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WHICH ORDERED SETS HAVE A COMPLETE LINEAR EXTENSION?

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Introduction. It is a well known and useful fact [**4**] that every (partially) ordered set P has a *linear extension* L (that is, a totally ordered set (chain) on the same underlying set as P and satisfying $a \leqq b$ in L whenever $a \leqq b$ in P). It is just as well known that an ordered set P can be embedded in an ordered set P' which, in turn, has a *complete* linear extension L' (that is, a linear extension in which every subset has both a supremum and an infimum); just take L' to be the "completion by cuts" of L. However, an arbitrary ordered set P need not, itself, have a complete linear extension (for example, if P is the chain of integers or, for that matter, if P is any noncomplete chain). It is natural to ask which ordered sets have a complete linear extension?

A routine (although not entirely trivial) argument will show that a completely distributive, complete lattice does have a complete linear extension. It is a plausible conjecture that every complete lattice has a complete linear extension. The purpose of this paper is twofold: first, to prove that the conjecture is true for a rather wide class of ordered sets; second, to illustrate that the conjecture is, in general, false.

A positive answer even for certain simple-minded examples involves some care. For instance, consider the complete lattice P consisting of the disjoint (cardinal) sum of two copies of the real chain together with a least element. Formally, take

$$P = \{(0, x) | x \in [0, 1[\} \cup \{(x, 0) | x \in [0, 1[\} \cup \{(1, 1)\}$$

with the componentwise ordering, where $[0, 1[$ denotes the unit interval of real numbers with the usual ordering. Then the linear extension with $(0, x) \leqq (x, 0) \leqq (0, y)$, for all $x, y \in [0, 1[$ such that $x < y$, is complete. As an alternative, the partition

$$\{(0,0)\} + \sum_{n \geqq 0} \left(\left\{ (0, x) \,\middle|\, \frac{2^n - 1}{2^n} < x \leqq \frac{2^{n+1} - 1}{2^{n+1}} \right\} + \left\{ (x, 0) \,\middle|\, \frac{2^n - 1}{2^n} < x \leqq \frac{2^{n+1} - 1}{2^{n+1}} \right\} \right) + \{(1, 1)\}$$

also induces a complete linear extension of P.

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In fact, the results presented here concern structures more general than lattices. We call an ordered set P *chain-complete* if every maximal chain of P is complete; P is *locally chain-complete* if, for each $x \leqq y$ in P, the interval $[x, y] = \{z \in P | x \leqq z \leqq y\}$ is chain-complete. If P itself is a chain and P is locally chain-complete then we shall simply say that P is *locally complete*. Of course, an ordered set P is locally chain-complete if and only if each of its maximal chains is locally complete. Moreover, P is locally chain-complete if and only if $P' = P \cup \{0, 1\}$ is chain-complete, where 0 is the least element of P' and 1 is the greatest element of P'. Thus, the problem of devising a locally complete linear extension for a locally chain-complete ordered set is, in substance, equivalent to the problem of devising a complete linear extension for a chain-complete ordered set. Still, it is not at all clear that this is identical to the problem of constructing a complete linear extension for a complete lattice. (For instance, does a chain-complete ordered set have a complete linear extension if its completion by cuts has a complete linear extension?) Recall that a *regular* ordinal α is an ordinal α which is isomorphic to the least ordinal cofinal in α, that is, $\underset{\sim}{\alpha} \cong \mathrm{cf}\ \underset{\sim}{\alpha}$. Let P^d stand for the dual of P.

In summary the main results of this paper are

THEOREM 1. *Every locally chain-complete ordered set in which all antichains are finite has a locally complete linear extension.*

THEOREM 2. *Every countable, locally chain-complete ordered set has a locally complete linear extension.*

THEOREM 3. *Let $\underset{\sim}{\kappa}$ and $\underset{\sim}{\lambda}$ be nonisomorphic, uncountable, regular ordinals. Then $\underset{\sim}{\kappa}^d \times \underset{\sim}{\lambda}$ is locally chain-complete and has no locally complete linear extension. Moreover, the ordered set obtained from $\underset{\sim}{\kappa}^d \times \underset{\sim}{\lambda}$ by adjoining a least element and a greatest element is a complete (distributive) lattice and has no complete linear extension.*

According to Theorem 1 every complete lattice without infinite antichains has a complete linear extension; according to Theorems 2 and 3 every countable complete lattice has a complete linear extension and there is a complete lattice of cardinality $\underset{\sim}{\omega}_2$ which has no complete linear extension. We do not know whether a complete lattice of cardinality $\underset{\sim}{\omega}_1$ must have a complete linear extension.

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Examples. 1. *Completely distributive, complete lattices.* Disjoint sums and direct products of complete chains provide the simplest examples of

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chain-complete ordered sets. A complete linear extension of the disjoint sum of a family of complete chains is easy to manufacture by imposing a complete linear ordering on the index set and then by taking the lexicographical sum of the components (complete chains). The direct product of a family of complete chains is, of course, a completely distributive, complete lattice; moreover, it, too, has a complete linear extension but the argument requires a little care.

Let $(C_i|i \in I)$ be a family of complete chains and let P stand for the direct product $\prod_{i \in I} C_i$ with elements $\bar{x} = (x_i)$, where $x_i \in C_i$ $(i \in I)$. We suppose that the index set I is well-ordered and then order the elements of P lexicographically, that is, $\bar{a} = (a_i) \leqq (b_i) = \bar{b}$ if, for the "first" $i \in I$ with $a_i \neq b_i$, $a_i < b_i$ in C_i. We claim that this lexicographic order prescribes a complete linear extension of P. That it is a linear extension is obvious. To see that it is complete it is enough to show that every subset A of P has an infimum in the lexicographic order. For this consider the element $\bar{a} = (a_i)$ defined by

$$a_i = \inf_{C_i}\{x_i| \ \bar{x} = (x_j) \in A \text{ and } x_j = a_j \text{ for every } j < i\}$$

(with the convention that the infimum in C_i of the empty set is the greatest element of C_i). Then $\bar{a} = \inf A$ in the lexicographic order of P.

Actually, every completely distributive, complete lattice has a complete linear extension. The proof involves three steps.

(i) Let A be a subset of the direct product $P = \prod_{i \in I} C_i$ of a family $(C_i|i \in I)$ of complete chains. If A is closed in the product topology of P (where each C_i is equipped with the interval topology) then the restriction of the lexicographic order of P induces a complete linear extension of A. We show, that under these conditions, every subset X of A has an infimum in A with respect to the lexicographic order of P. To this end we define the sequences (X_i) and (x_i), $i \in I$, as follows:

$$X_i = \{(y_j) \in A \,|\, y_j = x_j \text{ for all } j < i\}$$

and

$$x_i = \inf_{\pi_i(X_i)} \pi_i(X \cap X_i)$$

where π_i is the ith projection of P, and, subject again to the convention, that the infimum in $\pi_i(X_i)$ of the empty set is the greatest element of $\pi_i(X_i)$. Note that $X_{i_0} = A$ for the "first" element i_0 of I. Suppose that X_j and x_j are well-defined and $X_j \neq \emptyset$, for all $j < i$. If i is a successor ordinal (in I) then x_{i-1} is defined and $x_{i-1} \in \pi_{i-1}(X_{i-1})$ so $X_i \neq \emptyset$. Let i be a limit ordinal. Now, for each $j < i$, $X_j \neq \emptyset$ and X_j is a closed subset of P. As each of the chains C_i is complete, each is, in the interval topology, compact, so, $P = \prod_{i \in I} C_i$ is itself compact. It follows that $X_i = \bigcap_{j<i} X_j$ is nonempty; in particular, the X_i's are well-defined.

Now, the projection map π_i is continuous and $X_i \neq \emptyset$ is compact, so $\pi_i(X_i) \neq \emptyset$ is compact, whence it is closed in C_i and, therefore, $\pi_i(X_i)$ is a complete subchain of C_i. Except for the empty set, the infimum in $\pi_i(X_i)$ agrees with that in C_i, so x_i is also well-defined.

Finally, we claim that $\bar{x} = (x_i)$ is the infimum of X in the lexicographic order (of P). First, $\{(x_i)\} = \bigcap_{i \in I} X_i$ and, for each $i \in I$, $X_i \subseteq A$, so $(x_i) \in A$. Second, if $\bar{y} = (y_i) \in A$ and $x_j = y_j$ for $j < i$, then $\bar{y} \in X \cap X_i$ so, in view of the definition of x_i, $x_i \leqq y_i$.

(Note that if A is an arbitrary subset of P then its infimum in the lexicographic order of P need not coincide with the infimum of its closure (in the product topology).)

(ii) We now claim that every complete sublattice of the direct product $P = \prod_{i \in I} C_i$ of a family $(C_i | i \in I)$ of complete chains is closed in the product topology of P. The topology can be described abstractly in terms of the lattice operations: a generalized sequence $(\bar{x}_\alpha)$ of elements of P converges to $\bar{x} \in P$ (in the product topology) if and only if

$$\bar{x} = \limsup (\bar{x}_\alpha) = \liminf (\bar{x}_\alpha),$$

where

$$\limsup (\bar{x}_\alpha) = \inf_\alpha \{\sup\{\bar{x}_\beta | \beta \geqq \alpha\}\}$$

and the lim inf $(\bar{x}_\alpha)$ is defined dually. Now, if A is a complete sublattice of P then it is completely distributive whence it follows that A is closed in the product topology of P.

(iii) Finally, we assert that every completely distributive, complete lattice is a complete sublattice of the direct product of complete chains. (This is a well known (and nontrivial) result of [**3**]. Note that every complete sublattice of a power set $\mathbf{2}^S$ is the lattice of "initial segments" of some ordered set; such sublattices are generated by their supremum irreducible elements.)

2. *Disjoint sums of locally complete chains.* Again, disjoint sums and direct products of locally complete chains provide among the simplest examples of locally chain-complete ordered sets. While a direct product of locally complete chains need not have a locally complete linear extension (cf. Theorem 3), every disjoint sum P of a well-ordered family $(C_i | i \in I)$ of locally complete chains does have a locally complete linear extension. To prove this we associate with each C_i a triple (X_i, Y_i, Z_i) of subsets of C_i as follows: if $\sup C_i \in C_i$ set $X_i = C_i$, $Y_i = \emptyset = Z_i$; if $\sup C_i \notin C_i$ but $\inf C_i \in C_i$ set $X_i = \emptyset = Z_i$, $Y_i = C_i$; otherwise, consider a subsequence (z_{in}) of C_i and set

$$X_i = \leftarrow \inf (z_{in})], \quad Y_i = [\sup (a_{in}) \rightarrow,$$

and

$$Z_i = C_i = \bigcup_{n\in\mathbf{N}}[z_{i_n}, z_{i_n+1}[,$$

where $\leftarrow a]$, $[b \rightarrow$, and $[a, b[$ have the obvious meaning. Finally, the ordering induced by the partition

$$\sum_{i\in I^d} X_i + \left(\sum_{n\in\mathbf{N}} \sum_{i\in I} [z_{i_n}, z_{i_n+1}[\right) + \sum_{i\in I} Y_i,$$

where I^d is the dual of I, determines a complete linear extension of P.

It also follows that a disjoint sum of ordered sets, each of which has a locally complete linear extension, does itself have a locally complete linear extension.

3. *Well-founded ordered sets.* The well-ordered chains are perhaps the simplest examples of locally complete chains. Thus, an ordered set in which there is no infinite descending chain, that is, a *well-founded ordered set*, is also locally chain-complete. Every well-founded ordered set has a locally complete linear extension; in fact, every well-founded ordered set has a well-ordered linear extension. This is accomplished by decomposing the ordered set into "levels", each the subset consisting of the minimal elements not in the preceding levels, to obtain a partition (of the ordered set into antichains) indexed by a well-ordered set.

Proof of theorem 1. Loosely speaking, the proof consists in showing that, the ordering, induced on the blocks of a partition of P into antichains, is itself locally complete.

For now let P be an arbitrary ordered set. Let $\sim$ be an equivalence relation on P and define a relation $\leqq$ on the equivalence classes of P by $\bar{a} \leqq \bar{b}$ if and only if, there is an integer n, and there are sequences $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ of elements of P satisfying

$$a \sim x_1 \leqq y_1 \sim x_2 \leqq y_2 \sim x_3 \leqq y_3 \ldots x_{n-1} \leqq y_{n-1} \sim x_n \leqq y_n \sim b.$$

We call the equivalence $\sim$ *orderable* if $\leqq$ is an order on $\{\bar{a}|a \in P\}$; in this case, we refer to $(\{\bar{a}|a \in P\}, \leqq)$ as the *quotient* of P (*induced by* $\sim$) and denote it by $P/\sim$. Of course, an equivalence relation $\sim$ on P is orderable if and only if there is an order-preserving map f of P onto an ordered set (e.g., $P/\sim$) satisfying $f(a) = f(b)$ if and only if $a \sim b$.

LEMMA 1 (See [**1**]). *On every ordered set P there is an orderable equivalence relation $\sim$, whose equivalence classes are antichains of P, and, such that $P/\sim$ is a chain. Moreover, if $P/\sim$ is complete, then P has a complete linear extension.*

Proof. Let $E(P)$ denote the set of all orderable equivalence relations of P each of whose equivalence classes is an antichain of P. As the identity relation is orderable, $E(P) \neq \emptyset$. Furthermore, $E(P)$ itself is ordered by $\sim \leqq \sim'$ if and only if, for every $a, b \in P$, $a \sim b$ implies $a \sim' b$. Then $E(P)$ has a maximal member and, every such maximal member $\sim$ of $E(P)$ induces on P a quotient $P/\sim$, which is a chain. Finally, on each equivalence class $\bar{a} \subseteq P$ of $\sim$ we can obviously prescribe a complete linear extension $L(\bar{a})$. Then the transitive closure of the orderings induced on P, by the $L(\bar{a})$'s $(a \in P)$, and, the quotient order of $P/\sim$, yields a complete linear extension of P.

We shall require one other elementary fact.

LEMMA 2 (See [**2**]). *Let κ be a regular cardinal and let P be an ordered set of cardinality κ. If P contains no infinite antichains then P contains a chain of cardinality κ.*

Now, let P be a locally chain-complete ordered set with no infinite antichains. Let $\sim$ be an orderable equivalence relation on P whose equivalence classes are antichains (whence finite), and, such that $P/\sim$ is a chain. According to Lemma 1 we need only show that $P/\sim$ is locally complete.

If, to the contrary, $P/\sim$ is not locally complete, then there are disjoint subsets $\bar{A}$, $\bar{B}$ of $P/\sim$ such that, for each $\bar{a} \in \bar{A}$ and $\bar{b} \in \bar{B}$, $\bar{a} < \bar{b}$, and, neither sup $\bar{A}$ nor inf $\bar{B}$ exists in $P/\sim$. There are then regular ordinals α, β and sequences $(\bar{a}_i | i < \underset{\sim}{\alpha})$, $(\bar{b}_j | j < \underset{\sim}{\beta})$ such that $(\bar{a}_i | i < \underset{\sim}{\alpha})$ is cofinal in $\bar{A}$ and $(\bar{b}_j | j < \underset{\sim}{\beta})$ is coinitial in $\bar{B}$. Since, for each $i < \underset{\sim}{\alpha}$ and $j < \underset{\sim}{\beta}$, $\bar{a}_i < \bar{b}_j$, there exist elements $x_{ij} < y_{ij}$ in P satisfying $\bar{a}_i \leqq \bar{x}_{ij} \leqq \bar{a}_{i'}$ for some $i' < \underset{\sim}{\alpha}$, and $\bar{b}_{j'} \leqq \bar{y}_{ij} \leqq \bar{b}_j$ for some $j' < \underset{\sim}{\beta}$.

Let $\underset{\sim}{\alpha} = \underset{\sim}{\beta}$. According to Lemma 2, we may suppose (possibly after relabelling) that $(x_{ii} | i < \underset{\sim}{\alpha})$ and $(y_{ii} | i < \underset{\sim}{\alpha})$ are chains; moreover, since, for each $i < \underset{\sim}{\alpha}$, $x_{ii} < y_{ii}$, we conclude that

$$\{x_{ii} | i < \underset{\sim}{\alpha}\} \cup \{y_{ii} | i < \underset{\sim}{\alpha}\}$$

is a chain of P. As P is locally chain-complete, it follows that there is $c \in P$ satisfying $x_{ii} \leqq c \leqq y_{ii}$, for each $i < \underset{\sim}{\alpha}$; whence, $\bar{a} \leqq \bar{c} \leqq \bar{b}$ for each $\bar{a} \in \bar{A}$ and $\bar{b} \in \bar{B}$.

Let $\underset{\sim}{\alpha} < \underset{\sim}{\beta}$. We may suppose (possibly after relabelling) that, for fixed $i < \underset{\sim}{\alpha}$, $\bar{x}_{ij} = \bar{x}_{ij'}$, for all $j, j' < \underset{\sim}{\beta}$, and since $\bar{x}_{ij} \subseteq P$ is finite, that $x_{ij} = x_{ij'}$, for all $j, j' < \underset{\sim}{\beta}$. Set $x_i = x_{ij}$ for $i < \underset{\sim}{\alpha}$. Again by Lemma 2, we may suppose (possibly after relabelling) that $\{x_i | i < \underset{\sim}{\alpha}\}$, $\{y_{ij} | j < \underset{\sim}{\beta}\}$ $(i < \underset{\sim}{\alpha})$ are chains and that, for each $i < \underset{\sim}{\alpha}$ and $j < \underset{\sim}{\beta}$, $x_i < y_{ij}$. Let us suppose that, for each $i < \underset{\sim}{\alpha}$ there is $i < i' < \underset{\sim}{\alpha}$ and $j < \underset{\sim}{\beta}$ such that $x_{i'} \nleqq y_{ij}$. (Note that for each $i < i' < \underset{\sim}{\alpha}$ and $j < \underset{\sim}{\beta}$ there is $j < j' < \underset{\sim}{\beta}$ satisfying $\bar{y}_{i'j'} < \bar{y}_{ij}$ in $P/\sim$, so $y_{ij} \nleqq y_{i'j'}$ in P.) Then there is an antichain $\{y_{1j_1}, y_{i_1j_2}, y_{i_2,j_3}, \ldots, y_{i_kj_{k+1}}, \ldots\}$ in P satisfying

$$1 < i_1 < i_2 < \ldots < i_k < \ldots < \underset{\sim}{\alpha},$$
$$j_1 < j_2 < j_3 < \ldots < j_{k+1} < \ldots < \underset{\sim}{\beta},$$

and

$$x_{i_1} \nleqq y_{1j_1}, x_{i_2} \nleqq y_{i_1j_2}, x_{i_3} \nleqq y_{i_2j_3}, \ldots x_{i_{k+1}} \nleqq y_{i_kj_{k+1}}, \ldots .$$

It follows that, for some $i < \alpha$, $x_{i'} < y_{ij}$ for all $i' < \alpha$ and $j < \beta$. Finally, there is $c \in P$ such that $x_{i'} \leqq c \leqq y_{ij}$ for all $i' < \alpha$ and $j < \beta$ and, since $\{\bar{x}_{i'} | i' < \alpha\}$ is cofinal in $\bar{A}$ and $\{\bar{y}_{ij} | j < \beta\}$ is coinitial in $\bar{B}$, we conclude that $\bar{a} \leqq \bar{c} \leqq \bar{b}$ for each $\bar{a} \in \bar{A}$ and for each $\bar{b} \in \bar{B}$.

Proof of theorem 2. Let P be a locally chain-complete, ordered set. Let $\mathbf{I}_P$ denote the set of *initial segments* I of P (that is, for each $x, y \in P$, if $y \in I$ and $x \leqq y$ then $x \in I$) so that each convex subset S of I has a locally complete linear extension $L(S)$. (Recall, a subset A of an ordered set P is *convex* if, for every $x, y, z \in P$, $x \leqq z \leqq y$ and $x, y \in A$ implies $z \in A$.) Let $\mathbf{F}_P$ denote the set of *final segments* F of P (that is, for each $x, y \in P$, if $x \in F$ and $x \leqq y$ then $y \in F$) so that each convex subset S of F has a locally complete linear extension $L(S)$. (Note that a subset A of an ordered set P is convex if and only if it is the intersection of an initial segment with a final segment.)

LEMMA. *Let P be a locally chain-complete ordered set.*

(i) *If $I \in \mathbf{I}_P$ and $J \in \mathbf{I}_{P\backslash I}$ then $I \cup J \in \mathbf{I}_P$; moreover, if $I \in \mathbf{I}_P$ and $P\backslash I \in \mathbf{F}_P$ then $P \in \mathbf{I}_P$ and P has a locally complete linear extension.*

(ii) *Every countable union of members of $\mathbf{I}_P$ is a member of $\mathbf{I}_P$.*

Proof. (i) Let $I \in \mathbf{I}_P$ and $J \in \mathbf{I}_{P\backslash I}$. We show first that $I \cup J$ has a locally complete linear extension. If I has a maximal element a, say, then $I\backslash\{a\}$ is a convex subset of I and $L(I\backslash\{a\}) + \{a\} + L(J)$ induces a locally complete linear extension of $I \cup J$. If J has a minimal element b, say, then again $J\backslash\{b\}$ is a convex subset of J and $L(I) + \{b\} + L(J\backslash\{b\})$ induces a locally complete linear extension of $I \cup J$. If neither I has a maximal element nor J has a minimal element then, as P is locally chain-complete, there is $a \in I$ and $b \in J$ such that $a \nleqq b$ and $b \nleqq a$. (Otherwise, for each $a \in I$ and $b \in J$, $a \leqq b$ and, as $I \cup J$ is locally chain-complete, a maximal chain of $I \cup J$ would contain either a maximal element of I or a minimal element of J.) Let

$$A = \{x \in I | x > a\} \text{ and } B = \{y \in J | y < b\}.$$

Now, A is convex in I, B is convex in J and, for each $x \in A$, $y \in B$, x is noncomparable to y. We can then construct a locally complete linear extension $L(A \cup B)$ of $A \cup B$ (cf., the second example above). Finally, $L(I\backslash(A \cup \{a\})) + \{a\} + L(A \cup B) + \{b\} + L(J\backslash(B \cup \{b\}))$ induces a locally complete linear extension of $I \cup J$. In essentially the same way we can construct a locally complete linear extension for each convex subset of $I \cup J$. It follows that $I \cup J \in \mathbf{I}_P$.

(ii) We now show that for any countable increasing sequence $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ of members of $\underset{\sim}{\boldsymbol{I}}_P$, the union $I = \bigcup_n I_n$ is a member of $\underset{\sim}{\boldsymbol{I}}_P$. Indeed, for each $0 < n < \underset{\sim}{\omega}$, choose $a_n \in I_n \backslash I_{n-1}$ and set

$$A_n = \{x \in P | x < a_n\},$$
$$J_n = (I_n \cup A_{n+1}) \backslash (I_{n-1} \cup A_n)$$

(and $J_0 = I_0 \cup A_1$). Evidently, $I = \bigcup_n J_n$ and, for each $0 < n < \underset{\sim}{\omega}$, a_n is a minimal element of J_n. It follows that $J_n \backslash \{a_n\}$ is a convex subset of I_{n+1} so $\{a_n\} + L(J_n \backslash \{a_n\})$ is a locally complete extension of J_n. Then

$$L(J_0) + \{a_0\} + L(J_1 \backslash \{a_1\}) + \ldots + \{a_n\} + L(J_n \backslash \{a_n\}) + \ldots$$

is a locally complete extension of I. In a similar fashion we can verify that each convex subset of I also has a locally complete linear extension.

By duality $\underset{\sim}{\boldsymbol{F}}_P$, too, satisfies the claims of this lemma. In particular, if there is $I \in \underset{\sim}{\boldsymbol{I}}_P$ and $F \in \underset{\sim}{\boldsymbol{F}}_P$ such that $I \cup F = P$ then $F \backslash I \in \underset{\sim}{\boldsymbol{I}}_{P \backslash I}$ and so P has a locally complete linear extension.

Let us now assume P is countable and that P has no locally complete linear extension. As P is countable and both $\underset{\sim}{\boldsymbol{I}}_P$ and $\underset{\sim}{\boldsymbol{F}}_P$ are closed with respect to countable increasing sequences, it follows that, with respect to set inclusion, both $I_0 = \sup \underset{\sim}{\boldsymbol{I}}_P$ and $F_1 = \sup \underset{\sim}{\boldsymbol{F}}_P$ exist, that

$$N_{1/2} = P \backslash (I_0 \cup F_1) \neq \emptyset,$$

and that

$$\underset{\sim}{\boldsymbol{I}}_{N_{1/2}} = \emptyset = \underset{\sim}{\boldsymbol{F}}_{N_{1/2}}.$$

Let $x_{1/2} \in N_{1/2}$, $\hat{P}_{1/2} = \{x \in N_{1/2} | x \leqq x_{1/2}\}$ and $\check{P}_{1/2} = \{y \in N_{1/2} | y \geqq x_{1/2}\}$. Then, according to the maximality of I_0 and F_1, we conclude that

$$\underset{\sim}{\boldsymbol{I}}_{P_{1/2}} = \emptyset = \underset{\sim}{\boldsymbol{F}}_{\check{P}_{1/2}} \text{ and } N_{1/4} = \hat{P}_{1/2} \backslash F_{1/2} \neq \emptyset \neq \check{P}_{1/2} \backslash I_{1/2} = N_{3/4},$$

where again $F_{1/2} = \sup F_{P_{1/2}}$ and $I_{1/2} = \sup I_{\check{P}_{1/2}}$. We can then choose $x_{1/4} \in N_{1/4}$, $x_{3/4} \in N_{3/4}$ and again construct $\hat{P}_{1/4}$, $\check{P}_{1/4}$, $\hat{P}_{3/4}$, $\check{P}_{3/4}$, $F_{1/4}$, $I_{1/4}$, $F_{3/4}$, $I_{3/4}$, and nonempty sets $N_{1/8}$, $N_{3/8}$, $N_{5/8}$, $N_{7/8}$. In this way we obtain a countable dense chain C consisting of the elements x_q in P, where q is a dyadic fraction in the unit interval. As P is locally chain-complete there is a maximal chain C' in P containing C which is uncountable. With this contradiction, the proof of Theorem 2 is complete.

Proof of theorem 3. Let $\underset{\sim}{\kappa}$, $\underset{\sim}{\lambda}$ be nonisomorphic, uncountable regular ordinals and let $P = \underset{\sim}{\kappa}^d \times \underset{\sim}{\lambda}$. Let us suppose that there is a locally complete, linear extension L of P.

For purposes of this proof call an initial segment I (respectively, final segment F) of P *bounded* if $\pi_{\underset{\sim}{\lambda}}(I)$ $(\pi_{\underset{\sim}{\kappa}}{}^d(F))$ is bounded in $\underset{\sim}{\lambda}$ $(\underset{\sim}{\kappa})$, where $\pi_{\underset{\sim}{\lambda}}$ $(\pi_{\underset{\sim}{\kappa}}{}^d)$ is the $\underset{\sim}{\lambda}$ $(\underset{\sim}{\kappa}^d)$ projection of P.

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Note that an initial segment I of P is bounded if and only if the final segment $P\backslash I$ is not bounded. Indeed, if I and F are bounded, initial and final segments, respectively, then $I \cup F \neq P$ so, if I is a bounded initial segment of P then $P\backslash I$ cannot be bounded. Now, suppose that neither I nor $P\backslash I = F$ is bounded. Then there is a sequence (x_i) in I and a sequence (y_k) in F satisfying

$$\pi_{\underset{\sim}{\lambda}}(x_i) > i \ (i \in \underset{\sim}{\lambda}) \text{ and } \pi_{\underset{\sim}{\kappa}}{}^d(y_k) > k \ (k \in \underset{\sim}{\kappa}).$$

If $\underset{\sim}{\lambda} < \underset{\sim}{\kappa}$ then, for some $l < \underset{\sim}{\kappa}$, $\pi_{\underset{\sim}{\lambda}}(y_k)$ is constant for all $l < k < \underset{\sim}{\kappa}$, say, $\pi_{\underset{\sim}{\lambda}}(y_k) = j$. But then j must be a bound of $\pi_{\underset{\sim}{\lambda}}(I)$ and so I is bounded. If $\underset{\sim}{\kappa} < \underset{\sim}{\lambda}$ a similar argument will show that F is bounded.

For each $a \in P$, set $I(a) = \{x \in P | x \leqq a \text{ in } L\}$ and, dually, set $F(a) = \{y \in P | y \geqq a \text{ in } L\}$. Obviously $I(a)$, respectively, $F(a)$, is an initial, respectively, final segment of L and, indeed, of P. Let

$$P_I = \{a \in P | I(a) \text{ is bounded}\} \text{ and}$$

$$P_F = \{a \in P | F(a) \text{ is bounded}\}.$$

Evidently, P_I is an initial segment of L, P_F is a final segment of L, and, for each $a \in P$, $a \in P_I$ if and only if $a \notin P_F$. (Note that $P\backslash I(a) = F(a)\backslash\{a\}$, so $I(a)$ is bounded if and only if $F(a)$ is not bounded.)

Now, as $P_F = P\backslash P_I$, the final segment P_F is bounded if and only if the initial segment P_I is not bounded. Therefore, either P_I or P_F is not bounded. We may suppose that P_I is not bounded and P_F is bounded. Then, for each $i \in \underset{\sim}{\lambda}$, let $f(i)$ be the least member of $\underset{\sim}{\kappa}$ such that $(f(i), i) \notin P_F$. We define a (countable) sequence (I_n) of bounded initial segments of L inductively as follows: $I_0 = I(f(0), 0)$ and

$$I_{n+1} = I_n \cup \bigcup_{0 \leqq j \leqq i} I(f(j), j),$$

where i is the least member of $\underset{\sim}{\lambda}\backslash\pi_{\underset{\sim}{\lambda}}(I_n)$. Evidently, each $I_n \subseteq P_I$ although, since $\underset{\sim}{\lambda} \cong \operatorname{cf} \underset{\sim}{\lambda} > \underset{\sim}{\omega}$, the sequence (I_n) is not cofinal in P_I. Also, there is no $a \in P_I$ such that $I(a) = \bigcup_n I_n$.

We show finally that $L\backslash \bigcup_n I_n$ has no minimal element from which it follows that $\sup_L \bigcup_n I_n$ cannot exist, contrary to the assumption that L is locally complete. Suppose that (k, i) is the minimal element of $L\backslash\bigcup_n I_n$. Then $(k, i) \in P_I$ and there is $k < l < \underset{\sim}{\kappa}$ such that $(l, i) \in \bigcup_n I_n$, so $(l, i) \in I_n$ for some n. It follows that $(f(i), i) \in I_{n+1}$ so $(k, i) \in I_{n+1}$ also, which is a contradiction. This completes the proof of Theorem 3.

In some respects this theorem is best possible: for any regular ordinal $\underset{\sim}{\kappa}$, both $\underset{\sim}{\kappa}^d \times \underset{\sim}{\kappa}$ and $\underset{\sim}{\kappa}^d \times \underset{\sim}{\omega}$ have a locally complete linear extension.

References

1. R. Bonnet and M. Pouzet, *Extension et stratification d'ensembles dispersés*, C.R. Acad. Sc. Paris *268* (1969), 1512–1515.

2. B. Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math. *63* (1941), 600–610.
3. G. N. Raney, *Completely distributive complete lattices*, Proc. Amer. Math. Soc. *3* (1952), 677–680.
4. E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fund. Math. *16* (1930), 386–389.

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