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TOUCHING CONVEX SETS IN THE PLANE

MEIR KATCHALSKI AND JÁNOS PACH

ABSTRACT. Two subsets of the Euclidean plane touch each other if they have a point in common and there is a straight line separating one from the other.

It is shown that there exists a positive constant c such that if $\mathcal{A}$ and $\mathcal{B}$ are families of plane convex sets with $|\mathcal{A}| \geq c \cdot k$ and $|B| \geq c \cdot k$ for some $k \geq 1$ and if every $A \in \mathcal{A}$ touches every $B \in \mathcal{B}$, then either $\mathcal{A}$ or $\mathcal{B}$ contains k members having nonempty intersection.

1. **Introduction and main results.** Two subsets of the Euclidean plane *touch* each other if they have a point in common and there is a straight line separating one from the other. A family $\mathcal{A}$ of sets is said to *k-thin* (or to form a k-fold packing) if no point of the plane is contained in more than k members of $\mathcal{A}$. Two families $\mathcal{A}$ and $\mathcal{B}$ are called *touching* if every member of $\mathcal{A}$ touches every member of $\mathcal{B}$.

It is easy to find two 1-thin families (*i.e.*, packings) of convex sets $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_i : i = 1, 2, \ldots\}$ that are touching (see Figure 1).

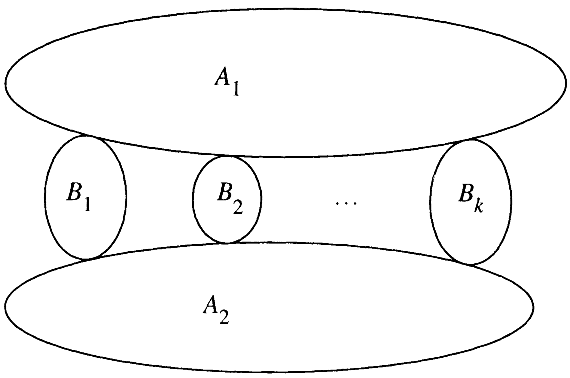

FIGURE 1

On the other hand, there is no touching pair of 3-member 1-thin families of sets since this would contradict the nonplanarity of $K_{3,3}$, a complete bipartite graph with 3 elements in each of its classes.

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There exist two touching 2-thin families of convex sets $\mathcal{A} = \{A_1, \ldots, A_6\}$, $\mathcal{B} = \{B_i : i = 1, 2, \ldots\}$, as depicted in Figure 2. (The sets A_j and A_{3+j} are only slightly different in the neighborhoods of their points of incidences with the B_i's.) At first glance one might think that there is a similar construction, when $\mathcal{A}$ consists of 4 pairs of sets and any two sets belonging to distinct pairs are disjoint. However, this possibility can be ruled out by the following result.

LEMMA 1. *Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, $\mathcal{B} = \{B_1, \ldots, B_k\}$ be touching families of plane convex sets, where the A_i's are pairwise disjoint. If every member of $\mathcal{B}$ touches the A_i's in the same counterclockwise cyclic order (A_1, A_2, A_3, A_4), then all members of $\mathcal{B}$ have a point in common.*

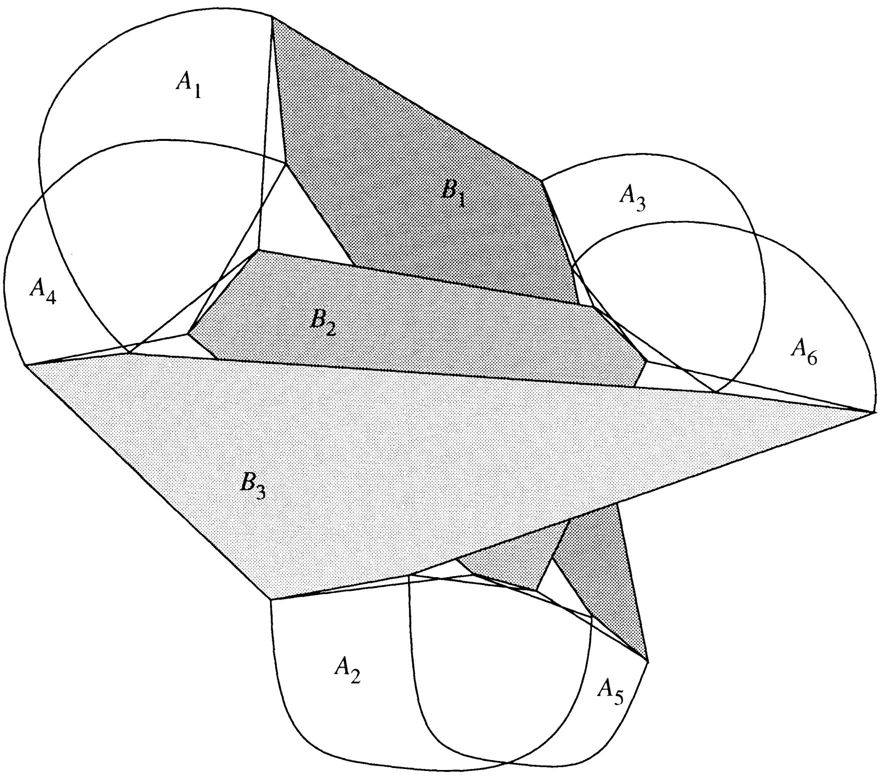

FIGURE 2

This suggests that the following assertion is true:

CONJECTURE. *If $\mathcal{A}$ and $\mathcal{B}$ are touching 2-thin families of plane convex sets, then one of them has at most 6 members.*

We can establish a more general (but less exact) statement by showing that if $\mathcal{A}$ and

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$\mathcal{B}$ are touching k-thin families of convex sets, then either $\mathcal{A}$ or $\mathcal{B}$ has relatively few members.

THEOREM 1. *There exists a positive constant c such that if $\mathcal{A}$ and $\mathcal{B}$ are touching families of plane convex sets with $|\mathcal{A}|, |\mathcal{B}| \geq c \cdot k$ for some $k \geq 1$, then either $\mathcal{A}$ or $\mathcal{B}$ contains k members having nonempty intersection.*

The proof is based on Lemma 1 and on the following result:

LEMMA 2. *Let $\mathcal{A}$ be any family of n plane convex sets with $n \geq 3 \cdot \binom{\ell}{3} \cdot k$ for some positive integers k and ℓ. Then either $\mathcal{A}$ has k members with nonempty intersection, or there is an ℓ-member subfamily $\mathcal{A}' \subseteq \mathcal{A}$ which is 2-thin.*

Assume for a moment that the Conjecture is true. Then Lemma 2 immediately implies that Theorem 1 holds with $c = 3 \cdot \binom{7}{3} = 105$. Indeed, let $\mathcal{A}$ and $\mathcal{B}$ be touching families of plane convex sets with $|\mathcal{A}|, |\mathcal{B}| \geq 105k$ and suppose, for the sake of contradiction, that neither $\mathcal{A}$ nor $\mathcal{B}$ contains k members having nonempty intersection.

By Lemma 2 there exist two 2-thin 7-member subfamilies $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{B}' \subseteq B$. But $\mathcal{A}'$ and $\mathcal{B}'$ must be touching, contradicting our Conjecture. Of course, the same argument can be applied if the Conjecture is true with any other value larger than 6, but it yields a worse constant c in Theorem 1.

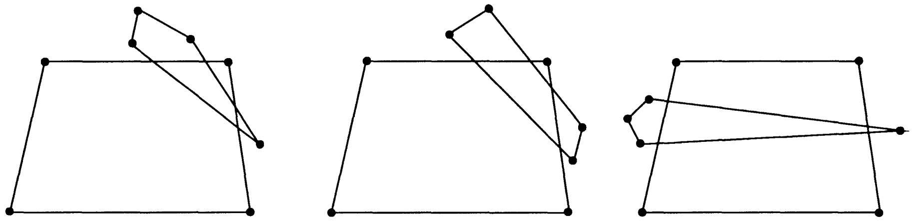
In Sections 2 and 3 we prove the lemmata and Theorem 1, respectively. Section 4 contains an application of our results to intersection patterns of convex sets. In the final section we discuss some related results and open problems.

2. **Proofs of the lemmata.** It is easy to see, replacing each set by a suitable subset, that there is no loss of generality if Lemma 1 is proved with the additional assumption that the families consist of compact sets.

PROOF OF LEMMA 1. First assume, without loss of generality, that every B_i is a quadrilateral which touches the A_j's at its vertices and put on each vertex of the quadrilateral the index j of the set A_j that contains it. Any two B_i's intersect since otherwise it is easy to verify that two A_j's intersect, a contradiction. Helly's theorem [2] implies that, if $\mathcal{B}$ does not have any 3-member subfamily which is 2-thin, then $\bigcap \mathcal{B} \neq \emptyset$. It is therefore sufficient to prove that no 3-member subfamily of $\mathcal{B}$ is 2-thin.

OBSERVATION. If we go along the boundary of the union of two or more B_i's, we obtain a sequence of numbers which does not contain a subsequence of type (i, j, i, j) with $i \neq j$. (The reason for this is that otherwise two A's intersect.) Call such a sequence *forbidden*.

This implies that two B_i's cannot intersect as in Figures 3(a) or 3(b) or 3(c).



(a) (b) (c)

FIGURE 3

Thus, two B_i's can intersect in one of the following ways (see Figure 4):

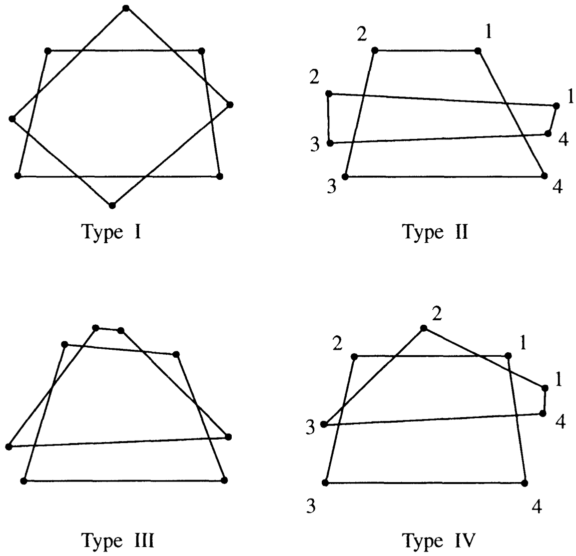


FIGURE 4

The sequence of numbers in types II and IV are determined uniquely up to a cyclic permutation.

Suppose that $B_i \cap B_j \cap B_i = \emptyset$. Then the intersection of two sets cannot be of types I or III of Figure 4, since this would imply that the intersection of another pair of sets is as in Figure 3(a), a contradiction.

Suppose that the intersection of two sets (say, B_j and B_i) is of type II. Then the essentially different possibilities for B_k are as in Figure 5. But (1) implies a forbidden $(1,2,1,2)$; (2) implies a forbidden $(2,3,2,3)$; (3) implies a forbidden $(1,2,1,2)$; (4) implies a forbidden $(4,1,4,1)$. If the intersection of every pair of B_i, B_j, B_k is of type IV, then the intersections are essentially as in Figure 6, implying a forbidden $(2, 3, 2, 3)$, a contradiction. Hence, $B_i \cap B_j \cap B_k \neq \emptyset$. ∎

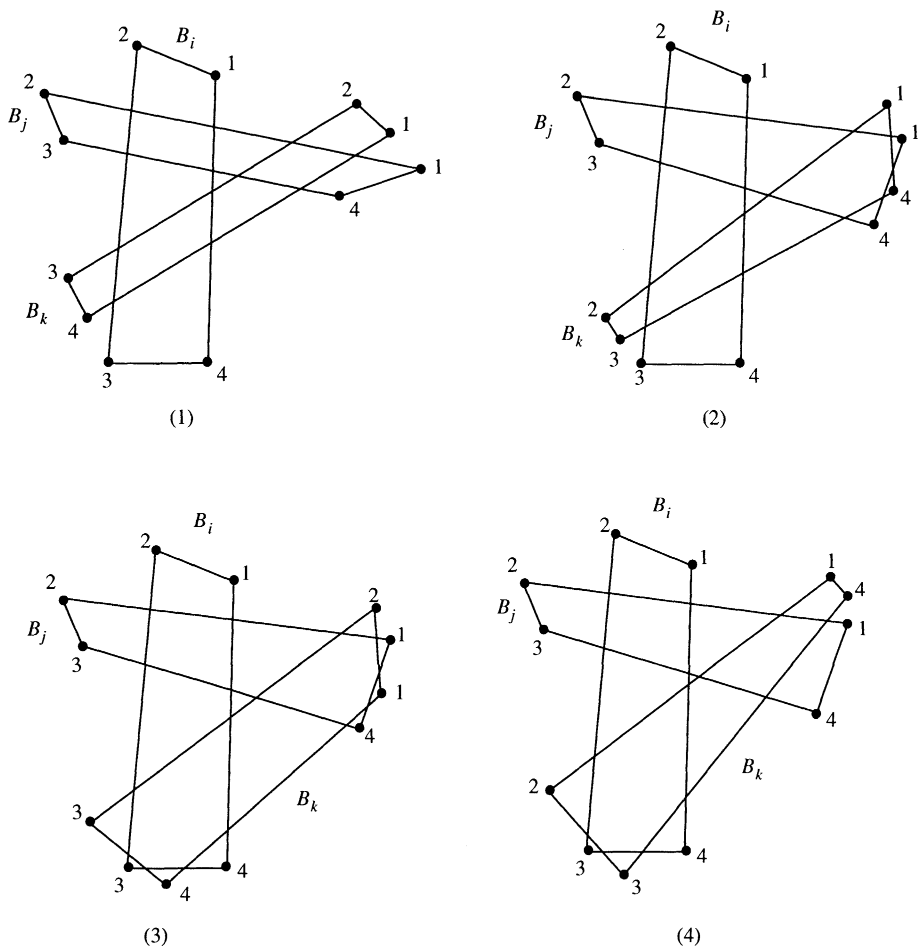


FIGURE 5

PROOF OF LEMMA 2. We may suppose that $k \geq 3$. Assume, for the sake of contradiction, that $\mathcal{A}$ has no k members with nonempty intersection, but every ℓ-member subfamily $\mathcal{B} \subseteq \mathcal{A}$ contains a triple $\mathcal{T}$ with $\bigcap \mathcal{T} \neq \emptyset$.

Let x denote the number of 2-thin triples $\mathcal{T}' \subseteq \mathcal{A}$, *i.e.*, those for which $\bigcap \mathcal{T}' = \emptyset$. By counting the number of pairs $(\mathcal{T}', \mathcal{B})$, where $\mathcal{T}'$ is a 2-thin triple of an ℓ-tuple $\mathcal{B}$, we obtain

$$x\binom{n-3}{\ell-3} \leq \left(\binom{\ell}{3} - 1\right)\binom{n}{\ell}.$$

On the other hand, by a result of Kalai [3], any $(k-1)$-thin family of plane convex sets has at least $\binom{n-k-3}{3}$ 2-thin triples. Thus,

$$\binom{n-k-3}{3}\binom{n-3}{\ell-3} \leq \left(\binom{\ell}{3}-1\right)\binom{n}{\ell},$$

so that

$$\binom{n-k-3}{3} \leq \binom{n}{3}\left(1-\frac{1}{\binom{\ell}{3}}\right).$$

However, if $n = ck$ with $c \geq 3\binom{\ell}{3}$, then

$$\binom{n-k-3}{3} > \frac{(n-k-5)^3}{3!} = \frac{n^3}{3!}\left(1-\frac{1}{c}-\frac{5}{ck}\right)^3$$
$$> \binom{n}{3}\left(1-\frac{3}{c}\right) \geq \binom{n}{3}\left(1-\frac{1}{\binom{\ell}{3}}\right),$$

a contradiction. ∎

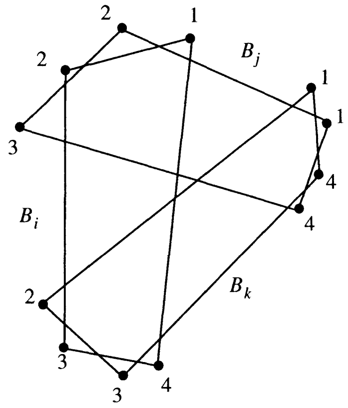


FIGURE 6

3. **Proof of Theorem 1.** As we have pointed out in the Introduction, it is sufficient to show that there is an integer ℓ such that there exist no two touching ℓ-member 2-thin families of planar convex sets.

Assume, in order to get a contradiction, that there are two such families $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = |\mathcal{B}| = \ell \geq 10^5$. There is no loss of generality in assuming that the sets are compact. By Ramsey's theorem,

(i) either $\mathcal{A}$ contains 4 pairwise disjoint members,

(ii) or there is a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, $|\mathcal{A}'| = 7$, whose members are pairwise intersecting.

If (i) holds, then choose $\lceil \ell/6 \rceil$ members of $\mathcal{B}$ that touch the 4 disjoint elements of $\mathcal{A}$ in the same cyclic order. By Lemma 1, all of these sets must have a point in common, contradicting our assumption that $\mathcal{B}$ is 2-thin. Similarly, we can assume that there are no 4 pairwise disjoint members in $\mathcal{B}$. Therefore, if (ii) holds, then one of the connected components of the complement of $\bigcup \mathcal{A}'$ contains at least $\ell/3$ members of $\mathcal{B}$.

Let C denote the closure of such a connected component. The boundary of C is composed of arcs, where each arc belongs to one or two members of $\mathcal{A}' = \{A_1, \ldots, A_7\}$. A *maximal* contiguous piece $\gamma \subseteq \mathrm{Bd}\, C$ is called an $\{i,j\}$*-arc* if

(a) every point of γ belongs to A_i or A_j, and

(b) γ has at least one point belonging to $A_i \cap A_j$.

It is easy to see that for $i \neq j$ there is only at most one $\{i,j\}$-arc. Indeed, if there were two such arcs γ and γ', then they would be separated from each other on $\mathrm{Bd}\, C$ by two points $p \in A_g$ and $p' \in A_h$ $\{g, h \notin \{i,j\}\}$. Let $q \in \gamma$, $q' \in \gamma'$ be points belonging to $A_i \cap A_j$. Since any two members of $\mathcal{A}'$ intersect each other, p and p' can be joined by a path in $A_g \cup A_h$. This path must meet the segment $qq' \subseteq A_i \cap A_j$ at some point r. But then r belongs to at least 3 members of $\mathcal{A}'$, contradicting our assumption that $\mathcal{A} \supseteq \mathcal{A}'$ is 2-thin (Figure 7).

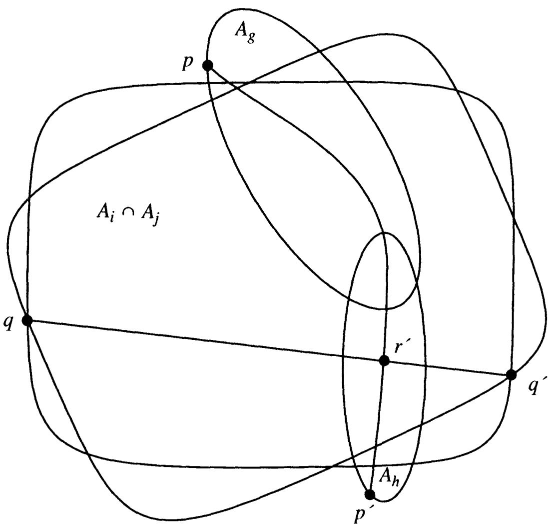


FIGURE 7

Let $\mathcal{B}'$ denote the family of all members of $\mathcal{B}$ which belong to C ($|\mathcal{B}'| \geq \ell/3$). All of them touch A_1, at some point, and for at least $1/6$ of them this point belongs to the same $\{1, i_1\}$-arc for some $2 \leq i_1 \leq 7$. Assume without loss of generality that $i_1 \neq 2$, and find a $\{2, i_2\}$-arc on the boundary of C which meets the largest number of these sets. Proceeding like this and changing the indices if necessary, we end up with an at least $|\mathcal{B}'|/6^4$-member subfamily $\mathcal{B}'' \subseteq \mathcal{B}'$, all of whose members meet A_j along the same $\{j, i_j\}$-arc γ_j $(1 \leq j \leq 4)$, where $i_1 \notin \{2,3,4\}$, $i_2 \notin \{3,4\}$ and $i_3 \neq 4$.

Let γ_j' denote the smallest subarc of γ_j containing all points of $\gamma_j \cap A_j$ $(1 \leq j \leq 4)$. Note that the arcs $\gamma_1', \ldots, \gamma_4'$ have pairwise disjoint interiors. Indeed, if *e.g.*, γ_1' and γ_2' had

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an interior point in common, then γ_1 would be a $\{1,2\}$-arc, contradicting the assumption $i_1 \neq 2$.

Since $\mathcal{B}'' \subseteq \mathcal{B}$ is a 2-thin family, any endpoint of an arc γ_j' is contained in at most two members of $\mathcal{B}''$. Thus, by removing two sufficiently small pieces from both ends of each γ_j', all of the resulting arcs γ_j'' $(1 \leq j \leq 4)$ will meet all but at most 8 members of $\mathcal{B}''$ (in the same cyclic order). Moreover, the convex hulls of these arcs $\operatorname{conv}\gamma_j'' \subseteq A_j$ will be pairwise disjoint. Hence, by Lemma 1 we can conclude that at least

$$|\mathcal{B}''| - 8 \geq \frac{|\mathcal{B}'|}{6^4} - 8 \geq \frac{\ell}{3 \cdot 6^4} - 8 \geq 3$$

members of $\mathcal{B}$ have a point in common, which is impossible, because $\mathcal{B}$ is 2-thin. This completes the proof of Theorem 1. ∎

4. **Intersection patterns of families of convex sets.** Given a family $\mathcal{A}$ of n convex sets, and a set of points T, we construct a family $\mathcal{A}^*$ as follows: For any $A \in \mathcal{A}$ let A^* be the convex hull of $A \cap T$, and let $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. T is said to be a *vertex representation* of $\mathcal{A}$ if $\mathcal{A}$ and $\mathcal{A}^*$ have the same intersection pattern, or equivalently, if for any subset $\mathcal{F}$ of $\mathcal{A}$, $\bigcap \mathcal{F} \neq \emptyset$ implies $\bigcap \mathcal{F}^* \neq \emptyset$.

Let $v(n)$ be the minimum value such that every 2-thin family of n closed convex sets in the plane has a vertex representation T with $|T| \leq v(n)$. In [5] it has been established that $v(n) \leq \frac{3n^2}{8}$ and that $v(n) \geq cn$ for every $c > 0$ provided that n is sufficiently large. This can be improved by

THEOREM 2. *There are positive constants c_1 and c_2 and a positive integer m such that for all n*

$$c_1 n^{\frac{4}{3}} \leq v(n) \leq c_2 n^{2-\frac{1}{m}}.$$

PROOF. The existence of n points and n lines in the plane such that the number of incidences between them is at least $cn^{\frac{4}{3}}$ (with $c > 0$ independent of n) has been established by Erdős (see the construction described in the proof of Theorem 6.18 in [1]). Replace every point by a closed disc of small positive radius ε, and move every line by ε parallel to itself. We obtain a 2-thin family C of n discs and n lines such that a disc touches a line if and only if the corresponding point and line were incident in the original construction. Hence there are at least $cn^{4/3}$ points at which some disc and some line in C touch each other, and each of them must be contained in every vertex representation of C. This implies that $v(2n) \geq cn^{\frac{4}{3}}$.

To prove the upper bound, notice that the members of any 2-thin family $\mathcal{A} = \{A_i : 1 \leq i \leq n\}$ can be replaced by smaller closed convex polygons (possibly segments or points) $A_i' \subseteq A_i$ so that $\mathcal{A}$ and $\mathcal{A}' = \{A_i' : 1 \leq i \leq n\}$ have the same intersection pattern and every vertex of A_i' is the only intersection point of A_i' with some A_j'. (For the details, see the proof of Theorem 7 in [5].) Clearly, the set of vertices of the members of $\mathcal{A}'$ is a vertex representation of $\mathcal{A}$.

Construct a graph G on the vertex set $\mathcal{A}'$ by connecting A_i' and A_j' with an edge if and only if they have exactly one point in common. By Theorem 1, G cannot contain a complete bipartite graph $K_{m,m}$ with $m \geq 3c$ vertices in its classes. Otherwise $\mathcal{A}'$ would not be 2-thin. Hence, using a well-known result [4] in extremal graph theory, G has at most $c_2 n^{2-\frac{1}{m}}$ edges. This implies that total number of vertices of all members of $\mathcal{A}'$ is at most $c_2 n^{2-\frac{1}{m}}$, completing the proof. ■

5. **Related problems and results.** Our proof of Theorem 1 yields a very poor upper bound for the smallest value of c for which the assertion is true. Much better bounds can be established, if we restrict our attention to some special families of plane convex sets. For instance, it is not hard to prove the following.

PROPOSITION. *If $\mathcal{A}$ and $\mathcal{B}$ are touching families of rectangles whose sides are parallel to the axes and $|\mathcal{A}|, |\mathcal{B}| \geq 2k$, then either $\mathcal{A}$ or $\mathcal{B}$ contains k members having nonempty intersection.*

PROOF. Assume that $k \geq 2$, since otherwise there is nothing to prove.

If any two rectangles of A intersect or if any two rectangles of $\mathcal{B}$ intersect then by a well-known property of rectangles with sides parallel to the axis either $\bigcap A \neq \emptyset$ or $\bigcap \mathcal{B} \neq \emptyset$.

If 3 A's are pairwise disjoint and 3 B's are pairwise disjoint then a planar $K_{3,3}$ can be formed which is impossible.

Assume therefore, without loss of generality that A_1 and A_2 are disjoint rectangles of $\mathcal{A}$ separated by the vertical line L and that no three rectangles of $\mathcal{B}$ are pairwise disjoint.

Let $C = \{L \cap B_i : B_i \in \mathcal{B}\}$. Since every rectangle B of $\mathcal{B}$ meets A_1 and A_2, it follows that C is a family of $2k$ closed segments on the line L and that no 3 segments of C are pairwise disjoint. This implies that there are 2 points a and b of L such that every segment of C meets the set $\{a, b\}$.

Consequently, one of the two points is in at least k of the segments and therefore in at least k rectangles of $\mathcal{B}$. ■

It would be interesting to obtain a similar improvement of Theorem 1, when $\mathcal{A}$ and $\mathcal{B}$ are families of translates of a fixed plane convex set C.

We can also prove the following statement, somewhat related to Theorem 1.

THEOREM 3. *For any natural number k there exists $g(k)$ such that, if $\mathcal{A}$ and $\mathcal{B}$ are touching families of compact convex sets in the plane with nonempty interiors, with their boundary curves in general position (i.e., no 3 of them pass through the same point), and $|A|, |\mathcal{B}| \geq g(k)$, then either $\mathcal{A}$ or $\mathcal{B}$ has two members whose boundaries meet in at least k points.*

At the moment we do not even see whether $g(k) = O(k)$.

It seems that all of the above results can be generalized to families of simply connected sets such that the intersection of any two of them is connected.

It follows immediately from Lemma 2 that any family of n plane convex sets has an at least $n^{1/4}$-member subfamily that is either 2-thin or all of its members have a point in common. Does there always exist a much bigger subfamily with this property? For a similar result see [6].

The analogue of Theorem 1 is clearly false in higher dimensions, because already in 3-space there are arbitrary large families of pairwise touching convex bodies such that no two of them share an interior point, see [7].

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