AN INTEGRAL EQUATION FROM PHYTOLOGY

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1. Introduction

We examine the equation

$$(1.1) \qquad f(\beta) = \int_0^{\frac{1}{2}\pi} K(\alpha, \beta) g(\alpha) d\alpha \qquad (0 \leqq \beta \leqq \tfrac{1}{2}\pi)$$

or, briefly,

$$f = Kg,$$

where

$$(1.2) \qquad K(\alpha, \beta) = \begin{cases} \cos\alpha \sin\beta & (\alpha \leqq \beta) \\ \cos\alpha \sin\beta \{1 + \frac{1}{2}\pi\ (\tan\theta(\alpha, \beta) - \theta(\alpha, \beta)\} & (\alpha \geqq \beta), \end{cases}$$

with

$$(1.3) \qquad \theta(\alpha, \beta) = \cos^{-1}\left(\frac{\tan\beta}{\tan\alpha}\right) \qquad (\text{principal value; } \alpha \geqq \beta).$$

This integral equation arises in connection with the problem of measuring foliage density of small plants and grasses by means of point quadrats; it is due to J. R. Philip [1]. Foliage density is defined to be the area of foliage per unit volume of space. In order to assess the foliage density within a certain spacial region, the phytologist pushes a point quadrat (which is a sharp needle, suitably mounted) through the region along a line inclined at an angle β to the horizontal, and records the number of contacts with foliage made by the point of the quadrat per unit length of travel: this figure determines $f(\beta)$. The unknown distribution of foliage angle is given by g: $g(\alpha)d\alpha$ is the contribution to foliage density due to foliage inclined at angles between α and $\alpha + d\alpha$ to the horizontal (it being supposed that the foliage slopes non-preferentially to all points of the compass). The practical problem is to find g from a knowledge of the values of $f(\beta)$ for a few values of β. In general the phytologist must work on the assumption that f is smooth; g is of course expected to be non-negative, but may be anything from constant to, say, a delta function.

The form of the kernel K is due to J. Warren Wilson and J. E. Reeve [4]. K is continuous over the square $[0, \frac{1}{2}\pi] \times [0, \frac{1}{2}\pi]$, but it is not symmetric: thus (1.1) is a Fredholm integral equation of the first kind whose

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L^2 theory would be covered by, say, the discussion in [3], §§ 3.15, 3.16. The purpose of the present note is to describe the L^1 theory, where an explicit formula for the solution ((3.5)) can be found by quite modest means when f is sufficiently smooth. The solution found is unique, but depends explicitly upon f and its first three derivatives; thus to estimate g, many values of $f(\beta)$ are required. Consequently, the solution is principally of theoretical interest, and is unsuitable for application to experimental data. We do not examine here what further conditions on f are necessary in order that g be non-negative, as required.

I must thank Dr. Philip for introducing me to the subject and for the benefit of several helpful discussions, and the referee for additional comments.

2. Range of K

The transform relation (1.1) is, in more detail,

$$\tfrac{1}{2}\pi f(\beta) = \tfrac{1}{2}\pi c_g \sin\beta + \sin\beta \int_\beta^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\,(\tan\theta(\alpha,\beta)-\theta(\alpha,\beta))\,d\alpha, \tag{2.1}$$

where

$$c_g = \int_0^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\,d\alpha.$$

In this section we shall take K to be the linear operator defined by (1.1) whose domain is $L^1(0, \frac{1}{2}\pi)$ (briefly, L^1); we assume that g is a function in L^1, and consider the consequent properties of its transform f. In this way *we find necessary conditions on f for the existence of solutions g in L^1.*

Notice that $0 \leqq \theta(\alpha,\beta) \leqq \frac{1}{2}\pi$; for fixed $\beta \neq 0$, $\theta(\alpha,\beta)$ increases from 0 to $\frac{1}{2}\pi$ as α increases from β to $\frac{1}{2}\pi$, while for fixed $\alpha \neq 0$, $\theta(\alpha,\beta)$ decreases from $\frac{1}{2}\pi$ to 0 as β increases from 0 to α.

LEMMA 1. *If $g \in L^1$, then*

$$\lim_{\beta\to 0} f(\beta) = \int_0^{\frac{1}{2}\pi} g(\alpha)\sin\alpha\,d\alpha, \quad \lim_{\beta\to\frac{1}{2}\pi} f(\beta) = \int_0^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\,d\alpha = c_g.$$

PROOF. Since

$$\tan\theta(\alpha,\beta) = \cot\beta\sqrt{\tan^2\alpha - \tan^2\beta}, \tag{2.2}$$

$\cos\alpha\tan\theta(\alpha,\beta)$ is an increasing function of α, and

$$0 \leqq \cos\alpha\tan\theta(\alpha,\beta) \leqq \cot\beta \quad \text{for} \quad 0 < \beta \leqq \alpha \leqq \tfrac{1}{2}\pi.$$

The result follows from (2.1).

LEMMA 2. *If $g \in L^1$, then $f'(\beta)$ exists for all β in $(0, \frac{1}{2}\pi)$, and*

$$\frac{\pi}{2}\frac{d}{d\beta}\left(\frac{f(\beta)}{\sin\beta}\right) = -\frac{1}{\sin^2\beta}\int_\beta^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\sqrt{\tan^2\alpha-\tan^2\beta}\,d\alpha. \tag{2.3}$$

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PROOF. Formula (2.3) follows formally from (2.1) by differentiation, since

$$\frac{\partial}{\partial\beta}\left(\tan\theta(\alpha,\beta)-\theta(\alpha,\beta)\right) = -\frac{\sqrt{\tan^2\alpha-\tan^2\beta}}{\sin^2\beta}. \tag{2.4}$$

To prove the lemma, let $h(\beta)$ denote the righthand side of (2.3). Taking $0<\sigma<\tau<\frac{1}{2}\pi$, integrate $h(\beta)$ over (σ,τ), inverting the order of integration in the double integral. (This is justified by Fubini's and Tonelli's theorems, under the assumption $g\in L^1$.) We find

$$\int_\sigma^\tau h(\beta)d\beta = \frac{\pi}{2}\frac{f(\tau)}{\sin\tau} - \frac{\pi}{2}\frac{f(\sigma)}{\sin\sigma}.$$

Hence

$$h(\tau) = \frac{\pi}{2}\frac{\partial}{\partial\tau}\left(\frac{f(\tau)}{\sin\tau}\right)$$

for almost all τ in $(0,\frac{1}{2}\pi)$. Since in fact $h(\tau)$ exists for all τ in $(0,\frac{1}{2}\pi)$, we can assume that $f'(\tau)$ likewise exists for all τ. The result follows.

LEMMA 3. *If* $g\in L^1$, *then* $\lim_{\beta\to 0} f'(\beta) = \lim_{\beta\to\frac{1}{2}\pi} f'(\beta) = 0$.

PROOF. (2.3) is equivalent to

$$\begin{aligned}\tfrac{1}{2}\pi f'(\beta) = \tfrac{1}{2}\pi c_g\cos\beta + \cos\beta\int_\beta^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\left(\tan\theta(\alpha,\beta)-\theta(\alpha,\beta)\right)d\alpha \\ -\operatorname{cosec}\beta\int_\beta^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\sqrt{\tan^2\alpha-\tan^2\beta}\,d\alpha.\end{aligned} \tag{2.5}$$

The value of the limit as $\beta\to\frac{1}{2}\pi$ follows without difficulty. To derive the limit as $\beta\to 0$, one first shows that

$$\lim_{\beta\to 0}\int_\beta^{\frac{1}{2}\pi} g(\alpha)\cos\alpha\,\theta(\alpha,\beta)d\alpha = \tfrac{1}{2}\pi c_g,$$

and then uses (2.5). We omit the details.

LEMMA 4. *If* $g\in L^1$, *then* $f''(\beta)$ *exists for almost all* β *in* $(0,\frac{1}{2}\pi)$, *determining a measurable function, and for such* β,

$$\tfrac{1}{2}\pi\cos^3\beta\left(f(\beta)+f''(\beta)\right) = \int_\beta^{\frac{1}{2}\pi}\frac{g(\alpha)\cos\alpha\,d\alpha}{\sqrt{\tan^2\alpha-\tan^2\beta}}. \tag{2.6}$$

The proof follows closely that of Lemma 2, so we omit it.

We conclude from Lemmas 2, 3 and 4 that *the range of* K *is contained in the class of functions* f *which are defined and have absolutely continuous first derivative on the open interval* $(0,\frac{1}{2}\pi)$, *with* $f'(0+0) = f'(\frac{1}{2}\pi-0) = 0$.

It can be shown that f has a third derivative if g is also absolutely continuous and satisfies certain integrability conditions.

3. Solution of the integral equation

The solution in L^1 is unique.

THEOREM 1. *The integral equation* (1.1) *has at most one solution* g *in* L^1, *if* f *is given.*

PROOF. Let g_1 and g_2 be two solutions of (2.1), in L^1. By Lemma 2,

$$\int_\beta^{\frac{1}{2}\pi} (g_1(\alpha)-g_2(\alpha)) \cos\alpha \sqrt{\tan^2\alpha-\tan^2\beta}\, d\alpha = 0 \tag{3.1}$$

for all β in $(0, \frac{1}{2}\pi)$. Make the change to variables x and y defined by

$$\tan^2\alpha = x, \quad \tan^2\beta = y, \tag{3.2}$$

and write

$$r(x) = \frac{h(\tan^{-1}\sqrt{x})}{2x^{\frac{1}{2}}(1+x)^{\frac{3}{2}}}, \quad h(\alpha) = g_1(\alpha)-g_2(\alpha);$$

(3.1) becomes

$$\int_y^\infty (x-y)^{\frac{1}{2}} r(x)dx = 0 \quad \text{for all } y \text{ in } (0, \infty).$$

Titchmarsh's convolution theorem [1] implies that $r(x) = 0$ for almost all x. The result follows.

In § 2 we have found necessary conditions on f for the existence of a solution $g \in L^1$. *We now find sufficient conditions, and obtain an explicit formula for the solution.* Formally, this is done as follows. The solution g satisfies the differentiated form (2.3); change to the variables x and y of (3.2), and introduce functions p and q by writing

$$p(y) = -\frac{\pi}{2}\sin^2\beta \frac{d}{d\beta}\left(\frac{f(\beta)}{\sin\beta}\right) \qquad (0 < \beta < \tfrac{1}{2}\pi;\ 0 < y < \infty),$$

$$q(x) = \frac{g(\alpha)}{2\tan\alpha\sec^3\alpha} \qquad (\beta < \alpha < \tfrac{1}{2}\pi;\ y < x < \infty).$$

Equations (2.3) becomes

$$p(y) = \int_y^\infty (x-y)^{\frac{1}{2}} q(x)dx. \tag{3.3}$$

Then

$$\int_t^\infty \frac{(y-t)^{-\frac{1}{2}}}{y^2} p(y)dy = \int_t^\infty q(x)dx \int_t^x \frac{(x-y)^{\frac{1}{2}}(y-t)^{-\frac{1}{2}}}{y^2}dy$$
$$= \int_t^\infty q(x)\frac{(x-t)}{x^{\frac{1}{2}}t^{\frac{3}{2}}}\frac{\pi}{2}dx,$$

so that

[1] [2], p. 325, Theorem 152.

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$$\frac{2t^{\frac{3}{2}}}{\pi}\int_t^\infty \frac{(y-t)^{-\frac{1}{2}}}{y^2}\,p(y)dy = \int_t^\infty du\int_u^\infty \frac{q(x)}{x^{\frac{1}{2}}}\,dx.$$

Therefore

$$(-1)^2\frac{q(t)}{t^{\frac{1}{2}}} = \frac{2}{\pi}\frac{d^2}{dt^2}\left(t^{\frac{3}{2}}\int_t^\infty \frac{(y-t)^{-\frac{1}{2}}}{y^2}\,p(y)dy\right)$$

$$= \frac{2}{\pi}\frac{d^2}{dt^2}\int_1^\infty \frac{(u-1)^{-\frac{1}{2}}}{u^2}\,p(tu)du$$

$$= \frac{2}{\pi}\int_1^\infty (u-1)^{-\frac{1}{2}}p''(tu)du$$

i.e.

$$q(t) = \frac{2}{\pi}\int_t^\infty (y-t)^{-\frac{1}{2}}p''(y)dy. \tag{3.4}$$

In terms of the original functions, this is

$$g(\alpha) = \tan\alpha\sec^3\alpha\int_\alpha^{\frac{1}{2}\pi}\frac{3\cos^2\tau\sin\tau(f(\tau)+f''(\tau))-\cos^3\tau(f'(\tau)+f'''(\tau))}{\sqrt{\tan^2\tau-\tan^2\alpha}}\,d\tau. \tag{3.5}$$

Rather than justify the above argument, it is simpler to start with (3.5), and show that it defines a solution of (2.1) under suitable conditions on f. If this is done (and we shall not elaborate the details here), we obtain

THEOREM 2. *Let f be such that f'' exists and is absolutely continuous on $[0, \frac{1}{2}\pi]$, and*

$$f'(0) = f'(\tfrac{1}{2}\pi) = 0.$$

Then $g(\alpha)$, given by (3.5), *exists for almost all α, and g is the solution of* (1.1) *belonging to L^1.*

Finally, to round off the discussion, we consider the circumstances in which a solution g of (1.1) can be found by solving one of the differentiated forms of the equation, (2.3) or (2.6). It is evident, for example, that a solution g of (2.3), which does not contain the constant c_g depending upon the solution, may not be a solution of (2.1). We state without proof

THEOREM 3. (i) *Let f' be absolutely continuous on $(0, \frac{1}{2}\pi)$. If g is a solution in L^1 of* (2.6), *it is also a solution of* (2.3) *if and only if $f'(\frac{1}{2}\pi-0) = 0$.*

(ii) *Suppose instead that f is absolutely continuous on $(0, \frac{1}{2}\pi)$. If g is a solution in L^1 of* (2.3), *it is also a solution of* (2.1) *if and only if $c_g = f(\frac{1}{2}\pi-0)$.*

References

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