_2}. Consequently, by continuity the curves $H_1(x)$ and $H_2(x)$ must intersect at least once in the interval $(y^{\lambda_1} \vee y^{\lambda_2,c}, y^{\lambda_2})$. Next we prove parts (i)–(v) of Lemma 4.

Part (i): H_1 is positive everywhere in I since p/ψ_0 and φ_0/ψ_0 are strictly decreasing. Standard differentiation and conditions $\lambda_2(R_2\psi_0)(x) = \psi_0(x)$ and $\lambda_2(R_2\varphi_0)(x) = \varphi_0(x)$ imply that the inequality $H_1'(x) > 0$ follows by the first equation of Lemma 1(i). At the lower boundary of I we get

$$\lim_{x\downarrow\mathfrak{l}} H_1(x) = \lim_{x\downarrow\mathfrak{l}} \frac{(p/\psi_0)'(x)}{(\varphi_0/\psi_0)'(x)} = \lim_{x\downarrow\mathfrak{l}} B_0^{-1}(\Psi_0(r + \lambda_2 - \mathcal{A})c)(x) = 0,$$

where the last equality follows since c, $(r - \mathcal{A})c \in L_r^1(I)$ by Assumption 1.

Part (ii): conditions (5), (6) and Lemma 1 imply that the inequality $K_1(x) > 0$ is equivalent to

$$\frac{c(x)}{\psi_2(x)}(\Psi_2\varphi_1)(x) + (\Phi_1 c)(x) > 0,$$

which holds by the nonnegativity of c. We also have

$$K_1'(x) = -\left(\frac{(\Psi_2\varphi_0)(x)}{(\Psi_2\psi_0)(x)}(\Phi_1\psi_0)(x) - (\Phi_1\varphi_0)(x)\right) H_1'(x),$$

so that $K_1'(x) < 0$ follows from part (i). Moreover, defining a function $F_1 : I \to \mathbb{R}_+$ by

$$F_1(x) = \frac{\varphi_0(x)(\Psi_2 p)(x) - p(x)(\Psi_2\varphi_0)(x)}{\varphi_0(x)(\Psi_2\psi_0)(x) - \psi_0(x)(\Psi_2\varphi_0)(x)},$$

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we can write K_1 as

$$K_1(x) = H_1(x)(\Phi_1\varphi_0)(x) + F_1(x)(\Phi_1\psi_0)(x) - (\Phi_1 p)(x).$$

Consequently, the limit of K_1 at the lower boundary of the state space can be seen to be

$$\begin{aligned}\lim_{x\downarrow\mathfrak{l}} K_1(x) &= \lim_{x\downarrow\mathfrak{l}} ((\Phi_1\varphi_0)(x)B_0^{-1}(\Psi_0(r-\mathcal{A})p)(x) + (\Phi_1\psi_0)(x)B_0^{-1}(\Phi_0(r-\mathcal{A})p)(x)\\ &\quad - (\Phi_1(r-\mathcal{A})p)(x))\\ &= \lim_{x\downarrow\mathfrak{l}} \frac{1}{\lambda_1 B_0}(\Phi_1(r-\mathcal{A})p)(x)\\ &= \lim_{x\downarrow\mathfrak{l}} \frac{1}{\lambda_1 B_0}\left(\frac{c'(x)\varphi_1(x) - c(x)\varphi_1'(x)}{S'(x)} + (\lambda_2-\lambda_1)(\Phi_1 c)(x)\right)\\ &= \infty.\end{aligned}$$

In the above calculation, the second equality follows from Lemma 1 and the last equality is true because c is nondecreasing, $\mathfrak{l}$ is a natural boundary (which implies that $\lim_{x\downarrow\mathfrak{l}} \varphi_1'(x)/S'(x) = \infty$), and $\lambda_1 < \lambda_2$.

Part (iii): by Assumption 1 we know that $H_2 > 0$ at least on the interval $(y^{\lambda_2,c}, \mathfrak{r})$. Assumption 1(iii) implies that H_2 is strictly increasing on this interval, because H_2 can be written as

$$H_2(x) = \frac{(\Phi_2\psi_0)(x)(\Psi_2 P)(x) - (\Phi_2 P)(x)(\Psi_2\psi_0)(x) + (\Phi_2\psi_0)(x)((P(x) - \lambda_2(R_2P)(x))/\varphi_2(x))}{(\Phi_2\psi_0)(x)(\Psi_2\varphi_0)(x) - (\Phi_2\varphi_0)(x)(\Psi_2\psi_0)(x)},$$

and thus,

$$\begin{aligned}&\frac{d}{dx}((\Phi_2\psi_0)(x)(\Psi_2\varphi_0)(x) - (\Phi_2\varphi_0)(x)(\Psi_2\psi_0)(x)) = 0,\\ &\frac{d}{dx}((\Phi_2\psi_0)(x)(\Psi_2 P)(x) - (\Phi_2 P)(x)(\Psi_2\psi_0)(x))\\ &\qquad = -\frac{P(x) - \lambda_2(R_2P)(x)}{\varphi_2(x)}\frac{d}{dx}(\Phi_2\psi_0)(x),\end{aligned}$$

so that the monotonicity of H_2 is determined by the monotonicity of $(P - \lambda_2(R_2P))/\varphi_2$. Using the resolvent identity (4) on p gives the form used in Assumption 1(iii).

Parts (iv) and (v): we have

$$\lim_{\varepsilon\downarrow 0}\frac{K_2(x+\varepsilon) - K_2(x)}{\varepsilon} = -\left(\frac{(\Phi_2\varphi_0)(x)}{(\Phi_2\psi_0)(x)}(\Phi_1\psi_0)(x) - (\Phi_1\varphi_0)(x)\right)\lim_{\varepsilon\downarrow 0}\frac{H_2(x+\varepsilon) - H_2(x)}{\varepsilon}$$

and similarly for $\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(K_2(x) - K_2(x-\varepsilon))$. The monotonicity of K_2 follows by part (iii).

Finally, we may prove part (v) and the positivity of K_2 on $(y^{\lambda_2,c}, \hat{y})$ simultaneously with the help of parts (ii) and (iv). Indeed, $\hat{y} \geq y^{\lambda_1}$ implies that

$$\begin{aligned}K_2(\hat{y}) - K_1(\hat{y}) &= H_2(\hat{y})(\Phi_1\psi_0)(\hat{y})\left(\frac{\varphi_0(\hat{y})}{\psi_0(\hat{y})} - \frac{(\Phi_2\varphi_0)(\hat{y})}{(\Phi_2\psi_0)(\hat{y})}\right)\\ &\quad + (\Phi_1\psi_0)(\hat{y})\left(\frac{(\Phi_2 p)(\hat{y})}{(\Phi_2\psi_0)(\hat{y})} - \frac{p(\hat{y})}{\psi_0(\hat{y})}\right) - (\Phi_1 g)(\hat{y}) + \frac{g(\hat{y})}{\psi_0(\hat{y})}(\Psi_1\psi_0)(\hat{y})\end{aligned}$$

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$$= (\Phi_1\psi_0)(\hat{y})\left(\frac{g(\hat{y})}{\psi_0(\hat{y})} - \frac{(\Phi_1 g)(\hat{y})}{(\Phi_1\psi_0)(\hat{y})}\right)$$
$$> 0,$$

and we know that $K_1(\hat{y}) > 0$ by part (ii). Now the monotonicity of K_2 on $(y^{\lambda_2,c}, \hat{y})$ implies that K_2 is positive on this interval as well.

A.4. Proof of Proposition 1

The proof is divided into four parts that we label (i), (ii), (iii) and (iv). In part (i) we construct a certain auxiliary function Γ and show that there is a correspondence between solutions to (22) and fixed points of Γ. In part (ii) we show that Γ has a fixed point and in part (iii) the fixed point is found to be unique. We prove that the obtained solution (x^*, y^*) to (22) satisfies $x^* < y^{\lambda_1} < y^*$ in part (iv).

1. The function H_1 is strictly increasing on $(\mathfrak{l}, \hat{y})$ and, by Assumption 1, H_2 is strictly increasing on $(y^{\lambda_2,c}, \hat{y})$. We also have $H_1(\mathfrak{l}) = 0 = H_2(y^{\lambda_2,c})$ and $H_1(\hat{y}) = H_2(\hat{y})$, so that H_2^{-1} is continuous and strictly increasing on $(0, H_1(\hat{y}))$ and $(H_2^{-1} \circ H_1)\colon [\mathfrak{l}, \hat{y}] \to [y^{\lambda_2,c}, \hat{y}]$ is a well-defined, continuous and strictly increasing function. The monotonicity properties of K_1 imply that K_1^{-1} is positive, continuous and strictly decreasing everywhere on I. The ordering $K_1(\hat{y}) < K_2(\hat{y}) < K_2(y^{\lambda_2,c}) < K_1(\mathfrak{l})$ now implies that the restriction of $(K_1^{-1} \circ K_2)$ to $[y^{\lambda_2,c}, \hat{y}]$ is a well-defined, continuous, strictly increasing function with the codomain $[\mathfrak{l}, \hat{y}]$. Thus, the mapping Γ defined by
$$\Gamma(x) = (K_1^{-1} \circ K_2 \circ H_2^{-1} \circ H_1)(x)$$
is a well-defined, continuous, strictly increasing function $\Gamma\colon [\mathfrak{l}, \hat{y}] \to [\mathfrak{l}, \hat{y}]$. Differentiating Γ yields
$$\lim_{\varepsilon\downarrow 0} \frac{\Gamma(x+\varepsilon) - \Gamma(x)}{\varepsilon} = \frac{(((\Phi_2\varphi_0)/(\Phi_2\psi_0))(\Phi_1\psi_0) - (\Phi_1\varphi_0))((H_2^{-1} \circ H_1)(x))}{(((\Psi_2\varphi_0)/(\Psi_2\psi_0))(\Phi_1\psi_0) - (\Phi_1\varphi_0))(\Gamma(x))} > 0,$$
and $\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(\Gamma(x) - \Gamma(x-\varepsilon))$ gives the same result. Thus, $\Gamma \in C^1(I)$ even though $H_2, K_2 \in C^0(I)$. It is also evident that $\Gamma'(x) > 0$ for all $x \in I$. We now see that the pair (22) has a solution (x^*, y^*) with $x^* \in [\mathfrak{l}, \hat{y})$, $y^* \in ([y^{\lambda_2,c}, \hat{y})$ if and only if x^* is a fixed point of the mapping Γ and $y^* = (H_2^{-1} \circ H_1)(x^*)$. Thus, it suffices to prove that Γ has a unique fixed point $x^* \in (\mathfrak{l}, \hat{y})$.

2. First note that $\Gamma(\hat{y}) < \hat{y}$ since $H_1(\hat{y}) = H_2(\hat{y})$, $K_1(\hat{y}) < K_2(\hat{y})$ and K_1 is strictly decreasing. We also have $\Gamma(\mathfrak{l}) > \mathfrak{l}$ because $(H_2^{-1} \circ H_1)(\mathfrak{l}) = y^{\lambda_2,c}$ and $K_2(y^{\lambda_2,c}) < K_1(\mathfrak{l})$. Combining these observations with the monotonicity of Γ implies that Γ must have a fixed point $x^* \in (\mathfrak{l}, \hat{y})$. Moreover, it is given by $\lim_{n\uparrow\infty} \Gamma^n(\hat{y})$.

3. We continue by proving $x^* < y^* = (H_2^{-1} \circ H_1)(x^*)$ as an intermediate step. Note that if $y^{\lambda_1} \le y^{\lambda_2,c}$ then $H_1(x) > H_2(x)$ for $x \in [y^{\lambda_2,c}, \hat{y})$ and, consequently, $x^* < (H_2^{-1} \circ H_1)(x^*)$. Now suppose that $y^{\lambda_1} > y^{\lambda_2,c}$. We have $H_2(y^{\lambda_2,c}) = 0 < H_1(y^{\lambda_2,c})$. If H_1 and H_2 do not intersect on $(y^{\lambda_2,c}, y^{\lambda_1})$, then again $H_1(x) > H_2(x)$ for $x \in [y^{\lambda_2,c}, \hat{y})$ and $x^* < (H_2^{-1} \circ H_1)(x^*)$. Now suppose that H_1 and H_2 do intersect on $(y^{\lambda_2,c}, y^{\lambda_1})$ and denote the largest

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such intersection point as $\check{y}$. We have

$$K_2(\check{y}) - K_1(\check{y}) = (\Phi_1\psi_0)(\check{y})\left(\frac{g(\check{y})}{\psi_0(\check{y})} - \frac{(\Phi_1 g)(\check{y})}{(\Phi_1\psi_0)(\check{y})}\right) < 0$$

since $\check{y} < y^{\lambda_1}$. It follows that $\Gamma(\check{y}) > \check{y}$ and, consequently, $x^* > \check{y}$. But the definition of $\check{y}$ implies that $H_1(x) > H_2(x)$ for $x \in [\check{y}, \hat{y})$ and so $x^* < (H_2^{-1} \circ H_1)(x^*)$. Thus, the inequality $x^* < (H_2^{-1} \circ H_1)(x^*)$ holds for all fixed points x^* of the function Γ.

Now suppose that $x' \in (\mathfrak{l}, \hat{y})$ is a fixed point of Γ. Then $y' = (H_2^{-1} \circ H_1)(x') > x'$ and we have

$$\begin{aligned}
\Gamma'(x') &= \frac{((\Phi_2\varphi_0)(y')/(\Phi_2\psi_0)(y'))(\Phi_1\psi_0)(y') - (\Phi_1\varphi_0)(y')}{((\Psi_2\varphi_0)(x')/(\Psi_2\psi_0)(x'))(\Phi_1\psi_0)(x') - (\Phi_1\varphi_0)(x')} \\
&= \frac{(\Phi_2\varphi_0)(y')(\Phi_1\psi_0)(y') - (\Phi_1\varphi_0)(y')(\Phi_2\psi_0)(y')}{(\Psi_2\varphi_0)(x')(\Phi_1\psi_0)(x') - (\Phi_1\varphi_0)(x')(\Psi_2\psi_0)(x')}\frac{(\Psi_2\psi_0)(x')}{(\Phi_2\psi_0)(y')} \\
&< \frac{\varphi_0(y')(\Phi_1\psi_0)(y') - \psi_0(y')(\Phi_1\varphi_0)(y')}{\varphi_0(x')(\Phi_1\psi_0)(x') - \psi_0(x')(\Phi_1\varphi_0)(x')}\frac{(\Psi_2\psi_0)(x')}{(\Phi_2\psi_0)(y')}\frac{(\Phi_2 1)(y')}{(\Psi_2 1)(x')} \\
&< \frac{\varphi_0(y')(\Phi_1\psi_0)(y') - \psi_0(y')(\Phi_1\varphi_0)(y')}{\varphi_0(x')(\Phi_1\psi_0)(x') - \psi_0(x')(\Phi_1\varphi_0)(x')}\frac{\psi_0(x')}{\psi_0(y')} \\
&< 1.
\end{aligned}$$

In the last inequality we used the fact that the function $x \mapsto \varphi_0(x)(\Phi_1\psi_0)(x) - \psi_0(x)(\Phi_1\varphi_0)(x)$ is decreasing for all x. Thus, by continuity, whenever the function Γ intersects the diagonal of $\mathbb{R}^2$ on $(\mathfrak{l}, \hat{y})$, the intersection must be from above. This implies that the fixed point x^* is unique in $(\mathfrak{l}, \hat{y})$.

4. Next we prove that x^* and y^* satisfy $x^* < y^{\lambda_1} < y^*$. In order to do this, we first define $\mathcal{T}\colon (y^{\lambda_2,c}, \hat{y}) \to (y^{\lambda_2,c}, \hat{y})$ by $\mathcal{T}(x) = (H_2^{-1} \circ H_1 \circ K_1^{-1} \circ K_2)(x)$. Using similar arguments as with Γ, it can be shown that $\mathcal{T}$ is well-defined, continuous and strictly increasing and y^* is the unique fixed point of $\mathcal{T}$. We also have $\mathcal{T}'(y^*) < 1$.

Now we reintroduce the cost scaling parameter $\alpha > 0$ as in the proof of Lemma 3. Owing to the uniqueness of x^* and y^* and $\Gamma'(x^*)$, $\mathcal{T}'(y^*) < 1$, it suffices to show that $d\Gamma_\alpha(y^{\lambda_1})/d\alpha > 0$ and $d\mathcal{T}_\alpha(y^{\lambda_1})/d\alpha < 0$ for all $\alpha \in (0, \alpha_c)$. Expanding the first derivative gives

$$\begin{aligned}
\frac{d}{d\alpha}\Gamma_\alpha(y^{\lambda_1}) = \Bigg(&\frac{-\alpha^{-2}K_{2,\alpha}((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})) + \alpha^{-1}(d/d\alpha)K_{2,\alpha}((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1}))}{K_1'(K_1^{-1}((K_{2,\alpha} \circ H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})/\alpha))} \\
&+ \frac{K_{2,\alpha}'((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1}))}{K_1'(K_1^{-1}((K_{2,\alpha} \circ H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})/\alpha))}\Bigg)\frac{d}{d\alpha}(H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})
\end{aligned}$$

By Assumption 1, Lemma 4, and $\alpha < \alpha_c$, we know that in the above expression the denominator and the numerator on the second line are negative. The numerator in the first line is negative as well. To see this, note that

$$-\alpha^{-2}K_{2,\alpha}((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})) + \alpha^{-1}\frac{d}{d\alpha}K_{2,\alpha}((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1}))$$

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$$= -\alpha^{-2}\left(\frac{g}{\psi_0}(\Phi_1\psi_0) - (\Phi_1 g) - \frac{g(\Phi_2\psi_0) - \psi_0(\Phi_2 g)}{\varphi_0(\Phi_2\psi_0) - \psi_0(\Phi_2\varphi_0)} \times \left(\frac{\varphi_0}{\psi_0}(\Phi_1\psi_0) - (\Phi_1\varphi_0)\right)\right)((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})).$$

φ_0/ψ_0 is strictly decreasing so

$$\varphi_0(\Phi_2\psi_0)(x) - \psi_0(\Phi_2\varphi_0)(x) > 0, \qquad \varphi_0(\Phi_1\psi_0)(x) - \psi_0(\Phi_1\varphi_0)(x) > 0.$$

On the other hand, $\alpha < \alpha_c$ is equivalent to $y^{\lambda_1} < (H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})$ and we know that $(H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1}) \in (y^{\lambda_2,c}, \hat{y})$ and $\hat{y} < y^{\lambda_2}$ so

$$(g(x)(\Phi_1\psi_0)(x) - \psi_0(x)(\Phi_1 g)(x))((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})) < 0,$$
$$(g(\Phi_2\psi_0) - \psi_0(\Phi_2 g))((H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})) > 0.$$

Next we show that $d(H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})/d\alpha < 0$. Indeed, for $x \in (y^{\lambda_2,c}, \hat{y})$, we get

$$\frac{d}{d\alpha}(H_{1,\alpha}^{-1} \circ H_{2,\alpha})(x) = \frac{-\alpha^{-2}H_{2,\alpha}(x) + \alpha^{-1}(d/d\alpha)H_{2,\alpha}(x)}{H_1'\left(H_1^{-1}\left(H_{2,\alpha}(x)/\alpha\right)\right)}$$
$$= -\frac{\frac{g(x)(\Phi_2\psi_0)(x) - \psi_0(x)(\Phi_2 g)(x)}{\varphi_0(x)(\Phi_2\psi_0)(x) - \psi_0(x)(\Phi_2\varphi_0)(x)}}{\alpha^2 H_1'\left(H_1^{-1}\left(H_{2,\alpha}(x)/\alpha\right)\right)}$$
$$> 0$$

and $H_{1,\alpha}^{-1} \circ H_{2,\alpha}$ is strictly increasing on $(y^{\lambda_2,c}, \hat{y})$, so for all $x \in (\mathfrak{l}, \hat{y})$, we have

$$\frac{d}{d\alpha}(H_{2,\alpha}^{-1} \circ H_{1,\alpha})(x) = \frac{d}{d\alpha}\left((H_{1,\alpha}^{-1} \circ H_{2,\alpha})^{-1}(x)\right) < 0$$

so that $d(H_{2,\alpha}^{-1} \circ H_{1,\alpha})(y^{\lambda_1})/d\alpha < 0$. Combining the observations yields $d\Gamma_\alpha(y^{\lambda_1})/d\alpha > 0$ for $\alpha \in (0, \alpha_c)$ and, consequently, $y^{\lambda_1} = \Gamma_{\alpha_c}(y^{\lambda_1}) > \Gamma_\alpha(y^{\lambda_1})$. But the unique fixed point x_α^* of Γ_α satisfies $\Gamma_\alpha'(x_\alpha^*) < 1$ so it must hold that $x_\alpha^* < y^{\lambda_1}$.

We may expand the derivative of $\mathcal{T}_\alpha(y^{\lambda_1})$ as

$$\frac{d}{d\alpha}\mathcal{T}_\alpha(y^{\lambda_1}) = \frac{d}{d\alpha}(H_{2,\alpha}^{-1} \circ H_{1,\alpha})((K_{1,\alpha}^{-1} \circ K_{2,\alpha})(y^{\lambda_1})) + (H_{2,\alpha}^{-1} \circ H_{1,\alpha})'((K_{1,\alpha}^{-1} \circ K_{2,\alpha})(y^{\lambda_1})) \times \frac{-\alpha^{-2}K_{2,\alpha}(y^{\lambda_1}) + \alpha^{-1}(d/d\alpha)K_{2,\alpha}(y^{\lambda_1})}{K_1'\left(K_1^{-1}\left(K_{2,\alpha}(y^{\lambda_1})/\alpha\right)\right)}(y^{\lambda_1}).$$

We have already seen that the first term is negative, the first half of the second term is positive and the second half is negative. Hence, $d\mathcal{T}_\alpha(y^{\lambda_1})/d\alpha < 0$ for $\alpha \in (0, \alpha_c)$ and, consequently, $y^{\lambda_1} = \Gamma_{\alpha_c}(y^{\lambda_1}) < \Gamma_\alpha(y^{\lambda_1})$. But the unique fixed point y_α^* of $\mathcal{T}_\alpha$ satisfies $\mathcal{T}_\alpha'(y_\alpha^*) < 1$ so it must hold that $y_\alpha^* > y^{\lambda_1}$. The proof is now complete.

A.5. Proof of Proposition 2

First note that the properties of the constants C_1, C_2, C_3 and the thresholds x^*, y^* imply that $\mathcal{K}_0 \in C^0(I)$. The rest of the proof is divided into four parts that we label (i), (ii), (iii) and (iv). We begin by showing in part (i) that $\mathcal{K}_0$ satisfies the DPP (24) on $[y^\infty, \mathfrak{r})$. Parts (ii), (iii) and (iv) contain the corresponding proofs for the intervals $[y^*, y^\infty)$, $[x^*, y^*)$ and $(\mathfrak{l}, x^*)$, respectively.

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1. Let $x \geq y^{\infty}$. Then $\mathcal{K}_0(x) = g(x) = \tilde{V}(x)$, where $\tilde{V}$ is the value function of the optimal stopping problem in continuous time (26). Here $\tilde{V}$ is r excessive and $p(x) - \lambda_2(R_2 p)(x) = c(x)$ by (4) so $\mathcal{K}_0(x) = \max\{g(x), \mathcal{K}_1(x), \mathcal{K}_2(x)\}$. Thus, the DPP holds on $[y^{\infty}, \mathfrak{r})$.

2. Let $y^* \leq x < y^{\infty}$. Now $\mathcal{K}_0(x) = g(x)$ and making use of the strong Markov property of linear diffusions, memoryless property of the exponential distribution, resolvent equation (4), and the assumption that $(P - \lambda_2(R_2 P))/\varphi_2$ is strictly increasing on $(y^{\lambda_2, c}, y^{\infty})$, we get

$$
\begin{aligned}
g(x) - \mathcal{K}_2(x) =& g(x) - \mathbb{E}_x \left[e^{-rU_2} \mathcal{K}_0(X_{U_2})(1(U_2 < \tau_{y^*}) + 1(U_2 \geq \tau_{y^*})) \right] + c(x) \\
=& g(x) - \mathbb{E}_x \left[e^{-rU_2} g(X_{U_2}) \right] + c(x) \\
& + \mathbb{E}_x \left[e^{-(r+\lambda_2)\tau_{y^*}} \right] \mathbb{E}_{y^*} \left[e^{-rU_2} (g(X_{U_2}) - \mathcal{K}_0(X_{U_2})) \right] \\
=& g(x) - \lambda_2(R_2 g)(x) + c(x) - \frac{g(y^*) - \lambda_2(R_2 g)(y^*) + c(y^*)}{\varphi_2(y^*)} \varphi_2(x) \\
\geq& 0.
\end{aligned}
$$

By similar arguments we obtain

$$
\begin{aligned}
g(x) - \mathcal{K}_1(x) &= g(x) - \lambda_1(R_1 g)(x) - \frac{\mathcal{K}_1(y^*) - \lambda_1(R_1 g)(y^*)}{\varphi_1(y^*)} \varphi_1(x) \\
&\geq \frac{g(y^*) - \mathcal{K}_1(y^*)}{\varphi_1(y^*)} \varphi_1(x).
\end{aligned}
$$

Recalling that $K_1(x^*) = K_2(y^*)$ and by Remark 2 and the proof of Lemma 4(ii), K_1 and K_2 can be written as

$$
\begin{aligned}
K_1(x) &= F_1(x)(\phi_1 \psi_0)(x) + H_1(x)(\phi_1 \varphi_0)(x) - (\phi_1 p)(x), \\
K_2(x) &= F_2(x)(\phi_1 \psi_0)(x) + H_2(x)(\phi_1 \varphi_0)(x) - (\phi_1 P)(x).
\end{aligned}
$$

We find that expanding the resolvent $\mathcal{K}_1(y^*) = \lambda_1(R_1 \mathcal{K}_0)(y^*)$ according to (2) yields

$$
\begin{aligned}
g(y^*) - \mathcal{K}_1(y^*) =& g(y^*) - F_2(y^*)\psi_0(y^*) - H_2(y^*)\varphi_0(y^*) + \lambda_1(R_1 p)(y^*) \\
& + \frac{F_1(x^*)\psi_0(x^*) + H_1(x^*)\varphi_1(x^*) - \lambda_1(R_1 p)(x^*)}{\varphi_1(x^*)} \varphi_1(y^*) \\
& + \frac{\lambda_1}{B_1} K_1(x^*) \left(\psi_1(y^*) - \frac{\psi_1(x^*)}{\varphi_1(x^*)} \varphi_1(y^*) \right) \\
=& \lambda_1(R_1 p)(y^*) - p(y^*) + \frac{p(x^*) - \lambda_1(R_1 p)(x^*)}{\varphi_1(x^*)} \varphi_1(y^*) \\
& + \frac{\lambda_1}{B_1} K_1(x^*) \left(\psi_1(y^*) - \frac{\psi_1(x^*)}{\varphi_1(x^*)} \varphi_1(y^*) \right).
\end{aligned}
$$

Letting $f(x) = c(x) + (\lambda_2 - \lambda_1)(R_1 c)(x)$ and invoking again the resolvent identity (4), we see that the above expression is nonnegative if and only if

$$
\frac{f(y^*)/\varphi_1(y^*) - f(x^*)/\varphi_1(x^*)}{\psi_1(y^*)/\varphi_1(y^*) - \psi_1(x^*)/\varphi_1(x^*)} \leq \frac{\lambda_1}{B_1} K_1(x^*),
$$

The left-hand side is continuously differentiable and $K_1(x^*)$ may be written according to Remark 2 as

$$K_1(x^*) = \frac{\lambda_2 - \lambda_1}{B_1}\left(\frac{c(x^*)}{\psi_2(x^*)}(\Psi_2\varphi_1)(x^*) + (\Phi_1 c)(x^*)\right).$$

The Cauchy mean value theorem then implies that

$$\begin{aligned}
&\frac{f(y^*)/\varphi_1(y^*) - f(x^*)/\varphi_1(x^*)}{\psi_1(y^*)/\varphi_1(y^*) - \psi_1(x^*)/\varphi_1(x^*)} \\
&\qquad \leq \sup_{x\in[x^*,y^*]} \frac{(f(x)/\varphi_1(x))'}{(\psi_1(x)/\varphi_1(x))'} \\
&\qquad = \sup_{x\in[x^*,y^*]} \frac{1}{B_1}\left((\Phi_1(r+\lambda_1-\mathcal{A})c)(x) + (\lambda_2-\lambda_1)(\Phi_1 c)(x)\right) \\
&\qquad = \frac{1}{B_1}(\Phi_1(r+\lambda_2-\mathcal{A})c)(x^*).
\end{aligned}$$

But

$$(\Phi_1(r+\lambda_2-\mathcal{A})c)(x^*) \leq \frac{\lambda_2-\lambda_1}{B_1}\left(\frac{c(x^*)}{\psi_2(x^*)}(\Psi_2\varphi_1)(x^*) + (\Phi_1 c)(x^*)\right)$$

is equivalent to

$$c'(x^*)\psi_2(x^*) - c(x^*)\psi_2'(x^*) \leq 0,$$

which is true since ψ_0/ψ_2 is strictly decreasing and c/ψ_0 is strictly decreasing by Assumption 1. This proves that $g(y^*) \geq \mathcal{K}_1(y^*)$ and, hence, $\mathcal{K}_0(x) \geq \mathcal{K}_1(x)$ for $x \in [y^*, y^\infty)$.

3. Let $x^* \leq x < y^*$. Now $\mathcal{K}_0(x) = \mathcal{K}_2(x)$. Denote by $\hat{\tau}$ the first exit time of X from the interval (x^*, y^*). We recall from [2, p.272] that

$$\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-(r+\lambda_1)\hat{\tau}} 1(\tau_{x^*} < \tau^{y^*})\right] &= \frac{\psi_1(y^*)\varphi_1(x) - \psi_1(x)\varphi_1(y^*)}{\psi_1(y^*)\varphi_1(x^*) - \psi_1(x^*)\varphi_1(y^*)}, \\
\mathbb{E}_x\left[\mathrm{e}^{-(r+\lambda_1)\hat{\tau}} 1(\tau_{x^*} > \tau^{y^*})\right] &= \frac{\psi_1(x)\varphi_1(x^*) - \psi_1(x^*)\varphi_1(x)}{\psi_1(y^*)\varphi_1(x^*) - \psi_1(x^*)\varphi_1(y^*)}.
\end{aligned}$$

Applying similar arguments as before, we see that $\mathcal{K}_2(x) \geq \mathcal{K}_1(x)$ is equivalent to

$$\begin{aligned}
&-f(x) + f(x^*)\frac{\psi_1(y^*)\varphi_1(x) - \psi_1(x)\varphi_1(y^*)}{\psi_1(y^*)\varphi_1(x^*) - \psi_1(x^*)\varphi_1(y^*)} \\
&+ (f(y^*) + g(y^*) - G_1(y^*))\frac{\psi_1(x)\varphi_1(x^*) - \psi_1(x^*)\varphi_1(x)}{\psi_1(y^*)\varphi_1(x^*) - \psi_1(x^*)\varphi_1(y^*)} \geq 0,
\end{aligned}$$

where again $f(x) = c(x) + (\lambda_2 - \lambda_1)(R_1 c)(x)$. Expanding $\mathcal{K}_1(y^*)$ as in part (ii), the above condition is found to be equivalent to

$$\frac{f(x)/\varphi_1(x) - f(x^*)/\varphi_1(x^*)}{\psi_1(x)/\varphi_1(x) - \psi_1(x^*)/\varphi_1(x^*)} \leq \frac{\lambda_1}{B_1}K_1(x^*),$$

which can be proven as in part (ii). Thus, $\mathcal{K}_0(x) \geq \mathcal{K}_1(x)$ for $x \in [x^*, y^*)$.

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In order to prove $\mathcal{K}_2(x) \geq g(x)$, we first note that the condition is equivalent to

$$\frac{F_2(x) - F_2(y^*)}{H_2(y^*) - H_2(x)} \leq \frac{\varphi_0(x)}{\psi_0(x)}.$$

On the other hand, $\varphi_0/\psi_0 > (\Phi_2\varphi_0)/(\Phi_2\psi_0)$ so it suffices to show that

$$\frac{F_2(x) - F_2(y^*)}{H_2(y^*) - H_2(x)} \leq \frac{(\Phi_2\varphi_0)(x)}{(\Phi_2\psi_0)(x)},$$

which in turn is equivalent to

$$\frac{(\Phi_2 P)(x)/(\Phi_2\psi_0)(x) - (\Phi_2 P)(y^*)/(\Phi_2\psi_0)(y^*)}{(\Phi_2\varphi_0)(x)/(\Phi_2\psi_0)(x) - (\Phi_2\varphi_0)(y^*)/(\Phi_2\psi_0)(y^*)} < H_2(y^*).$$

We know that $l_g < x^* < x < y^*$, $H_2(y) \leq 0$ for $y \in (l_g, y^{\lambda_2,c})$ and H_2 is strictly increasing on $(y^{\lambda_2,c}, y^*)$, so the Cauchy mean value theorem implies that

$$\begin{aligned}
&\frac{(\Phi_2 P)(x)/(\Phi_2\psi_0)(x) - (\Phi_2 P)(y^*)/(\Phi_2\psi_0)(y^*)}{(\Phi_2\varphi_0)(x)/(\Phi_2\psi_0)(x) - (\Phi_2\varphi_0)(y^*)/(\Phi_2\psi_0)(y^*)} \\
&\quad \leq \sup_{x \in [x^*, y^*)} \frac{((\Phi_2 P)(x)/(\Phi_2\psi_0)(x))'}{((\Phi_2\varphi_0)(x)/(\Phi_2\psi_0)(x))'} \\
&\quad = \sup_{x \in [x^*, y^*)} H_2(x) \\
&\quad = H_2(y^*),
\end{aligned}$$

proving the condition $\mathcal{K}_2(x) \geq g(x)$ for $x \in [x^*, y^*)$.

4. Let $x < x^*$. Now $\mathcal{K}_0(x) = \mathcal{K}_1(x)$ and $\mathcal{K}_1(x) \geq g(x)$ is equivalent to

$$\frac{g(x)}{\psi_0(x)} \leq \frac{\mathcal{K}_1(x^*)}{\psi_0(x^*)}.$$

The above inequality is true because $l_g < x^* < y^\infty$ so $g(x)/\psi_0(x) \leq g(x^*)/\psi_0(x^*)$ by Assumption 1 and $g(x^*) \leq \mathcal{K}_1(x^*)$ by part (iii). On the other hand,

$$\mathcal{K}_1(x) - \mathcal{K}_2(x) = c(x) - \frac{c(x^*)}{\psi_2(x^*)}\psi_2(x) \geq 0,$$

where the inequality follows by Assumption 1, since ψ_0/ψ_2 is strictly decreasing.

Combining the observations made in parts (i)–(iv), we see that $\mathcal{K}_0 = \max\{g, \mathcal{K}_1, \mathcal{K}_2\}$ so that $\mathcal{K}_0$ satisfies the DPP (24). Proving the smoothness properties of $\mathcal{K}_1$ and $\mathcal{K}_2$ is now completely analogous to the proof of Lemma 2.

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There were no competing interests to declare that arose during the preparation or publication process of this paper.

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