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Electrostatic surface waves in a magnetized plasma propagating in channels

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Dispersion relations of electrostatic surface waves propagating in magnetized plasmas contained in an infinite duct and in an infinite cylindrical column surrounded by vacuum are derived by means of a Vlasov equation and fluid equations, respectively. The kinematic boundary condition imposed on the distribution function, the specular reflection conditions on the four sides of a duct, can be satisfied by placing infinite number of fictitious surface charge sheets spaced by the duct widths. The Vlasov equation that includes these surface charge sheets is solved by summing up the contribution due to the infinite charge sheets. The method of placing appropriate fictitious surface charge sheets enables one to treat the surface waves in bounded plasmas of Cartesian structure with mathematical efficiency, kinetically. The kinetic duct dispersion relation is compared with the dispersion relation for the magnetized cylindrical plasma column. When the square duct cross-sectional area as well as the cylinder radius become infinity, both dispersion relations become the dispersion relation of the upper-hybrid wave.

Keywords: plasma waves

1. Introduction

We investigate two cases of electrostatic surface waves propagating in a magnetized plasma contained in a duct and in a cylinder interfaced with vacuum by using a Vlasov equation and fluid equations, respectively. The kinetic theory of a surface wave in a semi-infinite plasma, which is the simplest geometry for a bounded plasma, is well known (Barr & Boyd 1972; Alexandrov, Bogdankevich & Rukhadze 1984). Surface waves in a slab plasma have previously been studied kinetically (Lee & Lim 2007). Recently, the transverse magnetic mode of a surface wave in a streaming plasma in a duct was investigated by using the Vlasov equation non-relativistically (Lee & Cho 2022) and relativistically (Lee & Lim 2022). The above-mentioned references are concerned with unmagnetized plasmas. In this work, we study similar problems with regards to magnetized plasmas in a duct and in a cylinder.

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Works on the surface waves in magnetized cylindrical or duct plasmas are rather few. Trivelpiece & Gould (1959) investigated cylindrical surface waves in a plasma-filled wave guide in an infinite axial magnetic field. Krall & Trivelpiece (1973) derived the dispersion relations of the surface waves in cylindrical metallic wave guides of a magnetized plasma of an infinite axial magnetic field. The assumption of the infinite axial magnetic field simplifies the plasma dynamics since the electron motions are predominantly along the axial direction, but for finite values of the static magnetic field, the complexity of the solutions prohibit understanding the physical aspects of the waves. The slow wave approximation was employed (Krall & Trivelpiece 1973) for an electromagnetic wave, by neglecting the wave magnetic field as compared with the static magnetic field. Swanson (1989) dealt with waves in a magnetized plasma-filled cylindrical waveguide for finite values of the static magnetic field. In this work, we investigate the electrostatic surface waves in magnetized plasmas propagating in a duct and in a cylinder.

The primary importance of kinetic theory for a bounded plasma is to satisfy the kinematic boundary condition for the distribution function on the boundary surface, the specular reflection condition, which is assumed to hold on a sharp interface. Here, 'sharp' means that the density gradient across the interface is theoretically infinite. The boundary value problem of Vlasov–Maxwell equations plus the aforementioned kinematic condition is facilitated by introducing appropriate fictitious surface charge sheets as a means to satisfy the specular reflection condition (Lee & Cho 2022; Lee & Lim 2022). We consider an infinite duct formed by the intersections of four planes: $x = 0, a$ and $y = 0, b$, with $-\infty < z < \infty$. Then, the specular reflection conditions on the four planes, $x = 0, a$ and $y = 0, b$ are satisfied by introducing fictitious surface charge sheets in the Maxwell equations in the form (Lee & Cho 2022; Lee & Lim 2022)

$$\begin{aligned} S(x, y, z, t) = A_1 \sum_{n=0,1,2,\ldots} \delta(x \pm 2na) + A_2 \sum_{n=1,2,\ldots} \delta(x \pm (2n-1)a) \\ + B_1 \sum_{n=0,1,2,\ldots} \delta(y \pm 2nb) + B_2 \sum_{n=1,2,\ldots} \delta(y \pm (2n-1)b), \end{aligned} \tag{1.1}$$

where A and B terms are to be determined by satisfying the electric and dynamic boundary conditions connecting the plasma field and the vacuum field across the interface. Thus, we are furnished with a linear system of equations to determine the dispersion relation of the surface wave.

A hot magnetized plasma has a notoriously complex dielectric tensor which involves an infinite series of Bessel functions of various order. In the cold plasma approximation, the dielectric tensor reduces to a simple expression free from the Bessel function series. Consequently, we can derive simple-looking dispersion relations of the surface waves in a duct plasma, kinetically.

A cylindrical plasma has a macroscopic geometrical similarity with a square duct plasma. For comparison purposes, we work out the dispersion relations of surface waves for an infinite cold magnetized cylindrical plasma by using fluid equations. Krall & Trivelpiece (1973) investigated experimentally forced modes of surface waves in a cylindrical plasma. For a natural mode, the electric or dynamic boundary conditions which connect the plasma field and the vacuum field can be determined from the governing equations themselves that we choose to employ. Actually, surface waves result from satisfying the connection formula, and the latter can be extracted from the basic equations themselves that we use. If the density gradient across the plasma and the other side is very steep, the connection formula can be easily obtained by 'infinitesimal integration'

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across the interface, which is the operation performed on a certain relevant equation in the manner $\int_{-\epsilon}^{\epsilon}(\cdots)\mathrm{d}x$ (Lee & Cho 1997).

In § 2, the Vlasov equation is solved for a magnetized cold plasma, including the fictitious surface charge sheets in the Poisson equation. In §§ 3 and 4, the dispersion relation of the surface wave propagating in the duct is derived. In § 5, the dispersion relation of the surface wave in a cold magnetized cylindrical plasma is derived by using fluid equations. In § 6, we compare the dispersion relations obtained for the duct plasma and the cylindrical plasma. It is shown that both dispersion relations reduce to that of the upper-hybrid wave when both the side length of the duct and the cylinder radius become infinite.

2. Kinetic equation for the duct plasma

The basic equations are the linearized Vlasov equation and the Poisson equation for electrons. Ions are assumed to be stationary and only form the neutralizing background,

$$\frac{\partial}{\partial t}f(\boldsymbol{r},\boldsymbol{v},t)+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}}-\frac{e}{mc}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial f}{\partial \boldsymbol{v}}-\frac{e}{m}\boldsymbol{E}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}=0, \tag{2.1}$$

with $B_0=\hat{z}B_0$ and

$$\nabla\cdot\boldsymbol{E}=-4\pi e\int f\,\mathrm{d}^3v+S(x,y,z), \tag{2.2}$$

$$\nabla\times\boldsymbol{E}=0. \tag{2.3}$$

In (2.2), $S(x,y,z)$ represents the fictitious charge sheets as defined in (1.1). Additionally, $f(\boldsymbol{r},\boldsymbol{v},t)$ and $f_0(\boldsymbol{v})$ are the perturbation and the equilibrium distribution functions, respectively, and the rest of the symbols are standard.

We Fourier transform the above equations by performing $\int_{-\infty}^{\infty} d^3r\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\cdots)$ and by assuming $\partial/\partial t=-\mathrm{i}\omega$. Then, the wave has a phasor $\mathrm{e}^{\mathrm{i}k_zz-\mathrm{i}\omega t}$. The (k_z,ω) dependency in the Fourier amplitudes will be suppressed. Using (2.2) and (2.3), we can write

$$\boldsymbol{E}(\boldsymbol{k},\omega)=\mathrm{i}\frac{\boldsymbol{k}}{k^2}\left[4\pi e\int f(\boldsymbol{k},\omega,\boldsymbol{v})\,\mathrm{d}^3v+S(\boldsymbol{k})\right], \tag{2.4}$$

where $S(\boldsymbol{k})$ is the Fourier transform of (1.1):

$$\begin{aligned}S(\boldsymbol{k})=&\,\delta(k_y)[A_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nak_x}+A_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)ak_x}]\\&+\delta(k_x)[B_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nbk_y}+B_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)bk_y}],\end{aligned} \tag{2.5}$$

where A and B terms may be functions of k_z, ω, the double signs are summed over, and the notation $\varSigma_0$ and $\varSigma_1$ are the summations in (1.1). Introducing cylindrical coordinates in the velocity space such that $v_x=v_\perp\cos\varphi$, $v_y=v_\perp\sin\varphi$, (2.1) is written in the form

$$f(\boldsymbol{k},\boldsymbol{v},\omega)=\frac{e}{m\omega_c}\int_{-\infty}^{\varphi}\mathrm{d}\varphi'\exp[\varPhi(\varphi)-\varPhi(\varphi')]\left[E_z\frac{\partial f_0}{\partial v_z}+(E_x cos\varphi'+E_y\sin\varphi')\frac{\partial f_0}{\partial v_\perp}\right], \tag{2.6}$$

where $\omega_c=eB_0/mc$, the gyrofrequency, and

$$\varPhi(\varphi)=\frac{\mathrm{i}}{\omega_c}[(\omega-k_zv_z)\varphi-v_\perp k_x\sin\varphi+v_\perp k_y\cos\varphi]. \tag{2.7}$$

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The $\int \mathrm{d}\varphi'$ integral can be carried out by introducing a Bessel function series and we can write (2.4) in the form

$$E_z(\boldsymbol{k}, \omega) = \frac{\mathrm{i}k_z}{k^2 \varepsilon_L} S(\boldsymbol{k}), \tag{2.8}$$

where we used $E_x = (k_x/k_z)E_z$, $E_y = (k_y/k_z)E_z$, and the dielectric permittivity ε_L is

$$\varepsilon_L = 1 - \frac{i}{k^2}\frac{\omega_p^2}{\omega_c}\int_0^\infty v_\perp\, \mathrm{d}v_\perp \int_{-\infty}^\infty \mathrm{d}v_z \int_0^{2\pi} \mathrm{d}\varphi \int_{-\infty}^{\varphi} \exp\frac{\mathrm{i}}{\omega_c}[\Phi(\varphi) - \Phi(\varphi')] \\ \times \left[k_z \frac{\partial f_0}{\partial v_z} + \frac{\partial f_0}{\partial v_\perp}(k_x \cos\varphi' + k_y \sin\varphi')\right]. \tag{2.9}$$

We shall evaluate the velocity integral in the above equation for a cold plasma distribution function

$$f_0(v_\perp, v_z) = \frac{\delta(v_\perp)}{2\pi v_\perp}\delta(v_z). \tag{2.10}$$

The angular integrals should be performed first to remove the singularity $1/v_\perp$ in the velocity integral. Following the well-known steps involving Bessel function series expansion, ε_L becomes after $\int \mathrm{d}\varphi \int \mathrm{d}\varphi'$ integration,

$$\varepsilon_L = 1 + \frac{\omega_p^2}{k^2}\int_0^\infty v_\perp\, \mathrm{d}v_\perp \int_{-\infty}^\infty \mathrm{d}v_z \sum_n \frac{J_n^2(a_e)}{\omega - k_z v_z - n\omega_c}\left(k_z \frac{\partial f_0}{\partial v_z} + \frac{n\omega_c}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right), \tag{2.11}$$

where ω_p is the plasma frequency and $a_e = k_\perp v_\perp/\omega_c$, $k_\perp = \sqrt{k_x^2 + k_y^2}$.

In evaluating the $\partial f_0/\partial v_z$ term in (2.11), $\int dv_\perp$ can be straightforwardly carried out by taking the limit $a_e \to 0$, which gives $J_n(0) \to \delta_{n,0}$. Also, by integrating by parts with respect to $\partial/\partial v_z$, one obtains

$$-\frac{\omega_p^2}{k^2}\frac{k_z^2}{\omega^2}. \tag{2.12}$$

For the term $\partial f_0/\partial v_\perp$, integrating by parts and using

$$\frac{\partial}{\partial v_\perp} J_n^2(a_e) = 2\frac{k_\perp}{\omega_c} J_n(a_e) J_n'(a_e) \tag{2.13}$$

and the asymptotic relation

$$\lim_{a\to 0} \frac{2n}{a} J_n(a) J_n'(a) \to \frac{1}{2}(\delta_{n,1} - \delta_{n,-1}) \tag{2.14}$$

yield

$$-\frac{\omega_p^2}{k^2}\frac{k_\perp^2}{\omega^2 - \omega_c^2}. \tag{2.15}$$

Collecting the above results, the cold plasma dielectric permittivity is obtained as

$$\varepsilon_L = 1 - \frac{\omega_p^2}{k^2}\left(\frac{k_z^2}{\omega^2} + \frac{k_\perp^2}{\omega^2 - \omega_c^2}\right). \tag{2.16}$$

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Next, we calculate the electric displacement

$$\boldsymbol{D}(\boldsymbol{k}, \omega) = \boldsymbol{E}(\boldsymbol{k}, \omega) + \frac{4\pi i}{\omega}\boldsymbol{J}(\boldsymbol{k}, \omega), \tag{2.17}$$

where

$$\boldsymbol{J}(\boldsymbol{k}, \omega) = -\mathrm{e}\int \mathrm{d}^3 v\, \boldsymbol{v} f(\boldsymbol{k}, \omega, \boldsymbol{v}) \tag{2.18}$$

is the current. Using (2.6) and (2.18), we write the x component in (2.17) as an integral

$$\begin{aligned} D_x = E_x - \frac{4\pi e^2}{m}\frac{i}{\omega_c\omega}\int_0^\infty v_\perp^2\,\mathrm{d}v_\perp \int_{-\infty}^{\infty} dv_z \int_0^{2\pi}\mathrm{d}\varphi\cos\varphi \\ \times \int_{-\infty}^{\varphi}\mathrm{d}\varphi' \exp\frac{i}{\omega_c}[(\omega - k_z v_z)(\varphi-\varphi') - v_\perp k_x(\sin\varphi - \sin\varphi') \\ + v_\perp k_y(\cos\varphi - \cos\varphi')] \times \left[E_z\frac{\partial f_0}{\partial v_z} + \frac{\partial f_0}{\partial v_\perp}(E_x\cos\varphi' + E_y\sin\varphi')\right]. \end{aligned} \tag{2.19}$$

In the above integral, the $\partial f_0/\partial v_z$ term vanishes due to the δ-function, $\delta(v_\perp)$. Integrating by parts with respect to $\partial/\partial v_\perp$, (2.19) becomes

$$D_x = E_x - \frac{4\pi e^2}{m}\frac{2i}{\omega_c\omega}\int_0^{2\pi}\mathrm{d}\varphi\cos\varphi\,\mathrm{e}^{\mathrm{i}(\omega/\omega_c)\varphi}\int_{-\infty}^{\varphi}\mathrm{d}\varphi'\,\mathrm{e}^{-\mathrm{i}\varphi'(\omega/\omega_c)}(E_x\cos\varphi' + E_y\sin\varphi'). \tag{2.20}$$

In the above equation, integrals are elementary and can be easily carried out. Entirely analogous algebra yields the expression for D_y. Summarizing the results, we write

$$D_x = \varepsilon_x E_x = \mathrm{i}\frac{k_x}{k^2}\frac{\varepsilon_x}{\varepsilon_L}S(\boldsymbol{k}), \tag{2.21}$$

$$D_y = \varepsilon_y E_y = \mathrm{i}\frac{k_y}{k^2}\frac{\varepsilon_y}{\varepsilon_L}S(\boldsymbol{k}), \tag{2.22}$$

where we used (2.8), and the dielectric coefficients $\varepsilon_{x,y}$ are

$$\varepsilon_x = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} + \frac{\mathrm{i}k_y}{k_x}\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2}, \tag{2.23}$$

$$\varepsilon_y = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} - \frac{\mathrm{i}k_x}{k_y}\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2}. \tag{2.24}$$

We have shown that, for a cold plasma, $\varepsilon_{x,y}$ can be obtained without going through the Bessel function series expansion. Equations (2.8), (2.21) and (2.22) will be connected to the corresponding vacuum side field components to derive the boundary equations.

3. Boundary equations for the duct plasma waves

To apply the boundary conditions on the interface, the electric field and the displacement vector components expressed in the Fourier $\boldsymbol{k}$ space should be inverted to the fields in the ordinary $\boldsymbol{r}$ space by performing $\int_{-\infty}^{\infty}\mathrm{d}k_x\int_{-\infty}^{\infty}\mathrm{d}k_y\,\mathrm{e}^{\mathrm{i}k_x x + \mathrm{i}k_y y}(\cdots)$. The integrals involve infinite series through the surface charge $S(\boldsymbol{k})$ (see (2.5)), but the infinite series are nicely

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summed at the particular positions corresponding to $x = 0, a$ and $y = 0, b$ (Lee & Cho 2022; Lee & Lim 2022). Thus, we apply the boundary conditions along the two infinite lines: $(x, y, z) = (0, 0, z)$ and (a, b, z) with $-\infty < z < \infty$. The two lines correspond to the two seams of the duct which are diagonally opposite. In the following, we present only the wide steps of the development; the details are referred to the earlier reports (Lee & Cho 2022; Lee & Lim 2022):

$$\begin{aligned} E_z(0,0,z) &= \lim_{x=0,y=0}\int_{-\infty}^{\infty}\mathrm{d}k_x\,\mathrm{e}^{\mathrm{i}k_x x}\int_{-\infty}^{\infty}\mathrm{d}k_y\,\mathrm{e}^{\mathrm{i}k_y y}\frac{\mathrm{i}k_z}{k^2\varepsilon_L} \\ &\quad\times[\delta(k_y)(A_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nak_x}+A_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)ak_x}) \\ &\quad+\delta(k_x)(B_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nbk_y}+B_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)bk_y})] \\ &= \int_{-\infty}^{\infty}\mathrm{d}k_x\frac{\mathrm{i}k_z}{(k_x^2+k_z^2)\varepsilon_L(k_x)}(A_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nak_x}+A_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)ak_x}) \\ &\quad+\int_{-\infty}^{\infty}\mathrm{d}k_y\frac{\mathrm{i}k_z}{(k_y^2+k_z^2)\varepsilon_L(k_y)}(B_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nbk_y}+B_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)bk_y}) \\ &= \int_{-\infty}^{\infty}\mathrm{d}k_x\frac{2\mathrm{i}k_z}{(k_x^2+k_z^2)\varepsilon_L(k_x)}(A_1S_1(ak_x)+A_2S_2(ak_x)) \\ &\quad+\int_{-\infty}^{\infty}\mathrm{d}k_y\frac{2\mathrm{i}k_z}{(k_y^2+k_z^2)\varepsilon_L(k_y)}(B_1S_1(bk_y)+B_2S_2(bk_y)), \end{aligned} \tag{3.1}$$

where

$$\varepsilon_L(k_y)=\varepsilon_L(k_x,k_y)|_{k_x=0},\quad \varepsilon_L(k_x)=\varepsilon_L(k_x,k_y)|_{k_y=0}, \tag{3.2a,b}$$

$$S_1(\chi)=\tfrac{1}{2}+\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{e}^{4\mathrm{i}\chi}+\cdots\quad S_2(\chi)=\mathrm{e}^{\mathrm{i}\chi}+\mathrm{e}^{3\mathrm{i}\chi}+\cdots, \tag{3.3a,b}$$

$$\begin{aligned} E_z(a,b,z) &= \int_{-\infty}^{\infty}\mathrm{d}k_x\frac{2\mathrm{i}k_z}{(k_x^2+k_z^2)\varepsilon_L(k_x)}(A_1S_2(ak_x)+A_2S_1(ak_x)) \\ &\quad+\int_{-\infty}^{\infty}\mathrm{d}k_y\frac{2\mathrm{i}k_z}{(k_y^2+k_z^2)\varepsilon_L(k_y)}(B_1S_2(bk_y)+B_2S_1(bk_y)), \end{aligned} \tag{3.4}$$

$$\begin{aligned} D_x(0,0,z) &= \lim_{x=0,y=0}\int_{-\infty}^{\infty}\mathrm{d}k_x\,\mathrm{e}^{\mathrm{i}k_x x}\int_{-\infty}^{\infty}\mathrm{d}k_y\,\mathrm{e}^{\mathrm{i}k_y y}\frac{\mathrm{i}k_x}{k^2}\frac{\varepsilon_x(k_x,k_y)}{\varepsilon_L} \\ &\quad\times[\delta(k_y)(A_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nak_x}+A_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)ak_x}) \\ &\quad+\delta(k_x)(B_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nbk_y}+B_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)bk_y})] \\ &= \int_{-\infty}^{\infty}\mathrm{d}k_x\frac{\mathrm{i}k_x}{k_x^2+k_z^2}\frac{\nu}{\varepsilon_L(k_x)}(A_1\varSigma_0\,\mathrm{e}^{\pm\mathrm{i}2nak_x}+A_2\varSigma_1\,\mathrm{e}^{\pm\mathrm{i}(2n-1)ak_x}) \\ &= \mathrm{i}A_1\int_{-\infty}^{\infty}\mathrm{d}k_x\frac{k_x}{k_x^2+k_z^2}\frac{\nu}{\varepsilon_L(k_x)}, \end{aligned} \tag{3.5}$$

where

$$\nu = 1-\frac{\omega_p^2}{\omega^2-\omega_c^2}. \tag{3.6}$$

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Analogously, we obtain

$$D_x(a, b, z) = -\mathrm{i}A_2 \int_{-\infty}^{\infty} \mathrm{d}k_x \frac{k_x}{k_x^2 + k_z^2} \frac{\nu}{\varepsilon_L(k_x)}, \tag{3.7}$$

$$D_y(0, 0, z) = \mathrm{i}B_1 \int_{-\infty}^{\infty} \mathrm{d}k_y \frac{k_y}{k_x^2 + k_z^2} \frac{\nu}{\varepsilon_L(k_y)}, \tag{3.8}$$

$$D_y(a, b, z) = -\mathrm{i}B_2 \int_{-\infty}^{\infty} \mathrm{d}k_y \frac{k_y}{k_x^2 + k_z^2} \frac{\nu}{\varepsilon_L(k_y)}. \tag{3.9}$$

Vacuum solution

We have $\nabla^2 \boldsymbol{E} = 0$ in the vacuum region, which is solved by

$$\boldsymbol{E} \sim \exp \mathrm{i}k_z \times \exp(\pm k_x x) \times \exp(\pm k_y y), \tag{3.10}$$

with the important constraint $k_x^2 + k_y^2 = k_z^2$. We can construct the vacuum solutions as follows:

vacuum region (i). $x < 0$, $y < 0$, where

$$E_z(i) = H\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{e}^{k_x x}\,\mathrm{e}^{k_y y}, \tag{3.11}$$

$$E_x(i) = -\mathrm{i}H \frac{k_x}{k_z}\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{e}^{k_x x}\,\mathrm{e}^{k_y y}, \tag{3.12}$$

$$E_y(i) = -\mathrm{i}H \frac{k_y}{k_z}\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{e}^{k_x x}\,\mathrm{e}^{k_y y}; \tag{3.13}$$

vacuum region (ii). $x > a$, $y > b$, where

$$E_z(ii) = G\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{e}^{-k_x x}\,\mathrm{e}^{-k_y y}, \tag{3.14}$$

$$E_x(ii) = \mathrm{i}G \frac{k_x}{k_z}\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{e}^{-k_x x}\,\mathrm{e}^{-k_y y}, \tag{3.15}$$

$$E_y(ii) = \mathrm{i}G \frac{k_y}{k_z}\,\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{e}^{-k_x x}\,\mathrm{e}^{-k_y y}. \tag{3.16}$$

Putting $(x, y) = (0, 0)$ or (a, b) in the above equations gives the vacuum side boundary values of the relevant quantities.

4. Dispersion relation for the duct plasma surface wave

We enforce the following boundary conditions to connect the plasma and the vacuum fields: $[E_z] = 0$, $[D_x] = 0$, $[D_y] = 0$.

Along lines $(0, 0, z)$ and (a, b, z), $[E_z] = 0$ respectively gives

$$A_1 I_1 + A_2 I_2 + B_1 J_1 + B_2 J_2 = H, \tag{4.1}$$

$$A_1 I_2 + A_2 I_1 + B_1 J_2 + B_2 J_1 = G\,\mathrm{e}^{-k_x a}\,\mathrm{e}^{-k_y b}, \tag{4.2}$$

where

$$I_i = \int_{-\infty}^{\infty} \mathrm{d}k_x \frac{2\mathrm{i}k_z}{(k_x^2 + k_z^2)\varepsilon_L(k_x)} S_i(ak_x) \quad (i = 1, 2), \tag{4.3}$$

$$J_i = \int_{-\infty}^{\infty} \mathrm{d}k_y \frac{2\mathrm{i}k_z}{(k_y^2 + k_z^2)\varepsilon_L(k_y)} S_i(bk_y) \quad (i = 1, 2), \tag{4.4}$$

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$[D_x]=0$ gives

$$\mathrm{i}A_1\int_{-\infty}^{\infty}\mathrm{d}k_x\frac{k_x}{k_x^2+k_z^2}\frac{\nu}{\varepsilon_L(k_x)}=-\mathrm{i}H\frac{k_x}{k_z}, \tag{4.5}$$

$$-\mathrm{i}A_2\int_{-\infty}^{\infty}\mathrm{d}k_x\frac{k_x}{k_x^2+k_z^2}\frac{\nu}{\varepsilon_L(k_x)}=\mathrm{i}G\frac{k_x}{k_z}\,\mathrm{e}^{-k_xa}\,\mathrm{e}^{-k_yb}, \tag{4.6}$$

$[D_y]=0$ gives

$$\mathrm{i}B_1\int_{-\infty}^{\infty}\mathrm{d}k_y\frac{k_y}{k_x^2+k_z^2}\frac{\nu}{\varepsilon_L(k_y)}=-\mathrm{i}H\frac{k_y}{k_z}, \tag{4.7}$$

$$-\mathrm{i}B_2\int_{-\infty}^{\infty}\mathrm{d}k_y\frac{k_y}{k_x^2+k_z^2}\frac{\nu}{\varepsilon_L(k_y)}=\mathrm{i}G\frac{k_y}{k_z}\,\mathrm{e}^{-k_xa}\,\mathrm{e}^{-k_yb}. \tag{4.8}$$

Let us rewrite (4.3) and (4.4) in slightly different forms:

$$I_i=2\mathrm{i}k_z\mu\int_{-\infty}^{\infty}\mathrm{d}\kappa\,\frac{S_i(a\kappa)}{\kappa^2+\xi^2k_z^2}\quad (i=1,2), \tag{4.9}$$

$$J_i=2\mathrm{i}k_z\mu\int_{-\infty}^{\infty}\mathrm{d}\kappa\,\frac{S_i(b\kappa)}{\kappa^2+\xi^2k_z^2}\quad (i=1,2), \tag{4.10}$$

where

$$\mu=\frac{\omega^2-\omega_p^2}{\omega^2-\omega_P^2-\omega_c^2}, \tag{4.11}$$

$$\xi=\sqrt{\frac{(\omega^2-\omega_c^2)(\omega^2-\omega_p^2)}{\omega^2(\omega^2-\omega_P^2-\omega_c^2)}}=\sqrt{\left(1-\frac{\omega_p^2}{\omega^2}\right)\Big/\left(1-\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)}. \tag{4.12}$$

Integrals in (4.5)–(4.8) can be contour-integrated, giving the value $\mathrm{i}\pi$. Therefore, they become

$$\pi\eta A_1=\mathrm{i}H\frac{k_x}{k_z}, \tag{4.13}$$

$$\pi\eta A_2=\mathrm{i}G\frac{k_x}{k_z}\,\mathrm{e}^{-k_xa}\,\mathrm{e}^{-k_yb}, \tag{4.14}$$

$$\pi\eta B_1=\mathrm{i}H\frac{k_y}{k_z}, \tag{4.15}$$

$$\pi\eta B_2=\mathrm{i}G\frac{k_y}{k_z}\,\mathrm{e}^{-k_xa}\,\mathrm{e}^{-k_yb}, \tag{4.16}$$

where

$$\eta=\frac{\omega^2-\omega_p^2}{\omega^2-\omega_c^2}. \tag{4.17}$$

We have six unknowns, A_1, A_2, B_1, B_2, H, G in six equations (4.1), (4.2), (4.13)–(4.16). Eliminating H, G, A_1, A_2 gives

$$B_1(k_xI_1+k_yJ_1+\mathrm{i}\pi\eta\,k_z)+B_2(J_2k_y+I_2k_x)=0, \tag{4.18}$$

$$B_1(I_2k_x+J_2k_y)+B_2(I_1k_x+J_ik_y+\mathrm{i}\pi\eta k_z)=0, \tag{4.19}$$

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which give the dispersion relation in the form

$$k_x(I_1 \pm I_2) + k_y(J_1 \pm J_2) = -\mathrm{i}\pi\eta k_z. \tag{4.20}$$

The $+$ ($-$) sign corresponds to symmetric (anti-symmetric) mode. Since we have the constraint $k_x^2 + k_y^2 = k_z^2$, let us put

$$k_x = \frac{bk_z}{\sqrt{a^2+b^2}}, \quad k_y = \frac{ak_z}{\sqrt{a^2+b^2}}, \tag{4.21a,b}$$

so that the y-direction becomes ignorable when $b \to \infty$. Using the formula (Lee & Cho 2022; Lee & Lim 2022)

$$\int_{-\infty}^{\infty} \mathrm{d}\kappa \frac{S_i(a\kappa) \pm S_2(a\kappa)}{\kappa^2 + \xi^2 k_z^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa}{\kappa^2 + \xi^2 k_z^2} \frac{1 \pm \mathrm{e}^{\mathrm{i}a\kappa}}{1 \mp \mathrm{e}^{\mathrm{i}a\kappa}}. \tag{4.22}$$

Equation (4.20) takes the form

$$\frac{b}{\sqrt{a^2+b^2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa}{\kappa^2 + \xi^2 k_z^2} \frac{1 \pm \mathrm{e}^{\mathrm{i}a\kappa}}{1 \mp \mathrm{e}^{\mathrm{i}a\kappa}} + \frac{a}{\sqrt{a^2+b^2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa}{\kappa^2 + \xi^2 k_z^2} \frac{1 \pm \mathrm{e}^{\mathrm{i}b\kappa}}{1 \mp \mathrm{e}^{\mathrm{i}b\kappa}} = -\frac{\pi\eta}{k_z\mu}. \tag{4.23}$$

Picking up the simple pole at $\kappa = i\xi k_z$, (4.23) becomes

$$\frac{b}{\sqrt{a^2+b^2}} \tanh \frac{a}{2}\xi k_z + \frac{a}{\sqrt{a^2+b^2}} \tanh \frac{b}{2}\xi k_z = -\frac{\eta\xi}{\mu} = -\nu\xi, \tag{4.24}$$

for the lower sign (symmetric mode), where ν is defined by (3.6). For the upper sign (anti-symmetric mode), tanh is replaced by coth. Taking the limit $b \to \infty$ gives the slab dispersion relation

$$\tanh \frac{a}{2}\xi k_z = -\nu\xi. \tag{4.25}$$

Taking the limit $a \to \infty$ in the above equation gives the dispersion relation for semi-infinite plasma, which is $\nu\xi = -1$:

$$1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} + \sqrt{\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\right) \bigg/ \left(1 - \frac{\omega_p^2}{\omega^2}\right)} = 0, \tag{4.26}$$

which agrees with the electrostatic dispersion relation obtained from the cold fluid equation (Lee 1995). The slab dispersion relations obtained by taking $b \to \infty$ that contain the symmetric and anti-symmetric modes agree with the earlier results which were obtained by fluid equation (Gradov & Stenflo 1983).

5. Cylindrical plasma

In this section, we derive the surface wave dispersion relations in a cold magnetized plasma contained in infinite-length cylinder of radius 'a', interfaced with vacuum. We use

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the following fluid equations written in cylindrical coordinates:

$$-\mathrm{i}\omega \boldsymbol{v}(r,\theta,z) = \frac{e}{m}\nabla\phi - \omega_c \boldsymbol{v}\times\hat{z}, \tag{5.1}$$

$$-\mathrm{i}\omega n(r,\theta,z) + N(r)\nabla\cdot\boldsymbol{v} + v_r\frac{\partial N}{\partial r} = 0, \tag{5.2}$$

$$\nabla^2\phi = 4\pi e n. \tag{5.3}$$

Here, we assume that the equilibrium density $N(r)$ has density gradients in the radial direction. At the interface $r = a$, N changes abruptly with infinite gradient, otherwise is homogeneous. This artificiality is designed solely to derive the boundary conditions at the interface by 'infinitesimal integration' across the interface. Equation (5.1) yields

$$v_r = \frac{e}{m}\frac{\omega}{\omega^2-\omega_c^2}\left(\mathrm{i}\frac{\partial\phi}{\partial r} + \frac{\omega_c}{\omega}\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right), \tag{5.4}$$

$$v_\theta = \frac{e}{m}\frac{\omega}{\omega^2-\omega_c^2}\left(-\frac{\omega_c}{\omega}\frac{\partial\phi}{\partial r} + \mathrm{i}\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right), \tag{5.5}$$

$$v_z = \frac{\mathrm{i}}{\omega}\frac{e}{m}\frac{\partial\phi}{\partial z}. \tag{5.6}$$

Equations (5.2)–(5.6) give the wave equation written in terms of ϕ:

$$\frac{\partial}{\partial r}\left[\left(1-\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\frac{\partial\phi}{\partial r}\right] + \left(1-\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial}{\partial z}\left[\left(1-\frac{\omega_p^2}{\omega^2}\right)\frac{\partial\phi}{\partial z}\right]$$
$$+\frac{\mathrm{i}\omega_c}{\omega}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\frac{\partial\phi}{\partial\theta} + \left(1-\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0, \tag{5.7}$$

where $\omega_p^2 = 4\pi e^2 N(r)/m$. If we take the infinitesimal integration across the interface $r = a$ in the manner $\int_{-\epsilon}^{\epsilon}(\cdots)\,\mathrm{d}r$, we can extract useful boundary conditions. In this operation, we pick up only the perfect differentials that survive the limit $\epsilon \to 0$. Equation (5.4) yields

$$[\phi]_{r=a} = 0. \tag{5.8}$$

Integrating (5.7) yields

$$\left[\left(1-\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\frac{\partial\phi}{\partial r}\right]_{r=a} = 0, \tag{5.9}$$

where the square bracket denotes the jump across the interface. Putting $\omega_p =$ const. or zero in (5.7) gives the wave equation in the plasma or in the vacuum. We also restrict to an azimuthally symmetric field ($\partial/\partial\theta = 0$). We have

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0, \quad \text{in vacuum}, \tag{5.10}$$

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \xi^2\frac{\partial^2\phi}{\partial z^2} = 0, \quad \text{in plasma}, \tag{5.11}$$

where ξ is defined in (4.12).

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Now let us solve the above equations for an infinite cylindrical plasma. Since the z-direction is infinite, we can assume $\phi(z) \sim \mathrm{e}^{\mathrm{i}k_z z}$ and the plasma equation in (5.11) becomes

$$\frac{\partial^2\phi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \xi^2 k_z^2 \phi(r) = 0, \tag{5.12}$$

where we suppressed the z-dependent phase. Equation (5.12) is solved by the functions I_0 and K_0, the modified Bessel functions of first and second kind of order zero, respectively. Here, K_0 is rejected since it blows up at $r = 0$, so the plasma solution takes the form

$$\phi_p(r) = AI_0(k_z\xi r) \quad 0 < r < a. \tag{5.13}$$

In vacuum, the function I_0 should be rejected since it blows up at $r = \infty$. We have

$$\phi_v(r) = BK_0(k_z r) \quad r > a, \tag{5.14}$$

where A, B are constants to be determined by boundary conditions. Equations (5.8) and (5.9) give

$$AI_0(\xi k_z a) = BK_0(k_z a), \tag{5.15}$$

$$\nu\xi AI_0'(\xi k_z a) = BK_0'(k_z a), \tag{5.16}$$

where prime denotes the derivative with respect to the argument and ν is defined in (3.6). The above two equations yield the dispersion relation

$$\nu\xi \frac{I_0'(\xi k_z a)}{I_0(\xi k_z a)} = \frac{K_0'(k_z a)}{K_0(k_z a)}. \tag{5.17}$$

6. Discussion

We consider a square duct plasma whose surface wave dispersion relation can be obtained from (4.24) by putting $a = b$:

$$\sqrt{2}\tanh\frac{\xi}{2}ak_z = -\nu\xi. \tag{6.1}$$

Direct comparison of (5.17) and (6.1) requires numerical computation. Here, we wish to be satisfied only with an asymptotic evaluation of the two relations. Using the formula for Bessel functions

$$\frac{\mathrm{d}}{\mathrm{d}x}I_0(\alpha x) = \alpha I_1(\alpha x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}K_0(\alpha x) = -\alpha K_1(\alpha x), \tag{6.2a,b}$$

(5.17) becomes

$$\nu\xi \frac{I_1(\xi k_z a)}{I_0(\xi k_z a)} = -\frac{K_1(k_z a)}{K_0(k_z a)}. \tag{6.3}$$

If $k_z a \to \infty$, $I_0 \approx I_1$, $K_0 \approx K_1$, and therefore (6.3) becomes

$$\nu\xi = -1 \text{ or } \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\right)\sqrt{\frac{1 - \dfrac{\omega_p^2}{\omega^2}}{1 - \dfrac{\omega_p^2}{\omega^2 - \omega_c^2}}} = -1, \tag{6.4}$$

which is identical to (4.26).

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However, the square duct surface wave in (6.1) gives in the limit $k_z a \gg 1$, $\nu\xi = -\sqrt{2}$. Thus, they agree, asymptotically, except for the geometrical factor $\sqrt{2}$.

Scrutinizing (6.4) indicates that ω^2 should be in the range $\omega_c^2 < \omega^2 < \omega_p^2$. Solving (6.4) for ω^2 gives

$$\omega^2 = \tfrac{1}{2}(\omega_p^2 + \omega_c^2), \tag{6.5}$$

which is the upper-hybrid frequency.

By taking $a \to \infty$, we removed the cylindrical boundary, and (6.5) is the eigenmode of the upper-hybrid frequency oscillating in the semi-infinite plasma.

This work may find applications in a laboratory or astrophysical situation where electrostatic waves propagate through certain channels. Obtaining dispersion relations for waves in channels is of significant interest for the development of plasma electronics and studying the initial stage of development of instabilities and wave generation. In this work, we assumed the plasma–vacuum boundary is infinitely sharp, which might deviate from the actual experimental situation. However, models other than the sharp boundary, for example, the diffuse boundary model, prohibit analytic derivation of the surface wave dispersion relations.

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