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SOME DIFFERENTIAL EQUATIONS RELATED TO ITERATION THEORY

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0. Introduction. In connection with the translation equation

(T) $\quad F(F(x, s), t) = F(x, s + t)$

three differential equations arise together with a differential initial condition. They are satisfied by the differentiable solutions of (T) and of the initial condition

(I) $\quad F(x, 0) = x.$

These equations are attributed in **[9]** to E. Jabotinsky who seems to have been the first who treated these equations in connection with the theory of analytic iteration (see **[6]**, cf. **[7, 8]**, but see also **[1, 2, 3]**).

Gronau (see **[9]**) asked whether, conversely, it is true that all solutions of each of these "Jabotinsky differential equations", possibly with some further initial conditions added, are also solutions of the translation equation. In this paper we give counter examples but also partial positive answers to these questions. First we show how one obtains the Jabotinsky equations from (T) and (I).

We will do this for Banach space valued functions $F(x, t)$ where the "time" variable t varies in an open or half open real interval or an open connected complex set containing the zero. This of course contains the classical one dimensional case.

PROPOSITION 0. *Let X be a real or complex Banach space and I be a real interval (open or half closed) or an open connected set of the complex plane such that* 0 *is contained in I, further U, $U' \subseteq X$ open neighborhoods of the zero O in X, $U' \subseteq U$ and*

$$F: U \times I \to X$$

a function satisfying $F(U', I) \subseteq U$,

(I) $\quad F(x, 0) = x$

and

(T) $\quad F(F(x, s), t) = F(x, s + t)$

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whenever both sides of the equation (T) *are defined. If F is differentiable with respect to both variables x and t, then it satisfies the three "Jabotinsky equations"*

$$(1)\quad \frac{\partial F(x, t)}{\partial t} = \frac{\partial F(x, t)}{\partial x} \cdot G(x)$$

$$(2)\quad \frac{\partial F(x, t)}{\partial t} = G(F(x, t))$$

$$(3)\quad \frac{\partial F(x, t)}{\partial x} \cdot G(x) = G(F(x, t))$$

at least for all $x \in U'$ and $t \in I$, where for each of the three equations the differential initial condition

$$(4)\quad G(x) = \left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0}$$

holds.

Here $(\partial F(x, t)/\partial t)$ is (as usual) an element of the Banach space X for each $(x, t) \in U \times I$ and $(\partial F(x, t)/\partial x)$ is a linear map from X to X for each $(x, t) \in U \times I$. The $\cdot$ in (1) and (3) denotes the composition of the linear mapping $(\partial F(x, t)/\partial x)$ with G. In the case where X is $\mathbf{R}^n$ or $\mathbf{C}^n$

$$F(x, t) = (F_i(x_1, \ldots, x_n, t))_{i=1,\ldots,n}$$

is a column vector, $(\partial F(x, t)/\partial x)$ is the Jacobian matrix

$$\frac{\partial F(x, t)}{\partial x} = \left(\frac{\partial F_i(x, t)}{\partial x_j}\right)_{i,j=1,\ldots,n}$$

and $\cdot$ in (1) and (3) means (matrix) multiplication.

Proof. Differentiating equation (T) with respect to s, then putting $s = 0$ and using (I) yields (1) with (4). The same procedure but interchanging s and t in (T) (without using (I)) leads to (2) with (4). Equation (3) is a consequence of (1) and (2).

1. Real solutions of the Jabotinsky equations when G is nowhere 0. We give a negative answer to the question raised in **[9]**, whether all solutions of the equations (1), (2) or (3) together with the condition (or notation, depending on one's point of view) (4) are also solutions of the translation equation (T). This will be done by the following propositions which at the same time yield (under some restrictions on G, in particular that it is nowhere 0 on U) a general representation of the solutions of these equations in the real one dimensional case. In this section U and I are real intervals containing zero, U is open and I may be closed on one side. The composition $\cdot$ in equations (1) and (3) in this case is simply the multiplication.

1. *Solutions of equation* (1) (under the assumption that F is differentiable, G is continuous and $G(x) \neq 0$ on U).

(a) *General solution of*

$$(1) \quad \frac{\partial F(x, t)}{\partial t} = \frac{\partial F(x, t)}{\partial x} G(x).$$

Define f by

$$(5) \quad f'(x) = \frac{df(x)}{dx} = G(x)^{-1}$$

and H by

$$H(x, t) = f(x) + t$$

for $x \in U$ and $t \in I$. So one can see that equation (1) is equivalent to

$$\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial t} \\ \dfrac{\partial H}{\partial x} & \dfrac{\partial H}{\partial t} \end{vmatrix} = 0$$

which means that the functions F and H are functionally dependent. Since F is differentiable, H is continuously differentiable and

$$\frac{\partial H}{\partial t} = 1 \neq 0,$$

furthermore $U \times I$ is a region (possibly with a boundary line added to which the result extends by continuity), there exists a differentiable function φ (see [5]) such that

$$(6) \quad F(x, t) = \varphi(f(x) + t).$$

Conversely, every function F of the form (6), where φ is an arbitrary differentiable function and f satisfies (5), is a solution of (1).

(b) *Solutions of* (1) *satisfying the differential initial condition*

$$(4) \quad G(x) = \left.\frac{\partial F(x, t)}{\partial x}\right|_{t=0}$$

(equivalently, solutions of (1) where G is defined by (4)).

From the representation (6)

$$\frac{\partial F(x, t)}{\partial t} = \varphi'(f(x) + t)$$

follows. Therefore equation (4) implies

$$G(x) = \varphi'(f(x))$$

and, by (5),

$$1 = \varphi'(f(x))f'(x).$$

So we get $f'(x) \neq 0$, f invertible on one hand, on the other

$$\varphi(f(x)) = x + b \quad \text{with } b \in \mathbf{R},$$

hence $\varphi(y) = f^{-1}(y) + b$ and

$$(7) \qquad F(x, t) = f^{-1}(f(x) + t) + b.$$

Conversely, every function F of the form (7), where f is given by (5) and b is an arbitrary constant, is a solution of (1) and (4).

(c) *Solutions of* (1) *with the initial condition*

$$(\mathrm{I}) \qquad F(x, 0) = x.$$

From (I) and (6)

$$\varphi(y) = f^{-1}(y)$$

follows, hence

$$(8) \qquad F(x, t) = f^{-1}(f(x) + t)$$

which is a special case of (7). Conversely, every function (8), without any condition on f except differentiability and invertibility, is a solution of (1) and (I) with (4).

(d) PROPOSITION 1. *If G: $U \rightarrow \mathbf{R}$ is continuous and $G(x) \neq 0$ on U, then the general differentiable solution F: $U \times I \rightarrow \mathbf{R}$ of* (1) *is given by* (6), *where φ: $\mathbf{R} \rightarrow \mathbf{R}$ is an arbitrary differentiable function and f is defined by* (5). *Under the same conditions the general solution of* (1) *and* (4) *is given by* (7) *and the general solution of* (1) *and* (I) *is given by* (8); *thus every solution of* (1) *and* (I) *also satisfies* (4).

(e) *Conclusion*. Since (7) does not satisfy (T) if $b \neq 0$ (cf. also [**1**, **3**]), not every solution of (1), even with the relation (4), satisfies (T). However, every solution of (1) and (I) does satisfy (T) and (4).

2. *Solutions of* (2) (under the assumption that $(\partial F)/(\partial t)$ exists, G is continuous and $G(x) \neq 0$ on U).

(a) *General solution of*

$$(2) \qquad \frac{\partial F(x, t)}{\partial t} = G(F(x, t)).$$

Define f by (5) then equation (2) is equivalent to

$$f'(F(x, t))\frac{\partial F(x, t)}{\partial t} = 1$$

hence

(9) $f(F(x, t)) = t + h(x)$.

Therefore

(10) $F(x, t) = f^{-1}(h(x) + t)$

where h is an arbitrary function. Conversely, every function (10) with an arbitrary h and with f satisfying (5) is a solution of (2).

(b) *Solution of* (2) *with* (4). From (9) and (4) we get

$$f'(F(x, 0))G(x) = 1,$$

therefore, by (5) and (9),

$$G(x) = G(f^{-1}(h(x))).$$

So, if we suppose G to be strictly monotonic, then $h(x) = f(x)$ follows and

(8) $F(x, t) = f^{-1}(f(x) + t)$.

Conversely, every F of the form (8), without any restriction except differentiability and invertibility of f and without the assumption that G is strictly monotonic, satisfies (2), (I) and (T) with (4).

(c) *Solution of* (2) *with the initial condition* (I). The condition (I) implies $h(x) = f(x)$ for (10). Therefore the solutions again are of the form (8). By substituting $t = 0$ into (2) one also sees directly that (2) and (I) imply (4).

(d) PROPOSITION 2. *Let* G: $U \to \mathbf{R}$ *be continuous and* $G(x) \neq 0$ *on* U. *Then the general solution* F: $U \times I \to U$ *of* (2), *differentiable in its second variable, is given by* (10), *where* f *is defined by* (5) *and* h: $U \to \mathbf{R}$ *is an arbitrary function. Under the same conditions the general solution of* (2) *and* (I) *is given by* (8). *The same solution* (8) *is also obtained if* (2) *and* (4) *are supposed with strictly monotonic* G. *Finally*, (2) *and* (I) *always imply* (4).

(e) *Note*. If G is not strictly monotonic (but still continuous and nowhere 0 on U) then (T) (or (8)) does not necessarily follow, as the following examples show.

1. $F(x, t) = h(x) + t$ ($x \in U = \mathbf{R}$, $t \in I = \mathbf{R}$) does not satisfy (T) if $h(h(x) + s) \neq h(x) + s$ (e.g. if $h(x) = e^x$) but it satisfies (2) and (4) with $G(x) = 1$.

2. $F(x, t) = \arctan(t - \tan x)$ ($x \in U =]-\pi/2, \pi/2[$, $t \in I = \mathbf{R}$) does not satisfy (T) either, but it satisfies (2) and (4) with $G(x) = \cos^2 x$.

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(f) *Conclusion.* In general the solutions of (2) also fulfill (T) if the initial condition (I) holds, or if (4) holds and G is strictly monotonic (alternatively, 'G in (4) is strictly monotonic') but in general not without these conditions.

Z. Moszner has communicated to the authors the following result.

(g) PROPOSITION 2 (a). *The solutions of* (2) *fulfill also* (T) *if and only if the initial condition* (I) *holds for each* $x \in f^{-1}(h(U) + I)$.

Proof. Equation (T) for F of the form (10) is equivalent to

$$h(f^{-1}(h(x) + s)) = h(x) + s$$

which is equivalent to the condition

$$h(f^{-1}(u)) = u \text{ for } u \in h(U) + I$$

hence $h(z) = f(z)$ which means (I) for all $z \in f^{-1}(h(U) + I)$.

3. *Solutions of* (3) (under the assumption that $\partial F/\partial x$ exists, G is continuous and $G(x) \neq 0$ on U).

(a) *General solution of*

$$\frac{\partial F(x, t)}{\partial x} G(x) = G(F(x, t)). \tag{3}$$

Using the definition (5), from (3) we get

$$\frac{\partial}{\partial x} f(F(x, t)) = f'(F(x, t)) \frac{\partial F(x, t)}{\partial x} = f'(x).$$

Therefore we have

$$f(F(x, t)) = f(x) + g(t).$$

Hence

$$F(x, t) = f^{-1}(f(x) + g(t)) \tag{11}$$

with an arbitrary $g: I \to \mathbf{R}$. Conversely, each function F defined by (11) with an arbitrary g and with f given by (5) is a solution of (3).

(b) *Solutions of* (3) *and* (I). From (11) and (I) we have

$$g(0) = 0. \tag{12}$$

Conversely, every function F of the form (11) with $g(0) = 0$ satisfies (3) and (I).

(c) *Solutions of* (3), (I) *with* (4). Because of (4), $\partial F/\partial t$ must exist at $t = 0$ and thus the function g in (11) has to be differentiable at $t = 0$. From (11) and (4)

$$f'(F(x, 0))G(x) = g'(0)$$

follows, hence by (I) and (5)

(13) $g'(0) = 1.$

As before, equations (11) and (I) imply

(12) $g(0) = 0.$

Conversely, every function of the form (11) which satisfies (12) and (13), is a solution of (3), (4) and (I).

(d) PROPOSITION 3. *Let G be continuous and different from* 0 *on U. Then the general solution $F:U \times I \to U$ of* (3), *differentiable in its first variable, is given by* (11), *where f is defined by* (5) *and $g:I \to \mathbf{R}$ is an arbitrary function. This solution satisfies* (I) *if and only if $g(0) = 0$. Furthermore, if F is partially differentiable at $t = 0$ also in its second variable,* (3) *with* (4) *and* (I) *are satisfied if and only if* (11) *holds with* (12) *and* (13).

(e) *Conclusion*. The solutions of (3) are in general far from being solutions of the translation equation (T), even when they satisfy the initial condition (I) and the differential initial condition (4) ('G in (3) is given by (4)'). For instance,

$$F(x, t) = \ln(\exp(x/2) + t^3 + t)^2$$

satisfies (3), (4) and (I) on $\mathbf{R}$ with $G(x) = 2\exp(-x/2)$ but not (T).

With the representation (11) F is a solution of the translation equation (T) if and only if g is an additive function, i.e.,

$$g(t + s) = g(t) + g(s).$$

(f) *Note*. Not even all three Jabotinsky equations (1), (2), (3) and the differential initial condition (4) imply the translation equation (T). Indeed,

$$F(x, t) = x + t + 1$$

satisfies (1), (2), (3) and (4) with $G(x) \equiv 1$, but not (T). In general, from (10) and (11) (the solutions of (2) and (3)), with $x = x_0$, $c = h(x_0) - f(x_0)$, we get $g(t) = t + c$ and this, combined with (7) (the solution of (1) and (4)), gives

$$f^{-1}(f(x) + t) + b = f^{-1}(f(x) + t + c),$$

that is,

$$f(x + b) = f(x) + c.$$

The general solution of this equation is given by

$$f(x) = \frac{c}{b}x + p(x),$$

where p is an arbitrary periodic function with period b. So, if G is continuous and nowhere 0 and F differentiable, then the general common solution of (1), (2), (3) and (4) is given by

$$F(x, t) = f^{-1}(f(x) + t) + b,$$

where

$$f(x) = ax + p(x),$$

a, b being arbitrary constants and p an arbitrary differentiable function of period b such that $a + p'(x) \neq 0$ (e.g. $a = 2$, $p(x) = \cos x$). Indeed, functions of this form all satisfy (1), (2), (3), (4):

$$aF(x, t) - ab + p(F(x, t) - b) = ax + p(x) + t, \quad F(x, 0) = x + b$$

$$\frac{\partial F}{\partial x} = \frac{a + p'(x)}{a + p'(F(x, t))}, \quad \frac{\partial F}{\partial t} = \frac{1}{a + p'(F(x, t))},$$

$$G(x) = \left.\frac{\partial F}{\partial t}\right|_{t=0} = \frac{1}{a + p'(x)}, \quad \frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} G(x) = G(F(x, t)).$$

2. The first two Jabotinsky equations. In this section we will show that for Banach space valued functions the initial condition (I) implies that every solution of (1) or (2) is also a solution of the translation equation (T) under the supposition that the Cauchy problem for (1) or (2), respectively, has a unique solution.

Let X be a real or complex Banach space. In what follows we suppose that U is an open neighborhood of 0 in X and I is a real interval, open or half closed, or an open connected set of complex numbers, where I contains the number 0.

PROPOSITION 4. *Let $F: U \times I \to X$ be a solution of*

$$(1) \qquad \frac{\partial F(x, t)}{\partial t} = \frac{\partial F(x, t)}{\partial x} \cdot G(x)$$

with the initial condition

$$(\mathrm{I}) \qquad F(x, 0) = x.$$

Suppose further that the Cauchy problem

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x} \cdot G \text{ with } H(x, 0) = 0$$

has only the trivial zero solution (uniqueness condition of the Cauchy problem). Then F also satisfies the translation equation (T) *and the differential initial condition*

$$(4) \qquad G(x) = \left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0}$$

whenever $t, s, t + s \in I$ *and* $x, F(x, s) \in U$.

Proof. Although this result is standard, we give here an elementary proof. Define

$$H(x, t, s) = F(x, t + s) - F((F(x, t), s)$$

whenever $t, s, t + s \in I$ and $x, F(x, t) \in U$. With the notation F_1' and F_2' for the derivatives of H with respect to the first or second variable, respectively, taking into account equation (1) for F, we get

$$\begin{aligned}\frac{\partial H(x, t, s)}{\partial t} &= F_2'(x, t + s) - F_1'(F(x, t), s) \cdot F_2'(x, t) \\ &= F_1'(x, t + s) \cdot G(x) - F_1'(F(x, t), s) \cdot F_1'(x, t) \cdot G(x) \\ &= [F_1'(x, t + s) - F_1'(F(x, t), s) \cdot F_1'(x, t)] \cdot G(x) \\ &= \frac{\partial H(x, t, s)}{\partial x} \cdot G(x).\end{aligned}$$

Therefore H is a solution of the Cauchy problem

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x} \cdot G$$

with the initial condition

$$H(x, 0, s) = F(x, s) - F(F(x, 0), s) \equiv 0$$

due to (I). Hence, by supposition, $H(x, s, t) = 0$ whenever both terms of H are defined. So we get, as asserted,

$$F(x, t + s) = F(F(x, t), s)$$

whenever both sides of this equation are defined.

Now we show that, as in Proposition 1, (4) is implied by (1) and (I):

$$\left.\frac{\partial F(x, t)}{\partial x}\right|_{t=0} = \frac{\partial F(x, 0)}{\partial x} = I_X.$$

The first equality holds if $F(x, 0)$ and $\partial F/\partial x|_{t=0}$ exist and the second since $F(x, 0) = x$, so the derivative of $F(x, 0)$ is I_X, the identity operator on X. Therefore $t = 0$ in (1) indeed yields

$$\left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0} = \frac{\partial F(x, 0)}{\partial x} \cdot G(x) = G(x).$$

PROPOSITION 5. *Let $F(x, t)$ be a solution of*

$$\frac{\partial F(x, t)}{\partial t} = G(F(x, t)) \tag{2}$$

defined on $U \times I$, with the initial condition

$$F(x, 0) = x. \tag{I}$$

Then, if G fulfills the condition that the Cauchy problem

$$y' = G(y) \text{ with } y(0) = b$$

for each $b \in U$ has a unique solution (dependent on b), then F is a solution of the translation equation (T) *too, and the equation* (4) *follows.*

Proof. Also the proof that F satisfies (T) is standard (see [**4**]) but we give here an elementary proof in a somewhat different vein than that of Proposition 4: Let us denote the (by supposition unique) solution of

$$y'(t) = G(y(t)) \quad \text{and} \quad y(0) = b$$

by $y(t) = F(b, t)$ and introduce also the notation

$$z(t) = y(s + t) = F(b, s + t).$$

On the other hand, z is the unique solution of

$$z'(t) = G(z(t)) \quad \text{and} \quad z(0) = y(s)$$

because $y'(s + t) = G(y(s + t))$. So $z(t) = F(F(b, s), t)$ and (T) is satisfied:

$$F(F(b, s), t) = F(b, s + t) \text{ for all } b \in U.$$

We get (4) from (2) and (I) as follows:

$$\left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0} = G(F(x, 0)) = G(x).$$

Remark 5 (a). In generalisation of 1.2. (b) one can easily see that the definition or condition (4) for the solutions of (2) implies the initial condition (I) under the assumption that G is injective:

$$G(x) = \left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0} = G(F(x, 0))$$

hence $x = F(x, 0)$. (But see also Note 1.2. (e).)

3. The third Jabotinsky equation. As we have seen in 1.3, the equation

$$\frac{\partial F(x, t)}{\partial x} \cdot G(x) = G(F(x, t)) \tag{3}$$

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has not such a close relationship to the translation equation (T) as the two others. This will be more evident if one notices two other ways of deducing equation (3).

Remark 6 (a) (Commuting maps). Let U and I be as in Proposition 0 and the differentiable function

$$F:U \times I \to U$$

be commuting, that is,

$$\text{(C)} \quad F(F(x, s), t) = F(F(x, t), s)$$

and let F satisfy the initial condition (I). Differentiating (C) with respect to the variable s we get

$$\text{(14)} \quad F_1'(F(x, s), t) \cdot F_2'(x, s) = F_2'(F(x, t), s),$$

where as above F_1' and F_2' denote the derivatives of F with respect to the first or second variable, respectively. Putting $s = 0$ and using (I): $F(x, 0) = x$, this leads to equation (3) with $G(x) = F_2'(x, 0)$ (cf. (4)).

Remark 6 (b). In the one dimensional case, (3) can be deduced from (C) without using the initial condition (I) under the assumption that for at least one t_0 the derivatives

$$F_2'(F(x, t), t_0) \quad \text{and} \quad F_1'(x, t_0)$$

are different from 0 for all x and t. Without loss of generality, we can choose $t_0 = 0$ and suppose

$$\text{(15)} \quad F_2'(F(x, t), 0) \neq 0, \ F_1'(x, 0) \neq 0 \text{ for } x \in U, t \in I.$$

From (14) with $s = 0$ it follows that also

$$F_2'(x, 0) \neq 0$$

and

$$\text{(15')} \quad F_1'(F(x, 0), t) = \frac{F_2'(F(x, t), 0)}{F_2'(x, 0)} \neq 0$$

hold for $x \in U$, $t \in I$. Taking the derivative of (C) with respect to x we get

$$\text{(16)} \quad F_1'(F(x, s), t)F_1'(x, s) = F_1'(F(x, t), s)F_1'(x, t).$$

Putting $s = 0$ into (16) and taking (15′) into consideration we obtain

$$\frac{F_2'(F(x, t), 0)F_1'(x, 0)}{F_2'(x, 0)} = F_1'(F(x, t), 0)F_1'(x, t)$$

or

$$\frac{F_2'(F(x, t), 0)}{F_1'(F(x, t), 0)} = F_1'(x, t)\frac{F_2'(x, 0)}{F_1'(x, 0)}.$$

With the definition

$$G(x) = \frac{F_2'(x, 0)}{F_1'(x, 0)}$$

we see that F satisfies the third Jabotinsky equation

$$(3) \qquad \frac{\partial F(x, t)}{\partial x} G(x) = G(F(x, t))$$

whose solutions are given in Proposition 3.

Remark 6 (c). (Invariant transforms of differential equations, see [7]). Consider the following differential equation on a Banach space X

$$(17) \qquad \frac{dy}{dt} = G(y),$$

where G is defined on an open set $U \subseteq X$ with values in X and the solutions y have values in U dependent upon a real or complex variable t. By introducing a transformation $y = T(z)$, i.e., a diffeomorphism T from an open set $W \subseteq X$ onto U, equation (17) leads to

$$(18) \qquad \frac{dT(z)}{dz} \cdot \frac{dz}{dt} = G(T(z)).$$

PROPOSITION 7. *With y also z, defined by $y = T(z)$, is a solution of* (17) *if and only if*

$$(19) \qquad \frac{dT(z)}{dz} \cdot G(z) = G(T(z)).$$

This means that T is a solution of the Jabotinsky equation (3) where the parameter t does not appear explicitly.

Proof. If z is a solution of (17), then $dz/dt = G(z)$ together with (18) implies (19). Conversely, if (19) is fulfilled, then from (18) follows

$$\frac{dT(z)}{dz} \cdot \frac{dz}{dt} = \frac{dT(z)}{dz} \cdot G(z).$$

Since T is invertible, the derivative $dT(z)/dz$, which is a linear mapping from X to X, is an isomorphism for every $z \in W$, therefore

$$\frac{dz}{dt} = G(z).$$

As we have seen, the solutions of (3) are in general far from being solutions of the translation equation (T) (see 1.3), even when the initial condition (I) and the condition (4) are supposed.

As a weaker statement, in view of Remarks 6 (a) and 6 (b), one could conjecture that every parameter dependent solution F of (3) is a commuting map in the sense of Remark 6 (a). The results of 1.3 and [7] seem to confirm this conjecture. But the following representation of the general solution of (3) in the real one dimensional case shows that this conjecture does not hold when G is not different from zero at all points of its interval of definition (Examples 10, 12) even if G vanishes only at one point (Example 11 (c)).

THEOREM 8. *Let $G:U \to \mathbf{R}$ be continuous on the open interval $U \subseteq \mathbf{R}$ and let U_i be the at most countably many open intervals where $G(x) \neq 0$ and N_k the components of the set where $G(x) = 0$. Then all continuous solutions $F:U \times I \to U$ (differentiable in their first variable) of the equation*

$$(3) \qquad \frac{\partial F(x, t)}{\partial x} G(x) = G(F(x, t))$$

which satisfy $\partial F(x, t)/\partial x \neq 0$ for $x \in \cup U_j$, $t \in I$ have the form

$$F(x, t) = \begin{cases} f_j^{-1}(f_i(x) + g_i(t)) & \text{for } x \in U_i \\ h_k(x, t) & \text{for } x \in N_k \end{cases}$$

where f_i is a primitive function of $1/G$ in U_i, g_i is such that

$$f_i(U_i) + g_i(I) \subseteq f_j(U_j)$$

and h_k is an arbitrary continuous function (differentiable with respect to x) which has its values in the zero set $N = \cup N_k$ of G.

The functions g_i and h_k have to be chosen so that the function F defined in this way is continuous and differentiable with respect to x also at the boundary points of the intervals N_k.

(Of course, the h_k can be united into one function $(x, t) \mapsto h(x, t)$ on $N = \cup N_k$.)

Proof. For the proof of this theorem we can use the method of 1.3. Let F be a solution of (3) in $U \times I$ and consider $x \in U_i$, hence $G(x) \neq 0$ and, by the supposition

$$\frac{\partial F(x, t)}{\partial x} \neq 0,$$

(3) implies $G(F(x, t)) \neq 0$. Therefore, due to the continuity of F, there is a j such that $F(x, t) \in U_j$ for all $x \in U_i$ and $t \in I$. So, from (3) and from the definition of the f_i's we get

$$f_j'(F(x, t))\frac{\partial F(x, t)}{\partial x} = f_i'(x).$$

Therefore,

$$f_j(F(x, t)) = f_i(x) + g_i(t),$$

where g_i is a function with

$$f_i(x) + g_i(t) \in f_j(U_j) \quad \text{for } (x, t) \in U_i \times I,$$

which delivers the desired representation for $x \in U_i$. If $x \in N_k$ for some k then, by (3), $G(F(x, t)) = 0$ follows but $F(x, t)$ may take any value in the zero set N of $G(x)$ for $x \in N_k$ as long as it stays continuous and partially differentiable also at the boundaries.

On the other hand, every continuous function F partially differentiable in x which is defined in this way is a solution of (3).

Remark. Theorem 8 gives a method to construct solutions of equation (3). The main difficulties are to find conditions on the choice of the index j which is a function of i in the representation

$$F(x, t) = f_j^{-1}(f_i(x) + g_i(t))$$

for $x \in U_i$ and the choice of the functions g_i and h_k so that the constructed F is continuous and differentiable with respect to x, also at the boundary points of the U_i's and N_k's. Though the number of the intervals U_i for a given continuous G can at most be countable, the number of the N_k's (which are the components of the zero set N of G) in general can be uncountable. For instance, it is easy to construct a real function on the unit interval which is differentiable of any given order and takes the value zero on the Cantor discontinuum but is positive on the complement.

COROLLARY 9. *Under the same conditions as in Theorem* 8, *the continuous solutions* $F: U \times I \to U$ (*differentiable in the first variable*) *of equation* (3) *and the initial condition*

(I) $\quad F(x, 0) = x$

have the representation

$$F(x, t) = \begin{cases} f_i^{-1}(f_i(x) + g_i(t)) & \text{for } x \in U_i \\ h_k(x, t) & \text{for } x \in N_k \end{cases}$$

where the functions f_i, g_i *and* h_k *satisfy the same conditions as in Theorem* 8 *but, additionally*, $g_i(0) = 0$, $h_k(x, 0) = x$ *and* $h_k(x, t) \in N_k$ *for* $x \in N_k$, $t \in I$.

Proof. Only a slight modification of the proof of Theorem 8 is required. If $x \in U_i$ then by (I) $F(x, 0) = x \in U_i$, hence $F(x, t) \in U_i$ for all t, due to the continuity of F. Therefore, as in the proof of Theorem 8,

$$f_i'(F(x, t))\frac{\partial F(x, t)}{\partial x} = f_i'(x)$$

and

$$f_i(F(x, t)) = f_i(x) + g_i(t).$$

By (I), $g_i(0) = 0$. This yields the claimed representation of F for $x \in U_i$. For $x \in N_k$ the functions h_k have to satisfy $h_k(x, 0) = x$ and, due to the continuity of h_k, $h_k(x, t) \in N_k$ for $x \in N_k$, $t \in I$.

Note. If we want also (4) to hold then

$$g_i'(0) = 1 \quad \text{and} \quad \frac{\partial h_k}{\partial t}(x, 0) = 0 \quad \text{for all } i \text{ and } k.$$

Next we will give examples which illustrate how one can construct solutions of equation (3) using the method of Theorem 8. Some of them are not commuting although they satisfy also (I) and (4). This is shown, for instance, by the following example which has been communicated to the authors by Z. Moszner:

$$F(x, t) = x + x^2t^2$$

satisfies (3), (4) and (I) with $G(x) \equiv 0$ but is not a commuting map.

Another example, with a G not identically zero, is the following.

Example 10. Define

$$G(x) = \begin{cases} -x^2 & \text{for } x > 0 \\ 0 & \text{for } x \leqq 0 \end{cases}$$

which is continuously differentiable everywhere on $\mathbf{R}$. Then by Theorem 8 we get a representation for the continuous solutions of (3) with this G by

$$F(x, t) = \begin{cases} \dfrac{x}{1 + xg(t)} & \text{for } x > 0 \\ h(x, t) & \text{for } x \leqq 0 \end{cases}$$

where $1 + xg(t) > 0$ for $x > 0$ but g is an otherwise arbitrary continuous function, $h(x, t) \leqq 0$ for $x < 0$ and $h(0, t) = 0$ but h is an otherwise arbitrary continuous function, differentiable with respect to x.

If we choose

$$g(t) = t + t^2 \quad \text{and} \quad h(x, t) = x - x^2t^2$$

we get

$$F(x, t) = \begin{cases} \dfrac{x}{1 + x(t + t^2)} & \text{for } x > 0 \\ x - x^2t^2 & \text{for } x \leqq 0 \end{cases}$$

as a solution of (3) (with the above G) which is defined on the set

$$\{ (x, t) \in \mathbf{R} \times \mathbf{R} \mid x \leqq 0 \text{ or } [x > 0 \text{ and } 1 + x(t + t^2) > 0] \}$$

and differentiable there (also at $x = 0$). This function satisfies (I) and (4) but is not a solution of (C) because for $x \leqq 0$ we have $F(x, s) \leqq 0$ and

$$F(F(x, s), t) = x - x^2s^2 - x^2t^2 + 2x^3s^2t^2 - x^4s^4t^2,$$

which is not symmetric in s and t.

Here the domain of definition of F is not necessarily of the form $U \times I$. Examples 11 (b), (c), (e) will give non-commuting solutions with such rectangular domains.

Example 11 (a). Take $G(x) = x$ for $x \in \mathbf{R}$. According to Theorem 8 one gets

$$f(x) = \ln|x| \quad \text{for } x \neq 0$$

and

$$f^{-1}(x) = \pm\exp x$$

where the $\pm$ sign depends on the choice of the local inverse of $\ln|x|$. The general solution of equation (3) with $G(x) = x$ might be of the following form

$$F(x, t) = \begin{cases} \pm\exp(g_1(t))\, x & x > 0 \\ 0 & x = 0 \\ \pm\exp(g_2(t))\, x & x < 0 \end{cases}$$

where the functions g_1 and g_2 and the $\pm$ signs have to be chosen in such a way that $F(x, t)$ is continuous and partially differentiable in x (also at $x = 0$). For this purpose it is sufficient to demand that the partial derivatives

$$\frac{\partial F(x, t)}{\partial x} = \pm\exp(g_1(t)) \text{ for } x > 0$$

and

$$\frac{\partial F(x, t)}{\partial x} = \pm\exp(g_2(t)) \text{ for } x < 0$$

have the same limit when x tends to 0. Hence in both cases the same sign has to prevail and $g_1 \equiv g_2$. So we get the general solution of the equation

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$$\frac{\partial F(x, t)}{\partial x} x = F(x, t)$$

in the form

$$F(x, t) = \epsilon \exp(g(t))\, x \quad \text{for } x, t \in \mathbf{R}$$

where $\epsilon = +1$ or $\epsilon = -1$ and g is an arbitrary continuous real function. Each such solution is commuting.

Example 11 (b). Take $G(x) = -x^m/(m - 1)$ for integer $m \geqq 2$ (the factor $-(m - 1)^{-1}$ is there for sake of simplicity in the computation, afterwards we can omit it in equation (3)). By the same method, using the notations of Theorem 8,

$$U_1 = \{x \in \mathbf{R} \mid x > 0\}$$
$$U_2 = \{x \in \mathbf{R} \mid x < 0\},$$

we get

$$f_{1,2}(x) = x^{-(m-1)} \quad x \neq 0.$$

If $m - 1$ is odd, then as inverse of f_1 and f_2 we get the unique $(m - 1)$-st real root function

$$f_{1,2}^{-1}(y) = y^{-1/(m-1)}.$$

In the case that $m - 1$ is even, we have $f_{1,2}(x) > 0$ for $x \neq 0$, thus for $y > 0$

$$f_1^{-1}(y) = y^{-1/(m-1)}$$
$$f_2^{-1}(y) = -y^{-1/(m-1)}$$

where $y^{-1/(m-1)}$ denotes the unique positive $(m - 1)$-st real root function for positive real y.

Combining the cases $m - 1$ odd and $m - 1$ even, we get for $x \in U_1$ i.e., $x > 0$, if g_1 is chosen so that $1 + x^{m-1}g_1(t) > 0$ for $x > 0$,

$$\begin{aligned} F(x, t) &= f_j^{-1}(x^{-(m-1)} + g_1(t)) \\ &= \frac{f_j^{-1}(x^{-(m-1)})}{(1 + x^{m-1}g_1(t))^{1/(m-1)}} \\ &= \rho_1 x(1 + x^{m-1}g_1(t))^{-1/(m-1)} \end{aligned}$$

where ρ_1 is an $(m - 1)$-st real root of unity, either $+1$ or -1 depending on the choice of j.

Similarly, for $x \in U_2$, i.e., $x < 0$ we get

$$F(x, t) = \rho_2 x(1 + x^{m-1}g_2(t))$$

(also ρ_2 is an $(m - 1) - st$ real root of unity, either $+1$ or -1, and g_2 satisfies $1 + x^{m-1}g_2(t) > 0$ for $x < 0$). The function F is differentiable in the variable x at $x = 0$ if and only if

$$\lim_{x\to 0+} \frac{\partial F(x, t)}{\partial x} = \rho_1 \quad \text{and} \quad \lim_{x\to 0-} \frac{\partial F(x, t)}{\partial x} = \rho_2$$

are equal so we have $\rho_1 = \rho_2 = \rho$.

Therefore the general solution of the third Jabotinsky equation

$$\frac{\partial F(x, t)}{\partial x}x^m = F(x, t)^m, \quad m \geqq 2 \tag{20}$$

is given by

$$F(x) = \begin{cases} \rho x(1 + x^{m-1}g_1(t))^{-1/(m-1)} \\ \rho x(1 + x^{m-1}g_2(t))^{-1/(m-1)} \end{cases} \tag{21}$$

where ρ is an $(m - 1)$-st real unit root either $+1$ or -1, g_1 and g_2 are continuous functions with

$$\begin{matrix} 1 + x^{m-1}g_1(t) > 0 & x > 0 \\ 1 + x^{m-1}g_2(t) > 0 & x < 0 \end{matrix} \tag{22}$$

but otherwise arbitrary.

So, for example, for every continuous g_1 and g_2 with $g_1(t) \geqq 0$, $t \in \mathbf{R}$ and $g_2(t) \geqq 0$ if $m - 1$ is even or $g_2(t) \leqq 0$ if $m - 1$ is odd, respectively, and $t \in \mathbf{R}$, the representation (21) gives a solution of (20) which is defined on $\mathbf{R} \times \mathbf{R}$ and $(m - 1)$ times differentiable with respect to x.

None of these solutions is commuting if ρ can be chosen as -1 (i.e., $m - 1$ is even) and $g_1 \neq g_2$. In detail this will be discussed in the following example.

Example 11 (c). Now consider the solutions of (20) of Example 11 (b) in the representation (21) which satisfy the differential initial condition (4) with

$$G(x) = -x^m/(m - 1),$$

that is, the solutions of (20) with

$$G(x) = \left.\frac{\partial F}{\partial t}\right|_{t=0} = -x^m/(m - 1).$$

Here g_1 and g_2 have to be differentiable at $t = 0$.

Then for F, given by (21),

$$g_i(0) = 0 \quad \text{and} \quad g_i'(0) = \rho \quad \text{for } i = 1, 2 \tag{23}$$

are necessary and sufficient conditions that this solution fulfills the equation

$$(4') \quad \left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0} = -\frac{x^m}{m-1}.$$

In this case the interval U, where x varies, must be bounded, for otherwise there exist no functions g_1 and g_2 which satisfy (22) and (23).

We give a detailed example of such a solution which is not commuting. Take $m = 2k + 1$ with natural k then ρ can be $+1$ or -1. We choose $\rho = -1$ and $g_1(t) = -\sin t$, $g_2(t) = -1/2 \sin 2t$, so (23) is fulfilled.

$$F(x, t) = \begin{cases} -x(1 - x^{2k} \sin t)^{-1/2k} & \text{for } x > 0 \\ -x\left(1 - \dfrac{x^{2k}}{2} \sin 2t\right)^{-1/2k} & \text{for } x \leqq 0 \end{cases}$$

is defined at least for $-1 < x < 1$ and $t \in \mathbf{R}$. For $0 < x < 1/2$ we have $F(x, t) < 0$, $F(F(x, t), s)$ is defined, hence

$$F(F(x, t), s) = x\left(1 - x^{2k} \sin t - \frac{x^{2k}}{2} \sin 2s\right)^{-1/2k}$$

which is unequal $F(F(x, s), t)$. So this is an example for a noncommuting solution of the third Jabotinsky equation which satisfies (4) with $G(x) = -x^m/(m - 1)$. In this case $G(x) = 0$ only at the point $x = 0$.

The following is a significant example of a commuting solution of the third Jabotinsky equation of the form (20).

Example 11 (d). Consider, as in Example 11 (b), a solution of equation (3) with $G(x) = -x^m/(m - 1)$ that is, of (20) on $\mathbf{R}_+$:

$$F(x, t) = \rho x(1 + x^{m-1}g(t))^{-1/(m-1)}.$$

We can use the Taylor expansion for the root function

$$(24) \quad F(x, t) = \rho x \sum_{j=0}^{\infty} \binom{-1/(m-1)}{j} x^{(m-1)} g(t) j$$

$$= \rho x - \frac{\rho}{m-1} x^m g(t) + \dots .$$

This series converges for all $(x, t) \in \mathbf{R} \times \mathbf{R}$ with

$$|x^{m-1}g(t)| < 1.$$

It represents a function analytic in x which is a solution of (20) for real positive x and, by the principle of permanence of functional relations, it remains a solution of (20) on the whole domain where this function has an analytic continuation (even for analytic continuation on the complex domain). This analytic solution is commuting and satisfies (I) if and only if $\rho = 1$ and $g(0) = 0$, it satisfies (4), in this case (4′), if and only if

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$g(0) = 0$ and $g'(0) = \rho$. The function (24) is a solution of the translation equation (T) if and only if $\rho = 1$ and g is an additive function.

The next example once more illustrates the method of Theorem 8 and gives an example of a solution of (3) where the zero set of G is infinite but nowhere dense and the solution is defined on the entire plane $\mathbf{R} \times \mathbf{R}$ but is not commuting.

Example 12. Take $G(x) = \cos^2 x$ for $x \in \mathbf{R}$. The intervals U_ℓ where $G(x) \neq 0$ are

$$U_\ell = \left]-\frac{\pi}{2} + \ell\pi, \frac{\pi}{2} + \ell\pi\right[\quad \text{for integer } \ell.$$

The primitive function of $1/\cos^2 x$ is $\tan x$, therefore

$$f_\ell(x) = \tan x \quad \text{for } x \in U_\ell.$$

For its inverse we may take

$$f_\ell^{-1}(y) = \operatorname{Arc\,tan} y + \ell\pi$$

where Arc tan is the principal branch of arc tan, that is Arc tan $\mathbf{R} \subseteq U_0$. As a solution of (3) we choose

$$(25) \quad F(x, t) = \begin{cases} f_{\ell+1}^{-1}(f_\ell(x) + g_\ell(t)) & x \in U_\ell \\ x + \pi & x = \dfrac{\pi}{2} + k \cdot \pi \end{cases}$$

where the g_ℓ are arbitrary continuous functions and the k's are integers. The function F defined in this way is continuous on $\mathbf{R} \times \mathbf{R}$. It is partially differentiable in x for $x \in U_\ell$

$$\frac{\partial F(x, t)}{\partial x} = \frac{1}{\cos^2 x + (\sin x + \cos x g_\ell(t))^2}$$

and, since F is continuous also at the boundaries of U_ℓ and $\partial F/\partial x$ tends there to 1, the partial derivative exists on these boundaries too, and is equal to 1.

The function F satisfies also (4):

$$\left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0} = \cos^2 x \quad \text{for } x \in \mathbf{R}$$

if and only if $g_\ell(t)$ is differentiable at $t = 0$ and $g_\ell(0) = 0$ and $g_\ell'(0) = 1$ for all integers ℓ.

For $x \in U_\ell$ we have $F(x, t) \in U_{\ell+1}$ hence

$$\begin{aligned} F(F(x, t), s) &= f_{\ell+2}^{-1}(f_{\ell+1}[f_{\ell+1}^{-1}(f_\ell(x) + g_\ell(t))] + g_{\ell+1}(s)) \\ &= f_{\ell+2}^{-1}(f_\ell(x) + g_\ell(t) + g_{\ell+1}(s)), \end{aligned}$$

which is different from $F(F(x, s), t)$ if $g_\ell(t) \neq g_{\ell+1}(t)$. Therefore it suffices to choose the functions g_ℓ so that they are continuous, differentiable at $t = 0$, $g_\ell(0) = 0$, $g'_\ell(0) = 1$ for all integers ℓ and $g_j \neq g_{j+1}$ for at least one j. (For example, take $g_\ell(t) = (1/\ell) \cdot \sin \ell t$ for $\ell = \pm 1, \pm 2, \ldots$ and $g_0(t) = t$.)

As one can see form the examples above, in all those cases where the solutions of the third Jabotinsky equation do not commute, the zero set of G either contains a proper interval or the initial condition (I) does not hold.

We give now another positive result.

PROPOSITION 13. *Let $G: U \rightarrow \mathbf{R}$ be continuous, U_i the open intervals of U where $G(x) \neq 0$, N_k be the components of the set N where $G(x) = 0$. If $F: U \times I \rightarrow U$ is a solution of* (3) *and* (I), *as given in Corollary* 9, *and N is totally disconnected then F is commuting.*

Proof. Using the notations of Corollary 9 and Theorem 8 we have to consider two cases for $x \in U$.

(i) $x \in U_i$, then

$$F(x, t) = f_i^{-1}(f_i(x) + g_i(t)) \in U_i,$$

hence

$$F(F(x, t), s) = f_i^{-1}(f_i(x) + g_i(t) + g_i(s))$$
$$= F(F(x, s), t)$$

(ii) $x \in N_k$ for some N_k. Then by Corollary 9

$$F(x, t) \in N_k \text{ for all } t$$

follows. Since N is totally disconnected, N_k contains only the one point x, therefore

$$F(x, t) = x \text{ for all } t.$$

So (C) holds in this case too.

Remark. As Z. Moszner has kindly informed us, he has proved a similar theorem for $U = \mathbf{R}$, where the condition that N is totally disconnected is replaced by the assumption that the closure of $\mathbf{R}\backslash N$ is $\mathbf{R}$. Since the function G in (3) is supposed to be continuous, N is closed, hence closure $(\mathbf{R}\backslash N) = \mathbf{R}$ implies that N is totally disconnected.

Finally we give an example of a noncommuting solution of (3) in higher dimensions where $G(x)$ vanishes only at one point and the solution satisfies (4) and (I) too.

Example 14. Consider (3) in $\mathbf{R}^n$, $G(x) = x$ where

$$x = {}^t(x_1, \dots, x_n) \quad \text{and} \quad F(x, t) = {}^t(F_1(x, t), \dots, F_n(x, t))$$

are column vectors of $\mathbf{R}^n$. The derivative of F is the Jacobian

$$\frac{\partial F(x, t)}{\partial x} = \left(\frac{\partial F_i(x, t)}{\partial x_j}\right)_{i,j=1,\dots,n}.$$

Let

$$A(t) = \sum_{i=0}^{\infty} A_i \cdot t^i$$

be an analytic n by n matrix. Then $F(x, t) = A(t) \cdot x$ is a solution of the third Jabotinsky equation (3) in $\mathbf{R}^n$ with $G(x) = x$

$$\text{(26)} \quad \frac{\partial F(x, t)}{\partial x} \cdot x = F(x, t).$$

This F satisfies (I) if and only if $A_0 = E$ (the unit matrix), and it satisfies (4) if and only if $A_1 = E$. The function

$$F(x, t) = \left(\sum_{i=0}^{\infty} A_i \cdot t^i\right) \cdot x$$

is commuting if and only if

$$A_i \cdot A_j = A_j \cdot A_i \text{ for all } i \text{ and } j.$$

So, for example,

$$F(x, t) = x + t \cdot x + A_2 \cdot t^2 \cdot x + A_3 \cdot t^3 \cdot x$$

is a solution of (26), (4) and (I). But if $A_2 \cdot A_3 \neq A_3 \cdot A_2$ then this F does not satisfy (C).

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