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NEW FOUNDATIONS OF REASONING VIA REAL-VALUED FIRST-ORDER LOGICS

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Abstract. Many-valued logics in general, and real-valued logics in particular, usually focus on a notion of consequence based on preservation of full truth, typically represented by the value 1 in the semantics given in the real unit interval $[0, 1]$. In a recent paper (Foundations of Reasoning with Uncertainty via Real-valued Logics, *Proceedings of the National Academy of Sciences* 121(21): e2309905121, 2024), Ronald Fagin, Ryan Riegel, and Alexander Gray have introduced a new paradigm that allows to deal with inferences in *propositional* real-valued logics based on a rich class of sentences, multi-dimensional sentences, that talk about combinations of any possible truth values of real-valued formulas. They have proved a strong completeness result that allows one to derive exactly what information can be inferred about the combinations of truth values of a collection of formulas given information about the combinations of truth values of a finite number of other collections of formulas. In this paper, we extend that work to the first-order (as well as modal) logic of multi-dimensional sentences. We give a parameterized axiomatic system that covers any reasonable logic and prove a corresponding completeness theorem, first assuming that the structures are defined over a fixed domain, and later for the logics of varying domains. As a by-product, we also obtain a zero-one law for finitely-valued versions of these logics. Since several first-order real-valued logics are known not to have recursive axiomatizations but only infinitary ones, our system is by force akin to infinitary systems.

§1. Introduction. Typically the study of inference in many-valued logic answers the following question: given that all premises in a given set Γ are *fully* true, what other formulas γ can we see to be fully true as a consequence? This standard approach can be deemed unsatisfying because, when it comes to valid inference, it disregards almost all of the rich structure of truth values and concentrates only on preservation of the value 1 (or on the preservation of a set of designated truth values [13, 34]). A natural question involving all possible truth values would be instead: what information can be inferred about the combinations of truth values of a collection of formulas given information about the combinations of truth values of a finite number of other collections of formulas?

In fact, the recent paper [24] poses the above question not just for sets of single formulas but for sequences of *propositional* formulas taking any combinations of truth values considered as a single expression called a *multi-dimensional sentence* (in short, an MD-sentence). More precisely, an MD-sentence is a syntactic object of the form

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$\langle \sigma_1, \ldots, \sigma_k; S \rangle$ where S (called the *information set*) is a set of k-tuples of truth values for the sequence of formulas $\sigma_1, \ldots, \sigma_k$ (called the *components*). The semantic intuition is that $\langle \sigma_1, \ldots, \sigma_k; S \rangle$ should be true in an interpretation if the sequence of truth values that $\sigma_1, \ldots, \sigma_k$ take in that interpretation is one of the k-tuples in S.[1] The simplest case of MD-sentences so defined are those $\langle \sigma; S \rangle$ where σ is a single propositional formula and S is a set of truth values from $[0, 1]$, e.g. a singleton, an interval, a union of intervals or the rational numbers in $[0, 1]$.

In the context of fuzzy set theory, Pavelka introduced in [41] a formal system with fuzzy sets of axioms, many-valued inference rules. In this system, every formal proof comes with a degree, so, on one hand, Pavelka defined the *provability degree* of a formula as the supremum of the degrees of all its proofs. On the other hand, he defined the *truth degree* of a formula as the infimum of the set of values that it takes in each model. Then, he proved, as a generalization of the completeness theorem of classical logic, that these two degrees coincide for each formula. Subsequently, Vilém Novák extended Pavelka's logic and its completeness result to a first-order language in [36] and greatly developed this approach with the theory of fuzzy logic with evaluated syntax [37–39]. Petr Hájek gave in [28] a (partial) representation of fuzzy logic with evaluated syntax by means of an expansion of Lukasiewicz logic with a language enriched with a truth-constant $\overline{r}$ for each real number $r \in [0, 1]$ (he later showed that it sufficed to consider rational numbers and called the resulting system *Rational Pavelka logic*) and additional axioms, and proved a Pavelka-style completeness theorem that showed the equality of provability and truth degree for each formula. The enriched language of these systems allows to write sentences of the form $\overline{r} \to \varphi$ (which semantically means that the truth value of φ is at least r) and $\varphi \to \overline{s}$ (which semantically means that the truth value of φ is at most s) and hence allows to stipulate in a syntactical manner that the truth value that the formula φ has to take in a model belongs to a certain closed interval defined by rational numbers (or a union thereof). It is not clear whether this syntax also allows to express that the value of φ belongs to any arbitrary given subset S of $[0, 1]$, which instead can be directly expressed by design using MD-sentences of the form $\langle \varphi; S \rangle$.

The new approach to many-valued logics based on MD-sentences is relevant for AI due to the growing interest in any development that may contribute to augmenting the capabilities of learning-based methods in combination with reasoning methods, resulting in an integration that has been branded *neuro-symbolic*. In this setting, the expressive power of classical logic, with its defining restriction to crisp notions (that is, the bivalence principle that assumes every meaningful statement to be either completely true or completely false), becomes insufficient for the crucial goal of representing uncertain or vague knowledge and conclusions. Hence, several recent neuro-symbolic approaches employ real-valued logics instead, as one can see e.g. in logic tensor networks [6], probabilistic soft logics [2], Tensorlog [16], or Logical Neural Networks [32, 42].

[1] Observe that having this new language is not the same as simply having a wider collection of designated truth values and studying inference in that setting in the usual truth-preserving way. The general point is that the freedom afforded by selecting S arbitrarily lets us consider inferences that relate certain formulas having certain truth values to other formulas having other truth values in a totally unrestricted manner. For example, if we take the MD-sentence $\langle A, B, A\&B; [0, 1]^3 \rangle$ with semantics given by Product Logic (where $\&$ will be interpreted as the product t-norm) we might want to infer $\langle A, B, A\&B; S \rangle$ where $S \subseteq [0, 1]^3$ is the set of all triples $\langle s_1, s_2, s_3 \rangle$ such that $s_3 = s_1 \cdot s_2$.

Following these motivations, the goal of [24] was to axiomatize inference genuinely involving many truth values. The authors indeed have provided an axiomatization in terms of MD-sentences in a parametrized way that captures all of the most common *propositional* fuzzy logics and even logics that do not obey some standard restrictions (such as conjunction being commutative). However, many reasoning scenarios cannot be properly modeled only with the formal tools of a propositional language and need a more expressive setting. In fact, Logical Neural Networks (LNNs) are AI models that can only be properly formalized by means of the first-order MD-formulas that we introduce here. Most interesting reasoning problems for which one might wish to use LNNs require the expressive power of first-order logic (see the examples in [32, 42]), making the propositional formalism insufficient. Therefore, in the present article, we generalize the work in [24] to the *first-order* and *modal* contexts. Since it is already known that first-order and modal real-valued logics are not necessarily recursively enumerable for validity [43, 46] and one needs instead infinitary systems [29, 33] to deal with them,[2] our proposal is going to be necessarily more akin in applicability to an infinitary system than a finitary one. In the applications discussed for LNNs all one actually needs is a fixed *finite* domain (the universe of objects of a knowledge base), in which case one recovers recursivity (Remark 14).

The article is arranged as follows. First, in § 2, we give a fast overview of the necessary notions and results that we borrow from the propositional case studied in [24]. In § 3 we study the first-order (as well as modal) logic of multi-dimensional sentences (generalizing the definition of [24]) when the models considered all have the same fixed domain (which may be of any fixed cardinality, either finite or infinite). The key result is a completeness result that follows the strategy of that in [24] for the propositional case. In § 4 we show how our approach leads to parameterized axiomatizations of the valid finitary inferences of many prominent first-order real-valued logics. Since this includes several logics that are not recursively enumerable for validity, our system in general does not yield a recursive enumeration of theorems. In §5, we prove a zero-one law for finitely-valued versions of the logics dealt with in § 3. Finally, in § 6 we remove the restriction of a fixed domain and provide a completeness theorem for the first-order logic of multi-dimensional sentences on arbitrary domains.

§2. The propositional case: an overview. This section presents a brief summary of the key results and notions from [24]. Following that article, we take a (propositional) *multi-dimensional sentence* (in symbols, an MD-sentence) to be an expression of the form $\langle \sigma_1, \ldots, \sigma_k; S \rangle$ where $S \subseteq [0, 1]^k$. For a fixed k, we may speak of k-dimensional sentences.

The semantics of MD-sentences is as follows. By a *model* we mean an assignment $\mathfrak{M}$ from atomic sentences (propositional variables) of a propositional language $\mathcal{L}$ to truth values from $[0, 1]$. The usual real-valued logics (Lukasiewicz, Product, Gödel, etc.) all have inductive definitions indicating how to assign values to all formulas and hence the notion of the value of an arbitrary formula in the language $\mathcal{L}$ in a given model $\mathfrak{M}$ is well-defined. Fixing one such semantics (which means we will get different outcomes depending on the real-valued logic being considered), for an MD-sentence $\langle \sigma_1, \ldots, \sigma_k; S \rangle$, we say that $\mathfrak{M}$ *satisfies* this sentence (in symbols, $\mathfrak{M} \models \langle \sigma_1, \ldots, \sigma_k; S \rangle$)

[2] Infinitary presentations are not uncommon even for *propositional* many-valued logics, see e.g. [9, 31].

if $\langle s_1, \dots s_k\rangle \in S$ where s_i $(1 \leq i \leq k)$ is the value in $\mathfrak{M}$ of σ_i according to the semantics of the real-valued logic under consideration. Finally, given a set $\Gamma \cup \{\gamma\}$ of MD-sentences, we write $\Gamma \vDash \gamma$ if every model that satisfies all the sentences in Γ also satisfies γ; in this case we call '$\Gamma \vDash \gamma$' a *valid inference*.

Given these definitions one can consider Boolean combinations of MD-sentences. For example, take $\gamma_1 := \langle \sigma_1^1, \dots, \sigma_n^1; S_1\rangle$ and $\gamma_2 := \langle \sigma_1^2, \dots, \sigma_m^2; S_2\rangle$. Then, we may say that $\mathfrak{M} \models \gamma_1 \wedge \gamma_2$ iff $\mathfrak{M} \models \gamma_1$ and $\mathfrak{M} \models \gamma_2$. An interesting result from [24] is that MD-sentences are closed under Boolean combinations, in the sense that for any Boolean combination of such sentences there is an MD-sentence equivalent to such combination. Hence, the collection of MD-sentences is expressively quite robust.

Example 1. An easy example of a valid MD-sentence in, say, Gödel semantics, is the 3-dimensional sentence $\langle A, B, A \vee B; S\rangle$ where S is the set of all triples $\langle s_1, s_2, s_3\rangle$ where $s_1, s_2 \in [0, 1]$ and s_3 is the maximum of the set $\{s_1, s_2\}$.

Now it is natural to try to build a calculus that will capture exactly the valid finitary inferences involving MD-sentences. This is what we do next.
Axioms. We have only one axiom schema:

(1) $\langle \sigma_1, \dots, \sigma_k; [0, 1]^k\rangle$.

Observe that (1) is an axiom schema. That is, for example, $\langle p \wedge q, p \to r; [0, 1]^2\rangle$, $\langle p \vee (q \to r); [0, 1]\rangle$, and $\langle p, q, r; [0, 1]^3\rangle$ are all axioms. The idea of the schema is simply to assert that formulas always take some truth values.
Inference rules.

(2) From $\langle \sigma_1, \dots, \sigma_k; S\rangle$ infer $\langle \sigma_{\pi(1)}, \dots, \sigma_{\pi(k)}; S'\rangle$,

where $S' = \{\langle s_{\pi(1)}, \dots, s_{\pi(k)}\rangle \mid \langle s_1, \dots, s_k\rangle \in S\}$ and π is a permutation of $1, \dots, k$.

(3) From $\langle \sigma_1, \dots, \sigma_k; S\rangle$ infer

$$\langle \sigma_1, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_m; S \times [0, 1]^{m-k}\rangle.$$

(4) From $\langle \sigma_1, \dots, \sigma_k; S_1\rangle$ and $\langle \sigma_1, \dots, \sigma_k; S_2\rangle$ infer $\langle \sigma_1, \dots, \sigma_k; S_1 \cap S_2\rangle$.

(5) For $0 < r < k$, from $\langle \sigma_1, \dots, \sigma_k; S\rangle$ infer $\langle \sigma_1, \dots, \sigma_{k-r}; S'\rangle$, where $S' = \{\langle s_1, \dots, s_{k-r}\rangle \mid \langle s_1, \dots, s_k\rangle \in S\}$.

(6) From $\langle \sigma_1, \dots, \sigma_k; S\rangle$ infer $\langle \sigma_1, \dots, \sigma_k; S'\rangle$, when $S \subseteq S'$.

At this point, let us make a clarification about rule (4). In (4), $S_1 \cap S_2$ could, naturally, be empty. A very trivial example would be if we have the MD-sentences $\langle p; \{0.2\}\rangle$ and $\langle p; \{0.3\}\rangle$ for then $\{0.2\} \cap \{0.3\} = \emptyset$. This means that, if we have the MD-sentences $\langle p; \{0.2\}\rangle, \langle p; \{0.3\}\rangle$, we can infer the contradictory (in the sense of having no model) MD-sentence $\langle p; \emptyset\rangle$. Thus the set $\{\langle p; \{0.2\}\rangle, \langle p; \{0.3\}\rangle\}$ has itself no model.

Finally, before we introduce the last rule, let us define a piece of notation. For any j-ary connective $\circ$, from a real-valued logic and real numbers $s_1, \dots, s_j$ from $[0, 1]$ we can define the function $\hat{\circ}(s_1, \dots, s_j)$ giving as output what the connective $\circ$ indicates in a given real-valued logic for the values $s_1, \dots, s_j$. Given an MD-sentence $\langle \sigma_1, \dots, \sigma_k; S\rangle$, we say that a tuple $\langle s_1, \dots, s_k\rangle \in S$ is *good* if $s_m = \hat{\circ}(s_{m_1}, \dots, s_{m_j})$ whenever $\sigma_m = \circ(\sigma_{m_1}, \dots, \sigma_{m_j})$ (for any m_j-ary connective $\circ$ and for any m). In other words, a tuple of truth values in an MD-sentence is good if it respects the semantics under consideration of the connectives appearing in the MD-sentence (recall that for any real-valued logic we are fixing the semantics of the connectives). Notice that this is a local property of each tuple in S, in the sense that it does not depend on what other tuples are in the information set. Now, the last inference rule is

(7) From $\langle \sigma_1, \dots, \sigma_k; S\rangle$ infer $\langle \sigma_1, \dots, \sigma_k; S'\rangle$, where S' is the set of good tuples in S.

If there are no good tuples in S, then of course $S' = \emptyset$, and thus the formula we started with in the rule cannot have a model as it does not respect the semantics of the underlying real-valued logic.

A *proof* of an MD-sentence γ from a set Γ of MD-sentences in this system consists, as usual, of a finite sequence of MD-sentences such that the last member is γ and every element of the sequence is either an axiom, one of the member of Γ, or it follows from previous elements by one of the inference rules. We write $\Gamma \vdash \gamma$ to indicate that there exists a proof of γ from Γ.

The central result from [24] states that if Γ is a *finite* set of MD-sentences, we have that $\Gamma \vdash \gamma$ is equivalent to $\Gamma \vDash \gamma$. It is noteworthy that this technique provides a parameterized way of building calculi for MD-sentences with semantics for the standard real-valued logics (where the parameters give a particular semantic meaning to the connectives of the language); special extra steps need to be taken for the logic of probabilities, as discussed in [24]. The restriction to finite sets is necessary due to the finitary character of Lukasiewicz logic [28]. Finally, in [24] a decision procedure for validity in this system of MD-sentences for Gödel and Lukasiewicz semantics is introduced. Furthermore, the algorithm of the procedure is implemented and tested on various interesting cases.

Remark 2. Observe that there is nothing sacred about the t-norm algebras on $[0, 1]$: everything that has been said here could have been said for logics based on arbitrary fixed residuated lattices (see e.g. [28]). The reader could attempt to check this by themselves noticing that the definitions we have introduced only make use of algebraic properties of t-norm algebras on $[0, 1]$ that easily generalize to other lattice structures. This remark similarly applies to the remainder of this article.

§3. The logic of a fixed domain.

Throughout this section, let M be any fixed set, *finite* or *infinite*. Observe that for finite fixed domains, by means of eliminating quantifiers (turning a universal quantifier into a big conjunction and turning an existential quantifier into a big disjunction), we could use an approach that essentially reduces the problem to what was done in [24]. We work with a first-order relational vocabulary τ to simplify things (but everything we do can be adjusted to accommodate function and constant symbols).

3.1. First-order case (the logic of a fixed domain).

This part is devoted to provide an axiomatization of the logic of a fixed domain M (of any cardinality), in the sense of the valid inferences over all models with domain M.

Let us first give the basic notions for the semantics of real-valued first-order logics.

Definition 3. Given a vocabulary τ, a real-valued first-order *model* $\mathfrak{M}$ is a structure $\langle M, \langle R_{\mathfrak{M}}\rangle_{R\in\tau}\rangle$, where $M \neq \emptyset$ is called the *domain* and for an n-ary predicate $R \in \tau$, its interpretation in $\mathfrak{M}$ is a mapping $R_{\mathfrak{M}} \colon M^n \longrightarrow [0, 1]$.

Inductively, using the semantics of the real-valued logic in question, one can define the truth value of any formula for a sequence $\overline{a}$ of elements from M and write it as $\|\varphi[\overline{a}]\|_{\mathfrak{M}}$:

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- $\|P[\overline{a}]\|_{\mathfrak{M}} = P_{\mathfrak{M}}(\overline{a})$, for each $P \in Pred_\tau$;
- $\| \circ (\varphi_1, \dots, \varphi_n)[\overline{a}]\|_{\mathfrak{M}} = \hat{\circ}(\|\varphi_1[\overline{a}]\|_{\mathfrak{M}}, \dots, \|\varphi_n[\overline{a}]\|_{\mathfrak{M}})$, for n-ary connective $\circ$;
- $\|(\forall x)\varphi[\overline{a}]\|_{\mathfrak{M}} = \inf\{\|\varphi[\overline{a}, e]\|_{\mathfrak{M}} \mid e \in M\}$;
- $\|(\exists x)\varphi[\overline{a}]\|_{\mathfrak{M}} = \sup\{\|\varphi[\overline{a}, e]\|_{\mathfrak{M}} \mid e \in M\}$.[3]

Whenever the vocabulary includes the equality symbol $\approx$, its semantics is defined in the following way:

- $\|(x \approx y)[d, e]\|_{\mathfrak{M}} = 1$ iff $d = e$, for any $d, e \in M$.
- $\|(x \approx y)[d, e]\|_{\mathfrak{M}} = 0$ iff $d \neq e$, for any $d, e \in M$.

The definition of the truth value of a quantified formula as the infimum or the supremum of the truth values of its instances is customary in many-valued logics as a natural generalization of the semantics of quantifiers in classical logic.

A formula $\varphi(x_1, \dots, x_n)$ can be said to be *interpreted* in the model $\mathfrak{M}$ by the mapping $f_\varphi \colon M^n \longrightarrow [0, 1]$ defined as $\langle a_1, \dots, a_n\rangle \mapsto \|\varphi[a_1, \dots, a_n]\|_{\mathfrak{M}}$ (we also say that $\varphi(x_1, \dots, x_n)$ *defines* the mapping f_φ in the model $\mathfrak{M}$).

Now we can define the set $\mathrm{MD}(M)$ of ***MD-sentences*** with domain M. Given a natural number n, we denote by $[0, 1]^{M^n}$ the set of all functions from M^n to $[0, 1]$. Let $\mathrm{MD}(M)$ contain all sentences of the form $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ where $\overline{x}_{\varphi_i} := x_{i_1}, \dots, x_{i_{n_i}}$, and $S \subseteq [0, 1]^{M^{n_1}} \times \dots \times [0, 1]^{M^{n_k}}$. In the expression $\varphi_i(\overline{x})$, the free variables of φ_i (if any) will be exactly those in the list $\overline{x}_{\varphi_i}$. When $\overline{x}_{\varphi_i}$ is empty, φ_i is a sentence and what it gets assigned in a given S is simply a nullary function, in other words, an element of $[0, 1]$, as in the propositional case. If none of the formulas φ_i in the MD-sentence $\langle \varphi_1, \dots, \varphi_k; S\rangle$ contains free variables, then the situation is exactly as in the propositional case [24] and there is no need to mention in S the set M.

Example 4. Take a vocabulary τ with only two unary predicates P and U. Then, we can build the sentence $\langle Px, (\forall x)Ux; S\rangle$ where $S = \{\langle f, r\rangle \mid r \in [0.5, 0.8)$, f is a mapping with domain M and range included in the set $[0, 1]\}$. Intuitively, we want this sentence to be satisfied in a model $\mathfrak{M}$ with domain M if the truth value of $(\forall x)Ux$ is a real number in the interval $[0.5, 0.8)$ and the interpretation of the predicate P is a mapping from M into $[0, 1]$.

Next, take a sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$. Then, we may write

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$

if $\langle f_{\varphi_1}, \dots, f_{\varphi_k}\rangle \in S$. Notice that, if any of the φ_is is a sentence, then the corresponding f_{φ_i} is a constant function. If all the φs are sentences, this definition basically boils down to what appears in [24].

We introduce now a proof system associated to the domain M, called the ***MD-system of*** M, by considering the axioms and inference rules given in [24] for the propositional case and modifying only what is needed:

Axioms. We have only one axiom schema:

(1) $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}), [0, 1]^{M^{n_1}} \times \dots \times [0, 1]^{M^{n_k}}\rangle$ for all formulas $\varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k})$.

[3] The interpretation of universal quantifiers (resp. existential) as the infimum (resp. supremum) of the truth values of their instances can be traced back to Mostowski [35]; see e.g. [15, 28] for general studies of first-order many-valued logics that follow this idea.

Inference rules.

(2) From
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$
infer
$$\langle \varphi_{\pi(1)}(\overline{x}_{\varphi_{\pi(1)}}), \dots, \varphi_{\pi(k)}(\overline{x}_{\varphi_{\pi(k)}}); S'\rangle,$$
where $S' = \{\langle f_{\pi(1)}, \dots, f_{\pi(k)}\rangle \mid \langle f_1, \dots, f_k\rangle \in S\}$ and π is a permutation of $1, \dots, k$.

(3) From
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$
infer
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}), \varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \dots, \varphi_m(\overline{x}_{\varphi_m}); S \times [0,1]^{M^{n_{k+1}}} \times \dots \times [0,1]^{M^{n_m}}\rangle.$$

(4) From
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S_1\rangle$$
and
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S_2\rangle$$
infer
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S_1 \cap S_2\rangle.$$

(5) For $0 < r < k$, from
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$
infer
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_{k-r}(\overline{x}_{\varphi_{k-r}}); S'\rangle,$$
where $S' = \{\langle f_1, \dots, f_{k-r}\rangle \mid \langle f_1, \dots, f_k\rangle \in S\}$.

(6) From
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$
infer
$$\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S'\rangle$$
where $S \subseteq S'$.

Finally, before we introduce the last rule, let us define a piece of notation. Consider an arbitrary domain M and functions $f_1, \dots, f_j$ from some Cartesian products of M into $[0,1]$. Then, for any j-ary connective $\circ$ from a real-valued logic, we can define the function $\circ(f_1, \dots, f_j)$ as taking arguments componentwise as indicated by the output of the f_is ($i \in \{1, \dots, j\}$) and giving as output what $\circ$ indicates. Also, we need to generalize also the notion of good tuple. Indeed, given an MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$, we say that a tuple $\langle f_1, \dots, f_k\rangle \in S$ is *good* if

(a) $f_m = \circ(f_{m_1}, \dots, f_{m_j})$ whenever $\varphi_m(\overline{x}_{\varphi_m}) = \circ(\varphi_{m_1}(\overline{x}_{\varphi_{m_1}}), \dots, \varphi_{m_j}(\overline{x}_{\varphi_{m_j}}))$,

(b) $f_i(e_1, \dots, e_{n_j}) = \inf\{f_j(e_1, \dots, e_{n_j}, e) \mid e \in M\}$ whenever $\varphi_i(\overline{x}_{\varphi_i}) = \forall y\, \varphi_j(\overline{x}_{\varphi_j})$, for all $e_1, \dots, e_{n_j} \in M^{n_j}$,

(c) $f_i(e_1, \dots, e_{n_j}) = \sup\{f_j(e_1, \dots, e_{n_j}, e) \mid e \in M\}$ whenever $\varphi_i(\overline{x}_{\varphi_i}) = \exists y\, \varphi_j(\overline{x}_{\varphi_j})$, for all $e_1, \dots, e_{n_j} \in M^{n_j}$.

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(7) From

$$\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$

infer

$$\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S'\rangle,$$

where S' is the set of good tuples in S.

The following result establishing the soundness of the formal system is a simple exercise but it helps in building intuition on how the formalism works.

LEMMA 5. *The axioms and rules of the system are sound with respect to the semantics.*

PROOF. For axiom schema (1), given a model $\mathfrak{M}$ with domain M, and formulas $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k})$, evidently, $f_{\varphi_k} \in [0,1]^{M^{n_k}}$ by definition. Thus the first-order MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}), [0,1]^{M^{n_1}} \times \ldots \times [0,1]^{M^{n_k}}\rangle$ holds in $\mathfrak{M}$.

For rule (2), if $\mathfrak{M}$ is a model and $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ holds in $\mathfrak{M}$, clearly for any permutation π of $1, \ldots, k$, if $S' = \{\langle f_{\pi(1)}, \ldots, f_{\pi(k)}\rangle \mid \langle f_1, \ldots, f_k\rangle \in S\}$, we also have that $\langle \varphi_{\pi(1)}(\overline{x}_{\varphi_{\pi(1)}}), \ldots, \varphi_{\pi(k)}(\overline{x}_{\varphi_{\pi(k)}}); S'\rangle$ holds in $\mathfrak{M}$.

For rule (3), if $\mathfrak{M}$ is a model of $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$, it immediately follows that $\langle f_{\varphi_1}, \ldots, f_{\varphi_k}\rangle \in S$, and taking formulas $\varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \ldots, \varphi_m(\overline{x}_{\varphi_m})$, it is also obvious that $\langle f_{\varphi_{k+1}}, \ldots, f_{\varphi_m}\rangle \in [0,1]^{M^{n_{k+1}}} \times \ldots \times [0,1]^{M^{n_m}}$. Thus the first-order MD-sentence

$$\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}), \varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \ldots, \varphi_m(\overline{x}_{\varphi_m}); S \times [0,1]^{M^{n_{k+1}}} \times \ldots \times [0,1]^{M^{n_m}}\rangle$$

holds in $\mathfrak{M}$.

For rule (4), if we have both

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S_1\rangle$$

and

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S_2\rangle$$

then $\langle f_{\varphi_1}, \ldots, f_{\varphi_k}\rangle \in S_1$ and $\langle f_{\varphi_1}, \ldots, f_{\varphi_k}\rangle \in S_2$. Thus,

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S_1 \cap S_2\rangle,$$

as desired.

We leave the proofs of the soundness of rules (5)–(7) to the reader. The key observation for rule (7) is that S' retains only the elements of S corresponding to formulas that respect the semantics of the real-valued logic in question. $\dashv$

A *proof* of an MD-sentence γ from a set Γ of MD-sentences in this system consists, as usual, of a finite sequence of MD-sentences such that the last member is γ and every element of the sequence is either an axiom, one of the member of Γ, or it follows from previous elements by one of the inference rules. We write $\Gamma \vdash_M \gamma$ to indicate that there exists a proof of γ from Γ.

Before stating Lemma 8, let us introduce some terminology.

DEFINITION 6. *Given a set A of first-order formulas, we will say that A is* closed under subformulas *if for any formula $\varphi \in A$, every subformula of φ is also in A.*

DEFINITION 7. *We say that an MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ is* minimized*, i.e., whenever $\langle f_1, \ldots, f_k\rangle \in S$, there is a model of $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ such that for $1 \leq i \leq k$ the interpretation of $\varphi_i(\overline{x}_{\varphi_i})$ is f_i.*

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LEMMA 8. *Let* $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ *be the premise of Rule (7) and assume that* $G = \{\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k})\}$ *is closed under subformulas. Then, the conclusion* $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S'\rangle$ *is* minimized *and this is witnessed by models with domain* M.

PROOF. Assume that $\langle f_1, \ldots, f_k\rangle \in S'$. Since G is closed under subformulas, there is a subsequence of $\langle f_1, \ldots, f_k\rangle$ that determines interpretations on the domain M for the atomic formulas appearing in G, i.e., interpretations for the predicates of τ. But this subsequence then defines a model $\mathfrak{M}$ based on the domain M where the interpretations of $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k})$ are as indicated by $\langle f_1, \ldots, f_k\rangle$. This is because Rule (7) is designed to select only those sequences $\langle f_1, \ldots, f_k\rangle$ that respect the semantics of the underlying real-valued logic. ⊣

REMARK 9. Observe that Lemma 8 does not claim that any MD-sentence has a model. It is rather telling us that if the set $\{\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k})\}$ of traditional formulas in the MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ used as a premise in an application of Rule (7) is closed under subformulas, then if $S' \neq \emptyset$, the MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S'\rangle$ coming from Rule (7) has a model.

REMARK 10. Lemma 8 plays an important role in the completeness argument in this general framework. Roughly speaking, it relies on the fact that the set S' can encode a model for a series of formulas $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k})$ with domain M by a sequence of interpretations to the finite list of predicates appearing in such formulas in a way that is consistent with the semantics of the underlying real-valued logic. It is not difficult to see that, for a finite vocabulary τ, we can find a set S encoding all possible models with domain M. For example, if τ is the set $\{P_1, \ldots, P_k\}$ of predicates, then we can take S to be the set of all sequences $\langle f_1, \ldots, f_k\rangle$ of possible interpretations of the predicates from our list on the domain M.

Similarly to [24, Lemma 5.3], we obtain:

LEMMA 11. *The conclusion and premises of rules (2), (3), (4), and (7) are logically equivalent.*

PROOF. The equivalence of the premise and conclusion of Rule (2) is clear. For Rules (3) and (7), the fact that the premise logically implies the conclusion follows from soundness of the rules, as does the fact that the conjunction of the premises of Rule (4) logically implies the conclusion. We now show that for Rules (3) and (7), the conclusion logically implies the premise. For Rule (3), the equivalence follows from the soundness of Rule (5). For Rule (4), the conclusion logically implies the each of the premises, and hence the conjunction of the premises, because of the soundness of Rule (4).

We will sketch the argument for Rule (7). Let $\mathfrak{M}$ be a model such that

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle.$$

But then the interpretations $f_1, \ldots, f_k$ of the formulas $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k})$ in the model $\mathfrak{M}$ respect the semantics of the connectives and quantifiers according to the real-valued logic in question. Since $\langle f_1, \ldots, f_k\rangle \in S$ by hypothesis, we must have that $\langle f_1, \ldots, f_k\rangle \in S'$ where S' is as in Rule (7). Hence,

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S'\rangle,$$

as desired. On the other hand, if

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S' \rangle,$$

given the soundness of Rule (6), it follows that

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S \rangle.$$

$\dashv$

The following lemma is straightforward to show.

Lemma 12. *Minimization is preserved by the rules (2) and (4), i.e. if the premises of the rules are minimized, then their conclusions are too.*

Let $\Gamma \vDash_M \gamma$ mean that for each model $\mathfrak{M}$ with domain M, if $\mathfrak{M} \models \Gamma$ then $\mathfrak{M} \models \gamma$. We call the relation $\vDash_M$ the *MD-logic of* M. We can now reconstruct the soundness and completeness argument from [24] and obtain the following theorem that the MD-system of M is actually an axiomatization of the MD-logic of M.

Theorem 13 (Completeness of the logic of a fixed domain). *Let* Γ *be a* finite *set of MD-sentences and* γ *an MD-sentence. Then,* $\Gamma \vdash_M \gamma$ *iff* $\Gamma \vDash_M \gamma$.

Proof. To see that $\Gamma \vdash_M \gamma$ only if $\Gamma \vDash_M \gamma$, one proceeds, as usual, by induction on the length of the proof, i.e. we start by showing that the axiom schema is sound and that the rules preserve the truth of the MD-sentences. For example, every instance of the axiom schema is sound since every formula in the usual first-order sense is interpreted by some mapping on a given model based on the domain M.

To show completeness, we follow the argument on [24, p. 12] and thus only provide a sketch. The strategy is to transform Γ into an equivalent MD-sentence from which γ can be deduced. We may assume without loss of generality that Γ is non-empty, for otherwise we could replace it by an instance of Axiom (1).

Indeed, assume that we have a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of MD-sentences in which, for each $i \in \{1, \ldots, n\}$, γ_i is the MD-sentence $\langle \varphi_1^i(\overline{x}_{\varphi_1}), \ldots, \varphi_k^i(\overline{x}_{\varphi_{k_i}}); S_i \rangle$. Suppose further that γ is $\langle \varphi_1^0(\overline{x}_{\varphi_1}), \ldots, \varphi_k^0(\overline{x}_{\varphi_{k_0}}); S_0 \rangle$. Then, take the sets $\Gamma_i = \{\varphi_1^i(\overline{x}_{\varphi_1}), \ldots, \varphi_k^i(\overline{x}_{\varphi_{k_i}})\}$ and $\Gamma_0 = \{\varphi_1^0(\overline{x}_{\varphi_1}), \ldots, \varphi_k^0(\overline{x}_{\varphi_{k_0}})\}$. We take G to be the usual closure under subformulas of the set $\bigcup_{j \geq 0} \Gamma_j$.

G is a finite set and then we can follow step by step the argument in [24], applying our slightly modified Rules (3) and (7). In particular, we make use of Lemma 8 instead of [24, Lemma 5.2].

For each i such that $1 \leq i \leq n$, we set $H_i = G \setminus \Gamma_i$. Let r_i be the cardinality of H_i and suppose that $H_i = \{\theta_1(\overline{x}_{\theta_1}), \ldots, \theta_{r_i}(\overline{x}_{\theta_{r_i}})\}$. Then, by applying Rule (3), we can deduce the MD-sentence

$$\langle \varphi_1^i(\overline{x}_{\varphi_1}), \ldots, \varphi_k^i(\overline{x}_{\varphi_{k_i}}), \varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \ldots, \varphi_m(\overline{x}_{\varphi_m}); S \times [0,1]^{M^{n_{k+1}}} \times \ldots \times [0,1]^{M^{n_m}} \rangle$$

from $\langle \varphi_1^i(\overline{x}_{\varphi_1}), \ldots, \varphi_k^i(\overline{x}_{\varphi_{k_i}}); S \rangle$, i.e. γ_i, where the sequence $\varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \ldots, \varphi_m(\overline{x}_{\varphi_m})$ is $\theta_1(\overline{x}_{\theta_1}), \ldots, \theta_{r_i}(\overline{x}_{\theta_{r_i}})$. Now let ψ_i be the MD-sentence that results from applying Rule (7) to the conclusion of Rule (3) displayed above.

Let $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_p(\overline{x}_{\varphi_p})$ be some ordering of the formulas in G; then, since the set of first-order formulas that appear in ψ_i is exactly G, we may use Rule (2) to turn ψ_i into an equivalent MD-sentence of the form $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_p(\overline{x}_{\varphi_p}); T_i \rangle$, which we may denote by χ_i. Furthermore, since in deriving χ_i, we only appealed to rules (2), (3) and (7), by Lemma 11, this MD-sentence is logically equivalent to γ_i.

Assume that $T = T_1 \cap \ldots \cap T_n$ and define $\chi := \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_p(\overline{x}_{\varphi_p}); T\rangle$. From Lemma 8, each ψ_i is minimized since it comes from Rule (7) and

$$\{\varphi^i_1(\overline{x}_{\varphi_1}), \ldots, \varphi^i_k(\overline{x}_{\varphi_{k_i}}), \varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \ldots, \varphi_m(\overline{x}_{\varphi_m})\}$$

is closed under subformulas. Moreover, by Lemma 12, each χ_i is minimized and, hence, χ is minimized.

The MD-sentence χ can be derived from the MD-sentences χ_i by repeated applications of Rule (4). In fact, by Lemma 11, χ and $\{\chi_1, \ldots, \chi_n\}$ have the same logical consequences, and since χ_i is equivalent to γ_i, we have that $\{\chi_1, \ldots, \chi_n\}$ and $\{\gamma_1, \ldots, \gamma_n\} = \Gamma$ have the same logical consequences. Hence, $\chi \vDash \gamma$ given that $\Gamma \vDash \gamma$ by hypothesis. Furthermore, in order to show that $\Gamma \vdash \gamma$ we simply need to show that $\chi \vdash \gamma$ since $\Gamma \vdash \chi$ by the above reasoning.

Recall that γ is $\langle \varphi^0_1(\overline{x}_{\varphi_1}), \ldots, \varphi^0_k(\overline{x}_{\varphi_{k_0}}); S_0\rangle$ and χ is $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_p(\overline{x}_{\varphi_p}); T\rangle$, so by applying Rule (2) we can rearrange the order of the formulas $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_p(\overline{x}_{\varphi_p})$ so they start with $\varphi^0_1(\overline{x}_{\varphi_1}), \ldots, \varphi^0_k(\overline{x}_{\varphi_{k_0}})$ and infer from χ the MD-sentence $\chi' := \langle \varphi^0_1(\overline{x}_{\varphi_1}), \ldots, \varphi^0_k(\overline{x}_{\varphi_{k_0}}) \ldots; T'\rangle$. Using Lemma 11, we may see that χ and χ' are logically equivalent. Hence, $\chi' \vDash \gamma$ since $\chi \vDash \gamma$. Given that χ is minimized, it follows that χ' is too by Lemma 12. Using Rule (5), from χ' we may infer an MD-sentence χ'' of the form $\langle \varphi^0_1(\overline{x}_{\varphi_1}), \ldots, \varphi^0_k(\overline{x}_{\varphi_{k_0}}); T''\rangle$.

The final step in the proof is to show that $T'' \subseteq S_0$ (which uses minimization in a fundamental manner) for then we can use Rule (6) to infer γ from χ'', and hence we would have $\chi \vdash \chi' \vdash \chi'' \vdash \gamma$, which means that $\chi \vdash \gamma$ as desired.

Assume now that $\langle f_1, \ldots, f_k\rangle \in T''$ to show that $\langle f_1, \ldots, f_{k_0}\rangle \in S_0$. By definition of T'', there is a $\langle f_1, \ldots, f_{k_0}, \ldots, f_p\rangle \in T'$. Given that χ' is minimized, there is a model $\mathfrak{M}$ of χ' such that the interpretations of the formulas $\varphi^0_1(\overline{x}_{\varphi_1}), \ldots, \varphi^0_k(\overline{x}_{\varphi_{k_0}})$ are $f_1, \ldots, f_{k_0}$, respectively. Since $\chi' \vDash \gamma$, then $\mathfrak{M} \models \gamma$, and so $\langle f_1, \ldots, f_{k_0}\rangle \in S_0$. ⊣

There are some subtle points to consider around what we have done, which we will discuss in the next remarks. It is important to stress that we have axiomatized the logic of *all models* based on the set M, not the logic of one particular model $\mathfrak{M}$ based on M.

Remark 14. Let us look at the case of two-valued logic with equality (i.e. the classical first-order logic which, of course, is covered by our approach). Let M be a finite set (say of size n). Now, enumerate all the first-order validities of the form $(|M| = n) \to \varphi$ where φ is any first-order formula and $|M| = n$ is the first-order formula saying that the size of the domain M is exactly n. In the case of finite domains, one might modify the approach here by allowing only MD-sentences that are interval-based (in the sense of [24], that is, where the sets of truth values involved in S are unions of finitely-many rational intervals) or that come from such sentences by an application of Rule (7), making the set $\mathrm{MD}(M)$ countable, and then it is possible to show by essentially the argument in [24, Theorem 6.1] that validity is not only recursively enumerable but decidable on such domains for Lukasiewicz and Gödel logic.

Remark 15. Recall that satisfiability on countably infinite models is not recursively enumerable in two-valued first-order logic. Now take a first-order sentence φ and let $\varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}), \varphi$ be the list of all its subformulas. Fixing a countably infinite domain M, we may consider now the MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}), \varphi; S\rangle$ (call

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it ψ) where $S := \{0,1\}^{M^{n_1}} \times \ldots \times \{0,1\}^{M^{n_k}} \times \{1\}$. Take now the MD-sentence obtained by applying our Rule (7) to this sentence, $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}), \varphi; S' \rangle$ (call it ψ'). Observe that ψ and ψ' are equivalent. Furthermore, φ has a countably infinite model iff ψ is satisfiable iff ψ' is satisfiable. Finally, by minimization and the semantics of MD-sentences, ψ' is satisfiable iff S' is non-empty. Hence, the problem of whether an arbitrary S' is non-empty is not recursively enumerable.

Rule (7) implies that our formal system is not finitistic in the sense of metamathematics [30] since when infinite domains are involved it cannot all be formalizable in arithmetic, it goes into the realm of infinitary mathematics. In this sense it is akin to an infinitary proof system (although it does not involve infinitary formulas in the usual sense). Thus the system we have presented here is by necessity less 'usable' in practice than a finitary one but not than an infinitary one.

3.2. Propositional modal logic (of a fixed frame). Expansions of propositional many-valued logics with modalities are a topic of lively research (see e.g. [3, 10, 11, 14, 25, 26, 46] due to their richer expressive power that makes them more amenable for a variety of applications, as compared to purely propositional logics. Thus, it is natural to extend them to the setting of multi-dimensional sentences too.

For this subsection, fix a frame $\mathfrak{F} := \langle M, R \rangle$ where $R \subseteq M^2$ is a binary relation on a non-empty set M (finite or infinite, where we may call the elements M *worlds*).[4] Consider now a vocabulary τ consisting only of propositional variables as in modal logic and a base modal language with $\Box$ and $\Diamond$ (unlike classical logic, many-valued logics do not allow in general to define these two operators from one another). Now the set $\mathrm{MD}(M)$ of MD-sentences contains all the expressions of the form $\langle \varphi_1, \ldots, \varphi_k; S \rangle$ where each φ_i is a modal formula and $S \subseteq ([0,1]^M)^k$.

For each real-valued model $\mathfrak{M}$-based on $\mathfrak{F} = \langle M, R \rangle$, i.e., a structure where each propositional variable $p \in \tau$ is interpreted as a mapping $p_{\mathfrak{M}} \colon M \longrightarrow [0,1]$, we can define a notion of truth value at a world $w \in M$:

- $\|p[w]\|_{\mathfrak{M}} = p_{\mathfrak{M}}(w)$, for each $p \in \tau$;
- $\| \circ (\varphi_0, \ldots, \varphi_n)[w]\|_{\mathfrak{M}} =$
  $\circ(\|\varphi_0[w]\|_{\mathfrak{M}}, \ldots, \|\varphi_n[w]\|_{\mathfrak{M}})$, for n-ary connective $\circ$;
- $\|\Box\varphi[w]\|_{\mathfrak{M}} = \inf\{\|\varphi[v]\|_{\mathfrak{M}} \mid v \in M, \langle w, v\rangle \in R\}$;
- $\|\Diamond\varphi[w]\|_{\mathfrak{M}} = \sup\{\|\varphi[v]\|_{\mathfrak{M}} \mid v \in M, \langle w, v\rangle \in R\}$.

Every formula φ can be said to be *interpreted* in the model $\mathfrak{M}$ by the mapping $f_\varphi \colon M \longrightarrow [0,1]$ defined as $w \mapsto \|\varphi[w]\|_{\mathfrak{M}}$ (we also say that φ *defines* the mapping f_φ in the model $\mathfrak{M}$). Given an MD-sentence $\langle \varphi_1, \ldots, \varphi_k; S \rangle$, we write

$$\mathfrak{M} \models \langle \varphi_1, \ldots, \varphi_k; S \rangle$$

if the formulas $\varphi_1, \ldots, \varphi_k$ respectively define mappings $f_1, \ldots, f_k$ in the model $\mathfrak{M}$ and $\langle f_1, \ldots, f_k \rangle \in S$.

As with the first-order case, from the axioms and inference rules from [24] we need to modify only the following:

Axioms.

(1) $\langle \varphi_1, \ldots, \varphi_k; [0,1]^M \times \ldots \times [0,1]^M \rangle$ for any formulas $\varphi_1, \ldots, \varphi_k$.

[4] In this paper we consider only this classical notion of frame, although the literature of many-valued logics has also studied natural many-valued generalizations in which R would be taken as a mapping from M^2 to $[0, 1]$ (or to other more general structures of truth-degrees); see e.g. [10].

Inference rules.

(3) From

$$\langle \varphi_1, \ldots, \varphi_k; S \rangle$$

infer

$$\langle \varphi_1, \ldots, \varphi_k, \varphi_{k+1}, \ldots, \varphi_m; S \times [0,1]^M \times \ldots \times [0,1]^M \rangle,$$

and we also need to modify the notion of good tuple for Rule (7). Indeed, given an MD-sentence $\langle \varphi_1, \ldots, \varphi_k; S \rangle$, now we say that a tuple $\langle f_1, \ldots, f_k \rangle \in S$ is *good* if

(a) $f_m = \circ(f_{m_1}, \ldots, f_{m_j})$ whenever $\varphi_m = \circ(\varphi_{m_1}, \ldots, \varphi_{m_j})$,
(b) $f_i(w) = \inf\{f_j(e) \mid e \in M, \langle w, e \rangle \in R\}$ whenever $\varphi_i = \Box \varphi_j$, for all $w \in M$,
(c) $f_i(w) = \sup\{f_j(e) \mid e \in M, \langle w, e \rangle \in R\}$ whenever $\varphi_i = \Diamond \varphi_j$, for all $w \in M$.

As before, we get the following (since the interpretations of the propositional variables in τ is what determines a model over $\mathfrak{F}$):

Lemma 16. *Let $\langle \varphi_1, \ldots, \varphi_k; S \rangle$ be the premise of Rule (7) and assume that $G = \{\varphi_1, \ldots, \varphi_k\}$ is closed under subformulas in the usual sense. Then, the conclusion $\langle \varphi_1, \ldots, \varphi_k; S' \rangle$ is minimized.*

Once more, closely following the argument from [24], we may show that:

Theorem 17 (Completeness of the logic of a fixed frame). *For Γ a* finite *set of MD-sentences and γ an MD-sentence, $\Gamma \vdash_{\mathfrak{F}} \gamma$ iff $\Gamma \vDash_{\mathfrak{F}} \gamma$.*

The proofs of Lemma 16 and Theorem 17 are very similar (modulo some trivial modifications) to those of Lemma 8 and Theorem 13, respectively, and thus we omit them. One might think of modal formulas as first-order formulas in one variable, and then it is easy to see how the same arguments work.

Remark 18. *An interesting topic of research would be to extend this multidimensional approach to many-valued* first-order *modal logics. This can be done for a fixed frame and a fixed domain*

§4. Axiomatizations of prominent first-order (and propositional modal) real-valued logics. Recall that, in the context of classical first-order logic, by the Löwenheim–Skolem theorem, the first-order sentences which are true in all countably infinite models coincide with the sentences that are true in all infinite models. For if φ is true in all countably infinite models, then $\neg\varphi$ cannot have any infinite model since otherwise $\neg\varphi$ would have a countably infinite model by the Löwenheim–Skolem theorem. Moreover, the class of infinite models is axiomatizable in first-order logic: consider the theory formed by the sentences "there are at least n elements" for all natural numbers $n > 0$. Hence, the first-order sentences which are true in all infinite models are recursively enumerable.

Let us analyze now what happens in the real-valued case. In this section we will consider only the case of languages *without* equality. This is a very standard practice in mathematical fuzzy logic (e.g. [1, 7, 29, 33, 43, 44]). It is well-known that neither Lukasiewicz nor Product first-order logic have a recursively enumerable set of validities with the semantics given on $[0, 1]$ (see [43] and [1], respectively). In contrast, Gödel first-order logic is recursively axiomatizable [44], and both Lukasiewicz and Product logics can be axiomatized by the addition of an infinitary rule (see [7, 29] and [33], respectively).

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Proposition 19. *Let $\mathcal{L}$ be a first-order real-valued logic.*[5] *Suppose that we have a countable vocabulary without equality. Then, for any $\mathcal{L}$-sentences $\varphi_1, \dots, \varphi_k$ and any finite sequence $\langle r_1, \dots, r_k\rangle$ of reals from the interval $[0, 1]$, there is an $\mathcal{L}$-model where $\varphi_1, \dots, \varphi_k$ take values $r_1, \dots, r_k$ respectively if there is an $\mathcal{L}$-model with a countably infinite domain where $\varphi_1, \dots, \varphi_k$ take values $r_1, \dots, r_k$ respectively. Moreover, the converse of this implication holds even if the vocabulary has equality.*

Proof. Suppose there is an $\mathcal{L}$-model, $\mathfrak{M}$, where $\varphi_1, \dots, \varphi_k$ take values $r_1, \dots, r_k$ respectively. By [17, Thm. 31], if M is finite, one can build an $\mathcal{L}$-model with a countably infinite domain where $\varphi_1, \dots, \varphi_k$ take values $r_1, \dots, r_k$ respectively (in fact there is a mapping between the two models that preserves the truth values of all formulas). On the other hand, by [17, Thm. 30], if M is infinite, one can build an $\mathcal{L}$-model with a countably infinite domain where $\varphi_1, \dots, \varphi_k$ take values $r_1, \dots, r_k$ respectively (in such a way that the countable model can be chosen to be an elementary substructure of the original that preserves the truth values of all formulas). ⊣

From this proposition and Theorem 13 we immediately obtain that consequence from finite sets of premises in Lukasiewicz, Product, and Gödel first-order real-valued logic (without equality) is complete with respect to the MD-system of a countable domain:

Corollary 20. *Let M be a fixed countably infinite domain, let $\mathcal{L}$ be either Lukasiewicz, Product, or Gödel first-order real-valued logic without equality, and let $\vDash_{\mathcal{L}}$ be the corresponding consequence relation. For any finite set $\varphi_1, \dots, \varphi_k, \psi$ of $\mathcal{L}$-sentences, we have:*

$$\langle \varphi_1; \{1\}\rangle, \dots, \langle \varphi_k; \{1\}\rangle \vdash_M \langle \psi; \{1\}\rangle \textit{ iff } \varphi_1, \dots, \varphi_k \vDash_{\mathcal{L}} \psi.$$

Observe that Corollary 20 would fail in the presence of equality in the vocabulary. This is because general validity cannot be reduced to truth in any particular infinite (even if only countable) model. The reason is that, if ψ is the first-order sentence expressing that the size of the domain is 3 then $\neg\psi$ would hold in every infinite domain M, whereas this cannot be a valid sentence in any of the logics we are considering here since ψ holds in models with universes of size 3. Thus, we would have that $\nvDash_{\mathcal{L}} \neg\psi$ but $\vdash_M \neg\psi$.

The purpose of any completeness theorem is to obtain the equivalence between a universal statement (about validity) and an existential statement (about the existence of a proof). The claim of existence of a proof is a Σ_1 claim on the natural numbers when the proof system is arithmetizable. By Corollary 20 and since neither Lukasiewicz nor Product first-order logic has a recursively enumerable set of validities, our proof systems are not arithmetizable when the domain is infinite.

Remark 21. Observe that, even in the case of classical logic (without equality –the situation with equality is analogous and dealt with in §6), the axiomatization we have presented here (when the domain in question is infinite) cannot be recursive due to Rule (7), where most of the strength of the present approach resides (cf. Remark 15). Naturally, there are much more fine-tuned axiomatizations of classical logic and many of the real-valued logics under consideration here, but the sacrifice we have made in terms of the manageability of our proof system has been in the interest of generality, so we can encompass all these logics at once.

[5] For example, $\mathcal{L}$ might be Lukasiewicz, Product, or Gödel first-order logic or, more generally, any first-order extension of an algebraizable logic in the sense of [17].

REMARK 22. Readers not familiar with encoding syntax and proofs in set theory may skip this remark. By representing MD-sentences as sets and proofs as sequences of such sets (similarly as things are done in infinitary logic [18]), our notion of proof will be a Σ_1 predicate (in the Lévy hierarchy) over the set of all sets hereditarily of some sufficiently large cardinality κ (in fact cardinality $|2^\omega| + 1$ would suffice for the case of a countably infinite fixed domain). Therefore, we have completeness in the same sense as it can be obtained in infinitary proof systems. Let us sketch the details of this formalization. Suppose that we fix a countable domain M. To each formula ϕ we can assign a Gödel number $\ulcorner\phi\urcorner$ in the usual manner [30]. We may then assign to each MD-sentence $\langle\phi_1, \dots, \phi_k; S\rangle$ the "Gödel set" $\ulcorner\langle\phi_1, \dots, \phi_k; S\rangle\urcorner$ which is simply the set $\langle\ulcorner\phi_1\urcorner, \dots, \ulcorner\phi_k\urcorner; S\rangle$ (using the Kuratowski definition of ordered tuples). Take now the collection $H(|2^\omega| + 1)$ containing all sets x hereditarily of cardinality $< |2^\omega| + 1$ in the sense that x, its members, its members of members, etc., are all of cardinality $< |2^\omega| + 1$. Consider now the following set-theoretic structure: $\langle H(|2^\omega|), \in\restriction H(|2^\omega| + 1)\rangle$. All Gödel sets $\langle\ulcorner\phi_1\urcorner, \dots, \ulcorner\phi_k\urcorner; S\rangle$ are elements of $H(|2^\omega|+1)$. A collection $K \subseteq H(|2^\omega|+1)$ is said to be Σ_1 on $H(|2^\omega|+1)$ if it is definable in the structure $\langle H(|2^\omega| + 1), \in\restriction H(|2^\omega| + 1)\rangle$ by a set theoretic formula equivalent to one built from atomic formulas and their negations by means of the connectives $\wedge, \vee$, the restricted quantifier $\forall x \in y$ and the quantifier $\exists x$. One can check then that the notion of $\langle\ulcorner\phi_1\urcorner, \dots, \ulcorner\phi_k\urcorner; S\rangle$ being a provable formula in our system is Σ_1 on $H(|2^\omega| + 1)$ because it claims the existence of a finite sequence of MD-sentences such that $\langle\ulcorner\phi_1\urcorner, \dots, \ulcorner\phi_k\urcorner; S\rangle$ is the last element of such sequence and every MD-sentence in it has been obtained by applying one of a finite number of rules to previous elements.

REMARK 23. From the results in [46] we know that neither Lukasiewicz nor Product modal logics on the interval $[0, 1]$ have recursively enumerable finitary "global" consequence relations.[6] Hence, similarly to what we observed for the first-order case, the approach here does axiomatize the logics in question, but it gives recursive enumerability only when the frame is finite, not in general.

Part of the interest of the present approach is the uniformity it provides in axiomatizing the previously mentioned logics (which were known to be axiomatizable by other infinitary methods). We are essentially giving one recipe to deal with all cases. Moreover, none of our rules are explicitly infinitary and the infinitary component of our formulas is hidden in the sets S.

Finally, in general, we are clearly axiomatizing more levels of formal reasoning than it could be done before, for preservation of value 1 is a mere fraction of the possibilities that the present system actually handles. The system axiomatizes genuine real-valued reasoning in all of Gödel, Lukasiewicz, and Product first-order (and modal) logics.

§5. A zero-one law for MD-logics. Beginning with [19] in the context of graph theory, a natural question that one can consider in general is: what is the probability that a structure satisfies P when randomly selected among finite structures with the same domain for a suitable probability measure? Or, more interestingly, what do these probabilities converge to (if anything) as the size of the domain of the structures grows to infinite? Well-known and highly celebrated results show that when the properties

[6] This means that $\Gamma \vDash \phi$ if for all models based on frames from a given class, if Γ is true at all points (or worlds) of the model, then ϕ is similarly true in all of them.

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under consideration are expressible by formulas of a certain logic the probabilities converge to either 0 or 1 (and so we say that the formula is either *almost surely false* or *almost surely true*, respectively). After an early result for monadic predicate logic [12], the topic of logical zero-one laws was properly started independently in the papers by Glebskiĭ et al [27] and Fagin [23] for first-order classical logic on finite purely relational vocabularies.

In this section, we want to establish a zero-one law for certain MD-logics, namely those based on suitable finite subalgebras of $[0,1]$ (of the form $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$). For example, both Gödel and Lukasiewicz logic have multiple finitely-valued versions (though Product logic does not), and we will list some examples below. This restriction to the finite setting is because we wish to have, when our vocabularies are relational and finite, only a finite number of possible models on a given finite domain, in analogy to what happens in classical logic in [23] (or in the finitely-valued case already considered in [5]). Regarding infinitely-valued logics, the recent paper [4] contains a zero-one law for infinitely-valued Lukasiewicz logic and related systems.

Example 24 (The algebra of Lukasiewicz 3-valued logic). The algebra

$$Ł_3 = \langle \{0, \frac{1}{2}, 1\}, \wedge^{Ł_3}, \vee^{Ł_3}, \&^{Ł_3}, \rightarrow^{Ł_3}, 0, 1\rangle$$

such that

- $\wedge^{Ł_3}(x,y) = \min\{x,y\}$
- $\vee^{Ł_3}(x,y) = \max\{x,y\}$
- $\&^{Ł_3}(x,y) = \max\{0, x+y-1\}$
- $\rightarrow^{Ł_3}(x,y) = \min\{1, 1-x+y\}$

More generally, we may consider any Lukasiewicz n-valued logic by using the algebra $Ł_n$ on the carrier set $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$ and with the same definitions of operations.

Example 25 (The algebra of Gödel 4-valued logic). The algebra

$$\mathrm{G}_4 = \langle \{0, \frac{1}{3}, \frac{2}{3}, 1\}, \wedge^{\mathrm{G}_4}, \vee^{\mathrm{G}_4}, \&^{\mathrm{G}_4}, \rightarrow^{\mathrm{G}_4}, 0, 1\rangle$$

such that

- $\wedge^{G_4}(x,y) = \&^{G_4}(x,y) = \min\{x,y\}$
- $\vee^{G_4}(x,y) = \max\{x,y\}$
- *and for* $\rightarrow^{G_4}$:

$$\rightarrow^{G_4}(x,y) = \begin{cases} 1 & \textit{if } x \leq y \\ y & \textit{otherwise}. \end{cases}$$

As in the previous example, we may also consider any Gödel n-valued logic by using the algebra G_n on the carrier set $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$ and with the same definitions of operations.

Let us now recall some facts from classical finite model theory. Consider a purely relational vocabulary. A sentence is said to be *parametric* in the sense of Oberschelp in [40, p. 277] if it is a conjunction of sentences of the form

$$\forall x_1, \dots, x_k(\neq(x_1, \dots, x_k) \rightarrow \phi(x_1, \dots, x_k)),$$

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where $\neq (x_1, \ldots, x_k)$ is the conjunction of negated equalities expressing that $x_1, \ldots, x_k$ are pairwise distinct, and $\phi(x_1, \ldots, x_k)$ is a quantifier-free formula where in all of its atomic subformulas $Rx_{i_1} \ldots x_{i_k}$ we have that

$$\{x_{i_1}, \ldots, x_{i_k}\} = \{x_1, \ldots, x_k\}.$$

Moreover, for $k = 1$, any formula $\forall x_1 \phi(x_1)$, where ϕ is a quantifier-free formula, is parametric. For example,

$$\forall x \neg Rxx \wedge \forall x \forall y (x \neq y \rightarrow (Rxy \rightarrow Ryx))$$

is a parametric sentence, whereas

$$\forall x \forall y \forall z (\neq (x, y, z) \rightarrow (Rxy \wedge Ryz \rightarrow Rxz))$$

is not.

Oberschelp's extension [40, Thm. 3] of Fagin's zero-one law [23] says: *on finite models and finite purely relational vocabularies, for any class K definable by a parametric sentence, any first-order sentence φ will be almost surely true in members of K or almost surely false*. By "almost surely true" here we mean that the limit as n goes to ∞ of the fraction of structures in K with domain $\{1, \ldots, n\}$ that satisfy a given sentence φ is 1 (and "almost surely false" is defined analogously). Naturally, these fractions are well defined because there is only a finite number of possible structures on finite vocabulary on the domain $\{1, \ldots, n\}$. As we mentioned earlier, this fact is what motivates our restriction to finitely valued logics in this section. A very accessible presentation of Oberschelp's result is [22, Thm. 4.2.3].

An appropriate translation for our purposes from finitely-valued first-order logics into classical first-order logic is introduced in [3]. Namely, for any sentence ϕ of a first-order logic based on a finite set $A \subseteq [0, 1]$ of truth values and element $a \in A$, we have a first-order sentence $T^a(\phi)$ such that, for a certain theory Σ (which can be written as a parametric sentence in the sense of Oberschelp [40]), $T^a(\phi)$ is satisfied by a classical first-order model $\mathfrak{M}$ model of Σ iff there is a corresponding first-order real-valued model $\mathfrak{M}^*$ where ϕ takes value exactly a.

The idea is that, starting with a relational vocabulary τ containing countably many predicate symbols $P_1^n, P_2^n, P_3^n, \ldots$ for each arity n, we can introduce a vocabulary τ^* containing predicate symbols P_i^{na} for each $a \in A$ and each n (the intuition here is that P_i^{na} will hold of those objects for which P_i^n takes truth value a in a given model), and the following translation from [3] (where $\circ \in \{\vee, \wedge, \&, \rightarrow\}$):

$$
\begin{aligned}
T^a(P_i^n x_1 \dots x_n) &= P_i^{na} x_1 \dots x_n \quad (i \geq 1) \\
T^a(\circ(\psi_1, \dots, \psi_n)) &= \bigvee_{\substack{b_1, \dots, b_n \in A \\ \circ^{\boldsymbol{A}}(b_1, \dots, b_n) = a}} \bigwedge\nolimits_{1 \leq i \leq n} T^{b_i}(\psi_i) \\
T^a(\exists x\, \psi) &= \Big(\bigvee_{\substack{k \leq |A| \\ b_1 \dots b_k \in A \\ \max\{b_1, \dots, b_k\} = a}} \bigwedge_{i=1}^{k} \exists x\, T^{b_i}(\psi) \Big) \wedge \\
&\qquad \wedge \forall y \, \Big(\bigvee_{\substack{b \in A \\ b \leq a}} T^b(\psi(y/x)) \Big) \\
T^a(\forall x\, \psi) &= \Big(\bigvee_{\substack{k \leq |A| \\ b_1, \dots, b_k \in A \\ \min\{b_1, \dots, b_k\} = a}} \bigwedge_{i=1}^{k} \exists x\, T^{b_i}(\psi) \Big) \wedge \\
&\qquad \wedge \forall y \, \Big(\bigvee_{\substack{b \in A \\ a \leq b}} T^b(\psi(y/x)) \Big).
\end{aligned}
$$

Observe how the translations of quantified formulas exactly describe the semantics of quantifiers in these finitely-valued logics (i.e. existential as maximum of the truth values of instances of the formula and, dually, universal as minimum). We use classical disjunctions to run over all the possible choices of values $b_1, \dots, b_k \in A$ that would give value a as their maximum (resp. minimum) and then write the conjunction of the necessary conditions that make sure that these b_i's are indeed values of instances of ψ and any other instance would give a value smaller (resp. bigger) than a.

Next, we define the theory Σ given by:

$$\forall x_1, \dots, x_n \big(\bigvee_{a \in A} P_i^{na} x_1 \dots x_n \big),$$

$$\forall x_1, \dots, x_n (\neg (P_i^{na} x_1 \dots x_n \wedge P_i^{nb} x_1 \dots x_n)),$$

for $a, b \in A, a \neq b, P_i^n \in \tau$.

For any A-valued model $\mathfrak{M}$ for the vocabulary τ, we can introduce a classical model $\mathfrak{M}^*$ for the vocabulary τ^* such that for any $a \in A$, the value of ϕ in $\mathfrak{M}$ is a iff $\mathfrak{M}^* \models T^a(\phi)$. $\mathfrak{M}^*$ is built by taking the same domain, M, as in $\mathfrak{M}$ and letting the interpretation of P_i^{na} be the set of all elements from M^n such that the interpretation of P_i^n in $\mathfrak{M}$ assigns them value a. Observe that $\mathfrak{M}^*$ is a model of the theory Σ. By a similar process, from any model $\mathfrak{N}$ of Σ, we can extract an A-valued model $\mathfrak{M}$ such that $\mathfrak{N} = \mathfrak{M}^*$.

Proposition 26. *An MD-sentence $\langle \phi_1, \dots, \phi_n; S \rangle$ is almost surely true on A-valued models with finite domains iff $\bigvee_{\langle a_1, \dots, a_n \rangle \in S} (T^{a_1}(\phi_1) \wedge \dots \wedge T^{a_n}(\phi_n))$ is almost surely true on the finite models of Σ.*

Proof. Suppose that $\langle \phi_1, \dots, \phi_n; S \rangle$ is almost surely true on A-valued models with finite domains. But every finite model of Σ can be seen as an $\mathfrak{M}^*$ for some finite A-valued model $\mathfrak{M}$, and $\mathfrak{M}^* \models \bigvee_{\langle a_1, \dots, a_n \rangle \in S} (T^{a_1}(\phi_1) \wedge \dots \wedge T^{a_n}(\phi_n))$ iff $\mathfrak{M} \models \langle \phi_1, \dots, \phi_n; S \rangle$. Hence, $\bigvee_{\langle a_1, \dots, a_n \rangle \in S} (T^{a_1}(\phi_1) \wedge \dots \wedge T^{a_n}(\phi_n))$ is almost surely true on the finite models of Σ. The other direction follows by similar reasoning. ⊣

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Rewriting the theory Σ with some care, one can turn it into a parametric sentence when τ is finite. For example, suppose that τ contains only a binary predicate R. Then, Σ would have the form (for $a, b \in A, a \neq b$):

$$\forall x_1 \forall x_2 (\bigvee_{a \in A} R^a x_1 x_2),$$

$$\forall x_1 \forall x_2 (\neg (R^a x_1 x_2 \wedge R^b x_1 x_2)).$$

This can be put into parametric form by considering instead (for $a, b \in A, a \neq b$):

$$\forall x_1 (\bigvee_{a \in A} R^a x_1 x_1),$$

$$\forall x_1 \forall x_2 (x_1 \neq x_2 \to \bigvee_{a \in A} R^a x_1 x_2),$$

$$\forall x_1 (\neg (R^a x_1 x_1 \wedge R^b x_1 x_1)),$$

$$\forall x_1 \forall x_2 (x_1 \neq x_2 \to \neg (R^a x_1 x_2 \wedge R^b x_1 x_2)).$$

Theorem 27 (Zero-one law for MD-logics based on finite algebras). *For any finite relational vocabulary, any MD-logic based on a finite set of truth values, and any MD-sentence $\langle \phi_1, \dots, \phi_n; S \rangle$, we have that $\langle \phi_1, \dots, \phi_n; S \rangle$ is almost surely true in finite models or $\langle \phi_1, \dots, \phi_n; S \rangle$ is almost surely false in finite models.*

Proof. This is immediate by applying Oberschelp's version in [40] of the zero-one law in [23] and our previous observations. By Proposition 26, an MD-sentence $\langle \phi_1, \dots, \phi_n; S \rangle$ is almost surely true iff $\bigvee_{\langle a_1, \dots, a_n \rangle \in S} T^{a_1}(\phi_1) \wedge \dots \wedge T^{a_n}(\phi_n)$ is almost surely true on the parametric class defined by Σ. $\dashv$

Remark 28. One might wonder what is the relationship of Theorem 27 with the central result from [5]. Suppose we have a 1-dimensional sentence $\langle \phi; S \rangle$. Then, applying the zero-one law from [5], the value a_ϕ that ϕ takes almost surely is in S only if $\langle \phi; S \rangle$ is almost surely true. Furthermore, if $\langle \phi; S \rangle$ is almost surely true, then a_ϕ is in S because a_ϕ is the value that ϕ takes almost surely. Thus, in the 1-dimensional case, both zero-one laws are equivalent, but only the 1-dimensional case, and not the 2-dimensional case, is covered in [5]. Hence, the question really is whether for a *finitely-valued* logic we would have that each MD-sentence is equivalent to a 1-dimensional sentence. In [24] it is shown that there is a 2-dimensional MD-sentence not equivalent to any 1-dimensional MD-sentence in logics based on the full interval $[0, 1]$. Does the same hold for finitely-valued logics?

§6. The logic of all domains.

§6. The logic of all domains. In this section, we will be using the same notion of model as in Definition 3 and we will allow the presence of equality in the vocabulary. Now, for any given domain M, let us denote by $\mathcal{L}_{\mathrm{MD}}(M)$ the finitary part of $\vDash_M$, that is, the set of all pairs $\langle \Gamma, \theta \rangle$ where Γ is a finite set of MD-sentences, θ is an MD-sentence, and every model over M of Γ is a model of θ. In this section, we intend to take the next natural step and axiomatize the finitary part of the MD-logic of *all domains*, i.e. the logic $\bigcap_{M \text{ a domain}} \mathcal{L}_{\mathrm{MD}}(M)$. Let us denote this consequence relation simply as $\vDash$.

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What kinds of inferences can appear in $\bigcap_{M \text{ a domain}} \mathcal{L}_{\text{MD}}(M)$? Clearly, only those not mentioning any of the domains M, since otherwise the inference could be rather specific to a particular M. For example, an MD sentence where a domain $M' \neq M$ is mentioned in the set S does not make sense in models based on the domain M, or rather it is always false. Thus, we set the goal of axiomatizing all the valid inferences $\Gamma \vDash \theta$ where $\Gamma \cup \{\theta\}$ is a finite set of MD-sentences of the form $\langle \varphi_1, \ldots, \varphi_k; S\rangle$ with each φ_i being sentences in the usual sense of a first-order predicate language and, hence, S is simply a set of suitable tuples of truth values (thus without a mention of any domain).

Example 29. The MD-sentence $\langle \varphi_1, \varphi_2; S\rangle$ where $S = \{\langle 0.5, 0.7\rangle\}$ and $\varphi_1 = \forall x\, Px$ and $\varphi_2 = \forall x(Px \vee Ux)$ is an example of the kind of MD-sentence described above, where φ_1 and φ_2 are sentences in the usual first-order sense of not having any free individual variables.

Focusing on logical entailments between this kind of MD-sentences, we can restrict attention (without loss of generality) to the models based in the following countable list of domains (let us call these the *legal domains*):

(i) the infinite domain of natural numbers $\{1, 2, \ldots\}$,
(ii) for each natural number n, a domain D_n of size n (making sure that they are pairwise disjoint and also disjoint from $\{1, 2, \ldots\}$).

This is because we have the following:

Proposition 30. *Any MD-sentence $\langle \varphi_1, \ldots, \varphi_k; S\rangle$ (where, for each $1 \leq i \leq k$, φ_i is a first-order sentence in the usual sense) with an infinite model has a countable model too.*

Proof. Take $\mathfrak{M} \vDash \langle \varphi_1, \ldots, \varphi_k; S\rangle$, so $\|\varphi_i\|_{\mathfrak{M}} = s_i$ (for $1 \leq i \leq k$) for some $\langle s_1, \ldots, s_k\rangle \in S$. By Proposition 19, then if $\mathfrak{M}$ is infinite, there is a countable model $\mathfrak{M}'$ such that $\|\varphi_i\|_{\mathfrak{M}'} = s_i$ ($1 \leq i \leq k$) for $\langle s_1, \ldots, s_k\rangle$, and hence $\mathfrak{M}' \vDash \langle \varphi_1, \ldots, \varphi_k; S\rangle$, as desired. $\dashv$

Consequently, if we denote the finitary part of the consequence relation over legal domains by $\vDash^{\text{legal}}$, using Proposition 30, we can see that $\Gamma \vDash^{\text{legal}} \theta$ iff $\Gamma \vDash \theta$ (where $\Gamma \cup \{\theta\}$ is a finite set of MD-sentences of the form $\langle \varphi_1, \ldots, \varphi_k; S\rangle$ with each φ_i being sentences in the usual sense of a first-order predicate language). This means that we can focus on axiomatizing $\vDash^{\text{legal}}$ for the class of MD-sentences that we have described in Proposition 30 (even though proofs may involve manipulating all kinds of MD-sentences, like those we will introduce in the next paragraph). Therefore, in what follows, we will restrict ourselves to consider *legal models*, i.e., those based on a legal domain.

The idea is to assume MD-sentences to have the form $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ where each φ_i is a first-order formula whose free variables are $\overline{x}_{\varphi_i} = x_{i_1}, \ldots, x_{i_{n_i}}$ (for some $n_i \geq 0$), and $S \subseteq [0,1]^{\bigcup_{M \text{ is legal}} M^{n_1}} \times \ldots \times [0,1]^{\bigcup_{M \text{ is legal}} M^{n_k}}$.

Example 31. Take a vocabulary τ with one binary predicate R. Then, we can build the MD-sentence $\langle Rxy, \forall x \forall y(Rxy \to Ryx); S\rangle$ where

$$S = \{\langle f, 0.5\rangle \mid f \colon \bigcup_{M \text{ is legal}} M^2 \longrightarrow [0,1]\}.$$

We want this sentence to be satisfied in a legal model $\mathfrak{M}$ with domain M if the truth value of $\forall x \forall y(Rxy \to Ryx)$ is 0.5 and, furthermore, the interpretation of R in the

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model $\mathfrak{M}$ is the restriction to M of one of the functions f described in the definition of S (which in this case, happens trivially).

As expected, we may then write

$$\mathfrak{M} \models \langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$

if the formulas $\varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k})$ respectively define functions $f_1, \ldots , f_k$ on the domain M such that there are $\langle f'_1, \ldots , f'_k\rangle \in S$ for which $f_1, \ldots , f_k$ are the respective restrictions to the domain M.

We transform Axiom (1) into (1)*:

$$\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}), [0, 1]^{\bigcup_{M \text{ is legal}} M^{n_1}} \times \ldots \times [0, 1]^{\bigcup_{M \text{ is legal}} M^{n_k}}\rangle$$

for all formulas $\varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k})$.

Rules (2), (4), (5), and (6) from the original system are modified analogously into (2)*, (4)*, (5)* and (6)*. Rule (3) needs to be modified as:

(3)* From

$$\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}); S\rangle$$

infer

$$\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}), \varphi_{k+1}(\overline{x}_{\varphi_{k+1}}), \ldots , \varphi_m(\overline{x}_{\varphi_m}); S\times$$
$$[0, 1]^{\bigcup_{M \text{ is legal}} M^{n_{k+1}}} \times \ldots \times [0, 1]^{\bigcup_{M \text{ is legal}} M^{n_m}}\rangle.$$

Finally, Rule (7) is modified into Rule (7)* by changing the notion of good tuple. Indeed, given an MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}); S\rangle$, we will say that a tuple $\langle f_1, \ldots , f_k\rangle \in S$ is *good* if for some legal domain M

(a) $f_m \restriction M = \circ((f_{m_1} \restriction M), \ldots , (f_{m_j} \restriction M))$ whenever $\varphi_m(\overline{x}_{\varphi_m}) = \circ(\varphi_{m_1}(\overline{x}_{\varphi_{m_1}}), \ldots , \varphi_{m_j}(\overline{x}_{\varphi_{m_j}}))$,

(b) $(f_i \restriction M)(e_1, \ldots , e_{n_j}) = \inf\{(f_j \restriction M)(e_1, \ldots , e_{n_j}, e) \mid e \in M\}$ whenever $\varphi_i(\overline{x}_{\varphi_i}) = \forall y\, \varphi_j(\overline{x}_{\varphi_j})$, for all $e_1, \ldots , e_{n_j} \in M^{n_j}$,

(c) $(f_i \restriction M)(e_1, \ldots , e_{n_j}) = \sup\{(f_j \restriction M)(e_1, \ldots , e_{n_j}, e) \mid e \in M\}$ whenever $\varphi_i(\overline{x}_{\varphi_i}) = \exists y\, \varphi_j(\overline{x}_{\varphi_j})$, for all $e_1, \ldots , e_{n_j} \in M^{n_j}$.

Rule (7)* is clearly sound with respect to the relation $\models^{\text{legal}}$ since we are only considering models based on legal domains.[7] Given this system, we denote the corresponding provability relation simply as $\vdash$.

Remark 32. Observe that the complexity of identifying an application of Rule (7)* by constructing S' is the same, generally speaking, as in the case of a fixed countably infinite domain and Rule (7). This is because, for example, in the latter case, in order to identify which tuples are in S', one might still need to compute the infimum of an infinite set without any nice structure in general in the process of verifying the value of a universal quantification.

We will say that $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}); S'\rangle$ is *minimized* if when $\langle f_1, \ldots , f_k\rangle \in S'$, then there is a legal model of $\langle \varphi_1(\overline{x}_{\varphi_1}), \ldots , \varphi_k(\overline{x}_{\varphi_k}); S'\rangle$, $\mathfrak{M}$, such that for $1 \leq i \leq k$ the interpretation of $\varphi_i(\overline{x}_{\varphi_i})$ is $f_i \restriction M$.

[7] Notice that if in Rule (7)* we had written "for each legal domain" instead of "for some legal domain" in the definition of a good pair, the soundness argument would not work for the resulting rule.

Lemma 33 (Minimization Lemma). *Let* $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ *be the premise of Rule (7)** *and assume that* $G = \{\varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k})\}$ *is closed under subformulas in the usual sense. Then, the conclusion* $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S'\rangle$ *is minimized.*

Proof. Assume that $\langle f_1, \dots, f_k\rangle \in S'$. Since G is closed under subformulas, there is a legal domain M and a subsequence of $\langle g_1, \dots, g_j\rangle$ of $\langle f_1, \dots, f_k\rangle$ such that $\langle g_1 \restriction M, \dots, g_j \restriction M\rangle$ determines interpretations on M for the atomic formulas appearing in G, i.e., interpretations for the predicates of the vocabulary τ in question. But this subsequence then defines a legal model $\mathfrak{M}$ based on the domain M where the interpretations of $\varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k})$ are as indicated by $\langle g_1 \restriction M, \dots, g_j \restriction M\rangle$. $\dashv$

Lemma 34. *The conclusion and premises of rules (2)*, (3)*, (4)*, and (7)* are logically equivalent.*

Lemma 35. *Minimization is preserved by the rules (2)* and (4)*, i.e. if the premises of the rules are minimized, then their conclusions are too.*

With these key facts at hand, the soundness and completeness proof goes through basically as before:

Theorem 36 (Completeness of the logic of all legal domains). *Let* $\Gamma \cup \{\theta\}$ *be a finite set of MD-sentences in a first-order predicate language with equality. Then,* $\Gamma \vdash \theta$ *iff* $\Gamma \models^{legal} \theta$.

Corollary 37 (Completeness of the logic of all domains). *Let* $\Gamma \cup \{\theta\}$ *be a finite set of MD-sentences of the form* $\langle \varphi_1, \dots, \varphi_k; S\rangle$ *with each* φ_i *being a sentence in the usual sense of a first-order predicate language with equality. Then,* $\Gamma \vdash \theta$ *iff* $\Gamma \models \theta$.

Remark 38. The approach provided in this section allows us now to axiomatize, in particular, the valid finitary *consecutions* (i.e. pairs of the form $\langle \Theta, \theta\rangle$ where Θ is a finite set of first-order sentences and θ a first-order sentence such that the former logically entails the latter, see e.g. [15]) of each of Lukasiewicz, Product, Gödel, and real-valued logics with equality. This is analogous to what we did in Corollary 20. Hence, to deal with the presence of equality in the logic, we had to leave the realm of the *fixed* countable domain from Corollary 20 and, instead, study all domains that can be distinguished by the expressive power of a first-order language with equality (namely, all finite domains in addition to a countably infinite ones).

Another interesting consequence of our approach is that we can provide a finitary axiomatization of the valid inferences on finite models for any real-valued logic. Let the class of *legal* domains* be that of the legal domains minus the one countably infinite domain (so we are keeping only the finite domains). One can then modify the axiomatization given above by replacing the legal domains by the legal* ones. Clearly, $\Gamma \models^{\text{legal}^*} \theta$ iff $\Gamma \models^{\text{finite}} \theta$, where $\models^{\text{finite}}$ is the obvious logical consequence over all finite domains (notice that the legal domains are just a specific subset of all finite domains). Exactly as we did previously, we can obtain:

Theorem 39 (Completeness of the logic of all finite domains). *Let* $\Gamma \cup \{\theta\}$ *be a finite set of MD-sentences in a first-order predicate language with equality. Then,* $\Gamma \vdash \theta$ *iff* $\Gamma \models^{legal^*} \theta$ *iff* $\Gamma \models^{finite} \theta$.

By a well-known theorem of Trakhtenbrot [45], the validities of classical first-order logic on finite models are not recursively enumerable. In the real-valued setting, the result was generalized in [8] to a large class of logics. This entails that, once more, our axiomatization cannot possibly be recursive. In fact, we can observe that the problem of determining whether $S' = \emptyset$ in Rule (7)* of our axiomatization is not recursively enumerable, which explains why our system is not recursive. This is because we can reduce the problem of whether a sentence of classical first-order logic is valid in the finite to whether $S' = \emptyset$. Take a first-order sentence φ and let $\varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}), \varphi$ be the list of all its subformulas. Consider now the MD-sentence $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}), \varphi; S\rangle$ (call it ψ) where $S := \{0,1\}^{\bigcup_{M \text{ is legal}} M^{n_1}} \times \dots \times \{0,1\}^{\bigcup_{M \text{ is legal}} M^{n_k}} \times \{0\}$. Take now the MD-sentence obtained by applying our Rule (7) to this sentence, $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}), \varphi; S'\rangle$ (call it ψ'). Observe that ψ and ψ' are equivalent. Furthermore, φ is valid on all finite models iff $\neg\varphi$ has no finite model iff ψ is not satisfiable in a finite domain iff ψ' is not satisfiable in a finite domain. Finally, by minimization and the semantics of MD-sentences, ψ' is not satisfiable in a finite domain iff $S' = \emptyset$.

Remark 40. An alternative approach to the one followed in this section would have been to take instead of MD-sentences, 'MD-formulas' to be objects of the form $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ where S is a set of tuples of truth values. Then, given a first-order model $\mathfrak{M}$ and assignment variable v to the free individual variables in $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$, we say that $\mathfrak{M}$ *satisfies* $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ under the assignment v if

$$\langle \|\varphi_1[v(\overline{x}_{\varphi_1})]\|_{\mathfrak{M}}, \dots, \|\varphi_k[v(\overline{x}_{\varphi_k})]\|_{\mathfrak{M}}\rangle \in S.$$

With this modification, everything we have done in this section would work in a very similar manner manner as long as we modify Rule (7)* appropriately: given an MD-formula $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$, we will say that a tuple $\langle s_1, \dots, s_k\rangle \in S$ of truth values is *good* if for some model $\mathfrak{M}$ and variable assignment v for the signature of $\langle \varphi_1(\overline{x}_{\varphi_1}), \dots, \varphi_k(\overline{x}_{\varphi_k}); S\rangle$ based on a legal domain M,

(a) $s_m = \circ(s_{m_1}, \dots, s_{m_j})$ whenever $\|\varphi_m[v(\overline{x}_{\varphi_m})]\|_{\mathfrak{M}} = s_m$, $\|\varphi_{m_1}[v(\overline{x}_{\varphi_{m_1}})]\|_{\mathfrak{M}} = s_{m_1}$, etc., and $\varphi_m(\overline{x}_{\varphi_m}) = \circ(\varphi_{m_1}(\overline{x}_{\varphi_{m_1}}), \dots, \varphi_{m_j}(\overline{x}_{\varphi_{m_j}}))$,

(b) $s_i = \inf\{\|\varphi_j[v_{y\mapsto e}(\overline{x}_{\varphi_j})]\|_{\mathfrak{M}} \mid v_{y\mapsto e}, e \in M\}$ whenever $\varphi_i(\overline{x}_{\varphi_i}) = \forall y\, \varphi_j(\overline{x}_{\varphi_j})$, and $v_{y\mapsto e}$ is an assignment just like v except that the value of variable y is made e, and if $\varphi_j(\overline{x}_{\varphi_j})$ appears on the left-hand-side of our MD-formula, $\|\varphi_j[v(\overline{x}_{\varphi_j})]\|_{\mathfrak{M}} = s_j$,

(c) $s_i = \sup\{\|\varphi_j[v_{y\mapsto e}(\overline{x}_{\varphi_j})]\|_{\mathfrak{M}} \mid v_{y\mapsto e}, e \in M\}$ whenever $\varphi_i(\overline{x}_{\varphi_i}) = \exists y\, \varphi_j(\overline{x}_{\varphi_j})$, and $v_{y\mapsto e}$ is an assignment just like v except that the value of variable y is made e, and if $\varphi_j(\overline{x}_{\varphi_j})$ appears on the left-hand-side of our MD-formula, $\|\varphi_j[v(\overline{x}_{\varphi_j})]\|_{\mathfrak{M}} = s_j$.

With this new rule, once can reproduce the proof of the Minimization Lemma and the rest works in an analogous way.

§7. Conclusion. In this article, we have proposed a new paradigm for dealing with inference in first-order (and modal) real-valued logics. By means of the syntax of multi-dimensional sentences, we have obtained a high level of expressivity that goes beyond the usual preservation of full truth given by the value 1 and surpasses even the expressivity

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of rational Pavelka logic or other fuzzy logics with truth-constants (see e.g. [20,21]). As usual, there is a trade-off between expressivity and effectivity of any logical formalism. In our case, we have presented axiomatic systems that are not finitistic in the sense of metamathematics [30] because MD-sentences contain a hidden infinitary component (that is, the sets S), but yet these systems involve only *finitary* rules. We have proved corresponding completeness theorems in a similar sense as they had been obtained with ad hoc *infinitary* proof systems for some particular real-valued logics (see [29, 33]), but now in a general, uniform, parameterized way. However, it should be stressed that on finite domains our proof systems become finitistic and everything works as in the propositional case. Finally, sentences incorporating weights can be handled completely analogous to the way it is done in [24]. As open problems that we have not solved in this paper and remain as matters for future research we may mention the question whether one can extend, in the case of modal logics, the completeness theorem for the logic of a fixed frame (Theorem 17) to logics corresponding to meaningful classes of frames, and the problem of developing the multidimensional approach for first-order modal logics.

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