On Integral Transforms

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1. Titchmarsh and others [4] have studied integral transform pairs of the type

$$f(x) = \int_0^\infty k(xy)\, g(y)\, dy,$$

$$g(y) = \int_0^\infty l(xy)\, f(x)\, dx,$$

showing how such pairs may be generated as a consequence of the relation which holds between the Mellin transforms of the kernels.

There is an interesting allied formal process which, in some respects, may be regarded as a generalisation of this.

For a particular transformation T let the basic formulae be

$$H_T(x) = \int_{a_T}^{b_T} k_T(x, y)\, h(y)\, dy, \tag{1}$$

$$h(y) = \int_{c_T}^{d_T} l_T(x, y)\, H_T(x)\, dx. \tag{2}$$

Consider the function

$$f(s) = \int_\alpha^\beta g(s, t)\, i(t)\, dt. \tag{3}$$

Then, formally, we have

$$F_T(u) = \int_\alpha^\beta G_T(u, t)\, i(t)\, dt. \tag{4}$$

Suppose, however, that $G_T(u, t) \equiv \lambda(u)\; k_Q\{\mu(u), t\}$ and also that the path of integration in (4) is identical with, or capable of deformation without crossing a pole of the integrand into, the path from a_Q to b_Q required by the formula (1) for the transformation Q. Then (4) is the same as

$$\frac{F_T(u)}{\lambda(u)} = \int_{a_Q}^{b_Q} k_Q\{\mu(u), t\}\, i(t)\, dt = I_Q\{\mu(u)\}. \tag{5}$$

It follows from the substitution $v = \mu(u)$ (or $u = \mu^{-1}(v)$) that

$$I_Q(v) = \frac{F_T\{\mu^{-1}(v)\}}{\lambda\{\mu^{-1}(v)\}}, \tag{6}$$

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and hence

$$i(t) = \int_{c_Q}^{d_Q} l_Q(v, t) \frac{F_T\{\mu^{-1}(v)\}}{\lambda\{\mu^{-1}(v)\}} dv$$

$$= \int_{c_Q}^{d_Q} \frac{l_Q(v, t)}{\lambda\{\mu^{-1}(v)\}} dv \int_{a_T}^{b_T} k_T\{\mu^{-1}(v), s\} f(s)\, ds$$

$$= \int_{a_T}^{b_T} f(s)\, ds \int_{c_Q}^{d_Q} \frac{l_Q(v, t)\, k_T\{\mu^{-1}(v), s\}}{\lambda\{\mu^{-1}(v)\}} dv, \qquad (7)$$

assuming that the order of integration may be inverted.

If the function

$$j(s, t) = \int_{c_Q}^{d_Q} \frac{l_Q(v, t)\, k_T\{\mu^{-1}(v), s\}}{\lambda\{\mu^{-1}(v)\}} dv$$

can be evaluated explicitly, we have the transform pair

$$f(s) = \int_{a_Q}^{b_Q} g(s, t)\, i(t)\, dt, \qquad (8)$$

$$i(t) = \int_{a_T}^{b_T} j(s, t) f(s)\, ds. \qquad (9)$$

Use of an argument of this sort has been made by A. M. Efross [2], and later by A. Erdélyi, N. W. McLachlan and M. Parodi, in various places, to evaluate Laplace transforms.

2. As an example of the application of the process we consider the integral

$$F(p) = 2^{\nu} \int_0^{\infty} \left(\frac{p}{q}\right)^{\frac{\nu}{2}} K_{\nu}(\sqrt{(pq)})\, g(q)\, dq. \qquad (10)$$

Then [1, 5.16 (40)], provided that Rl $q > 0$,

$$f(t) = \tfrac{1}{2} t^{-\nu-1} \int_0^{\infty} e^{-q/4t} g(q)\, dq$$

$$= \tfrac{1}{2} t^{-\nu-1} G\left(\frac{1}{4t}\right), \qquad (11)$$

where F and G are the Laplace transforms of f and g. Hence

$$G(s) = 2^{-1-2\nu} s^{-1-\nu} f\left(\frac{1}{4s}\right),$$

and so

$$g(q) = \frac{2^{-1-2\nu}}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} s^{-1-\nu} e^{qs} ds \frac{1}{2\pi i} \int_{c_2 - i\infty}^{c_2 + i\infty} e^{p/4s} F(p)\, dp.$$

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The order of integration may be inverted, and we have

$$g(q) = \frac{2^{-1-\nu}}{2\pi i}\int_{c_2 - i\infty}^{c_2 + i\infty} \left(\frac{q}{p}\right)^{\frac{1}{2}\nu} I_\nu(\sqrt{(pq)})\, F(p)\, dp, \tag{12}$$

provided that Rl $\nu > -1$, since [1,5·5 (35)]

$$\frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} e^{qs + p/4s} s^{-1-\nu} ds = \left(\frac{4q}{p}\right)^{\frac{1}{2}\nu} I_\nu\,(\sqrt{(pq)}),$$

provided that Rl $\nu > -1$.

We have thus obtained essentially the pair

$$\phi(x) = \int_0^\infty K_\nu(\sqrt{(xy)})\, \psi(y)\, dy, \tag{13a}$$

$$\psi(y) = \frac{1}{4\pi i}\int_{c - i\infty}^{c + i\infty} I_\nu(\sqrt{(xy)})\phi\,(x)\, dx, \tag{13b}$$

in which y is a real positive variable, x a complex variable, and Rl $\nu > -1$.

A simple example of this transformation is supplied by the pair

$$\psi(y) = 1, \qquad \phi(x) = \frac{2\Gamma(1 + \frac{1}{2}\nu)\,\Gamma(1 - \frac{1}{2}\nu)}{x}.$$

3. It is of greater interest to consider an example in which different transformations play a part. A combination of the Laplace and Fourier transformations will produce a pair of formulae due to A. Erdélyi, but unpublished by him [3, p. 189]. The formulae are

$$\phi(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{M_{K,\,m}(ix)}{i^{m+\frac{1}{2}}\sqrt{x}}\, w(K)\, dK, \tag{14}$$

$$(m,\, x \text{ real};\ x > 0,\ m > -\tfrac{1}{2})$$

$$w(K) = e^{iK\pi}\frac{\Gamma(\frac{1}{2} + m + K)\Gamma(\frac{1}{2} + m - K)}{\{\Gamma(1 + 2m)\}^2}\int_0^\infty \frac{M_{-K,\,m}(-iy)}{(-i)^{m+\frac{1}{2}}\sqrt{y}}\,\phi(y)\, dy. \tag{15}$$

They may be derived by the process of section 1 above from the results

$$\mathcal{L}_p\left\{\frac{x^{m-\frac{1}{2}}}{i^{m+\frac{1}{2}}} M_{K,m}(ix)\right\} = \frac{\Gamma(1 + 2m)}{(p^2 + \frac{1}{4})^{\frac{1}{2}+m}} \exp\left(-\, 2iK \tan^{-1}\frac{1}{2p}\right) \tag{16}$$

provided that $m > -\frac{1}{2}$; and

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$$\frac{M_{ih,g}(ix)}{(ix)^{g+\frac{1}{2}}} = \frac{\Gamma(1+2g)}{2^{1+2g}\,\Gamma(\frac{1}{2}+g+ih)\,\Gamma(\frac{1}{2}+g-ih)}\int_{-\infty}^{+\infty} e^{\frac{1}{2}ix\,\mathrm{th}\,u - ihu}\,\frac{du}{\mathrm{ch}^{1+2g}u}, \quad (17)$$

which holds for real positive x and Rl $g > -\frac{1}{2}$.

Added in proof.

It appears that the transformation described by equations (13) is called the K-transformation and was introduced by C. S. Meijer in 1940. For further information and a table of transforms see Reference 1, Vol II.

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